Permutaciones espaciales Gaussianas y proceso puntual de bosones

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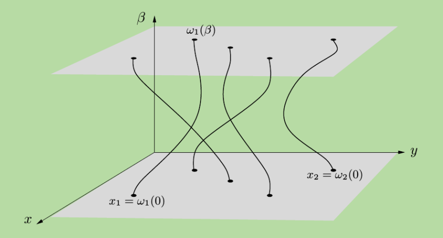


Picture by Daniel Ueltschi

Congreso Monteiro, Bahia Blanca 2019

En colaboración con Inés Armendáriz y Sergio Yuhjtman.

Proceso binomial en un segmento entero

Fijamos un entero positivo L y una probabilidad $p \in (0,1)$.

A cada entero en $[1,\ldots,L]$ le asignamos independientemente valor:

1 con probabilidad p.

0 con probabilidad $(1-p)$.

El número de 1's es Binomial(N,p).

El subconjunto $\chi \subset [1,\ldots,L]$ ocupado por los 1's es aleatorio:

Proceso puntual.

χ es una variable aleatoria en el espacio de subconjuntos de $[1,\ldots,L]$.

Proceso binomial en la caja $\Lambda = [1, L] \times [1, L] \subset \mathbb{Z}^2$

Lo mismo pero los N puntos se eligen uniformemente en una caja.

χ es el conjunto de puntos en la caja.

$|\chi|$ es el número de puntos elegidos.

$P(|\chi|) = k) = \binom{N}{k} p^k (1-p)^{n-k}$.

Configurationes con k puntos tienen probabilidad $p^k(1-p)^{n-k}$.

$\binom{N}{k}$ es el número de configurationes con k puntos.

Para subconjuntos $A, B \subset \Lambda$ las variables

$$|\chi \cap A| \text{ y } |\chi \cap B| \quad \text{son independientes}$$

$$|\chi \cap A| \sim \text{Binomial}(\tfrac{|A|}{L^2}\lambda)$$

Proceso de Poisson como límite de Binomial

Hacemos LN ensayos en los instantes $\frac{i}{N}$, $i = 0, \ldots, LN$.

Rescalamos $p = \lambda/N$

χ_N conjunto de puntos con valor 1.

Cuenta:

$$P(|\chi_N| = k) = \binom{LN}{k} (\tfrac{\lambda}{N})^k (1 - \tfrac{\lambda}{N})^{LN-k} \quad \to \quad e^{-\lambda L} \frac{(\lambda L)^k}{k!}$$

Cuando N es grande tenemos un número de puntos Poisson$(L\lambda)$.

Las cantidades de puntos en regiones disjuntas son independientes:

$$|\chi \cap A| \text{ y } |\chi \cap B| \quad \text{son independientes}$$

$$|\chi \cap A| \sim \text{Poisson}(|A|\lambda)$$

Definición de proceso de Poisson en un espacio general

Sea (D, m) un espacio de medida.

Por ejemplo $D = \mathbb{R}^d$ y $m(A) =$ volumen de A, medida de Lebesgue.

χ es un proceso de Poisson en subconjuntos localmente finitos de D si para A_i de medida finita, vale

$$|\chi \cap A_i| \quad \text{son independientes}$$

$$|\chi \cap A_i| \sim \text{Poisson}(m(A_i))$$

Ejemplo: proceso de Poisson de pares de puntos

$D = (\mathbb{R}^2)^2$ (pares de puntos de $\mathbb{R}^2$).

$$m(A \times B) = \int_A \int_B \lambda J(x,y) dy dx$$

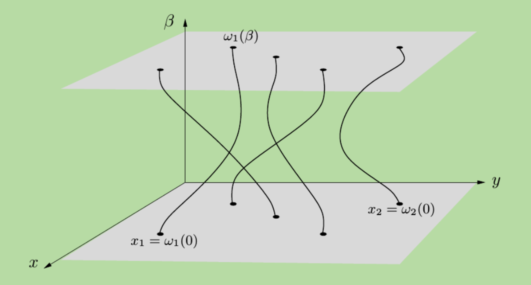
donde $J(x,y) = Ce^{-\|x-y\|^2/2}$ densidad de una Normal de media x calculada en y.

Construcción: X un proceso de Poisson en $\mathbb{R}^2$ con tasa λ y para cada $x \in X$ sea G_x una variable aleatoria con densidad $J(0,y)$ independientes e independientes de X.

El proceso es

$$\chi = \{(x, x + G_x) : x \in X\}$$

Mientras tanto, Feynmann en 1953 estudia el Gas de Bose...



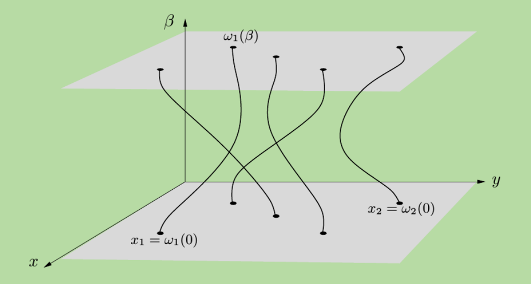
Picture by Daniel Ueltschi

Proceso de Poisson χ arriba (tiempo 0)

Mismo proceso χ abajo (tiempo t)

Brownianos que van desde los puntos de arriba a los puntos de abajo en tiempo t.

Mientras tanto, Feynmann en 1953 estudia el Gas de Bose...



Picture by Daniel Ueltschi

Si los Brownianos son independientes, queda es una biyección aleatoria

$$\sigma : \chi \to \chi$$

.

El par (χ, σ) es una *permutación espacial aleatoria.*

Permutaciones espaciales aleatorias en conjuntos acotados

$$P((\chi,\sigma)\in A) = \frac{1}{Z}\sum_{\sigma\in S_N}\int_{\Lambda^N} \mathbf{1}_{\{(\chi,\sigma)\in A\}}\; e^{-\alpha\sum_i \|x_i - x_{\sigma(i)}\|^2}\, dx_1 \ldots dx_N. \tag{1}$$

$\chi = \{x_1, \ldots, x_N\}$

Z : factor de normalización

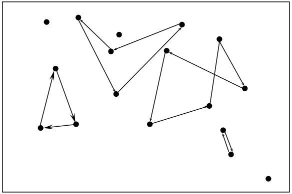
$\alpha > 0$ temperatura.

$S_N :=$ permutaciones de $\{1, \ldots, N\}$.

$\Lambda :=$ conjunto acotado de $\mathbb{R}^d$.

Permutaciones espaciales aleatorias en conjuntos acotados

$$P((\chi,\sigma)\in A) = \frac{1}{Z}\sum_{\sigma\in S_N}\int_{\Lambda^N} \mathbf{1}_{\{(\chi,\sigma)\in A\}}\; e^{-\alpha\sum_i \|x_i - x_{\sigma(i)}\|^2}\, dx_1 \ldots dx_N.$$



Realización típica de una permutación espacial aleatoria en un conjunto acotado Λ.

Trabajos precedentes

Puntos fijos, permutaciones aleatorias

Gandolfo, Ruiz, Ueltschi 2007 Points $= \mathbb{Z}^d$

Biskup Ritchammer 2015 Points $= \mathbb{Z}$ or PP in $d = 1$ quenched.

Armendáriz, F., Groisman, Leonardi 2015 Subcritical $\mathbb{Z}^d$

Marginal de puntos: Bosones

Macchi 1975

Fichtner 1991

Suto 1993 2002 Loop independence.

Shirai Takahashi 2003 Subcritical.

Tamura Ito 2006 2007 Supercritical: mixture of 2 processes

Eisenbaum 2008 Cox process, infinite divisibility

Marginal de permutaciones:

Ginivre 1971 cycle distribution

Benfatto Cassandro Merola Presutti 2005 macro cycle convergence

Betz Ueltschi 2006 2009 2010 2011 macro cycle Poisson-Dirichtlet

Elboim-Peled 2017 more Poisson-Dirichtlet

Permutaciones espaciales infinitas en volumen infinito $\mathbb{R}^d$

Vamos a construir

$$(\chi, \sigma) \text{ con distribución } \mu_\rho$$

χ: proceso puntual)

$\sigma : \chi \to \chi$ permutación

tal que μ_ρ is homogeneo y tiene densidad puntual ρ

μ_ρ es Gibbs (compatible) con la permutación espacial aleatoria en volumen finito (1)

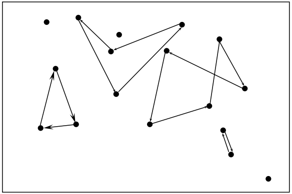
Biyección entre permutación espacial y sopa de ciclos espaciales

$(\chi, \sigma) \mapsto \Gamma$, the set of loops given by

Γ : sopa de ciclos.

La densidad factoriza para χ finito:

$$e^{-\alpha \sum_{x \in \chi} \|\sigma(x)-x\|^2} = \prod_{\gamma \in \sigma} e^{-\alpha \sum_{x \in \{\gamma\}} \|\gamma(x)-x\|^2}$$



Loops induced by a spacial random permutation in a box.

Sopa de ciclos Gaussiana en $\mathbb{R}^d$

D : Espacio de ciclos espaciales.

Medida de intensidad:

$$Q_\lambda^{\mathrm{ls}}(d[x_1, \ldots, x_k]) := \frac{\lambda^k}{k} \left(\frac{\alpha}{\pi}\right)^{d/2} e^{-\alpha \sum_{i=1}^k \|x_i - \gamma(x_i)\|^2} \, dx_1 \ldots dx_k,$$

Sopa de ciclos Gaussiana:

$$\Gamma_\lambda^{\mathrm{ls}} := \text{ Proceso de Poisson en } D \text{ con intensidad } Q_\lambda^{\mathrm{ls}},$$
$$\mu_\lambda^{\mathrm{ls}} := \text{ Distribución de } \Gamma_\lambda^{\mathrm{ls}}.$$

(Brownian loop soup. Lawler and Werner 2004, Lawler and Trujillo Ferreras 2007, Le Jan 2017.)

Densidad de puntos

$$\rho(\lambda) = \left(\frac{\alpha}{\pi}\right)^{d/2} \sum_{k\geq 1} \frac{\lambda^k}{k^{d/2}}$$

Densidad crítica:

$$\rho_c := \rho(1) = \left(\frac{\alpha}{\pi}\right)^{d/2} \sum_{k\geq 1} \frac{1}{k^{d/2}} \qquad \begin{cases} = \infty & \text{if } d \leq 2 \\ < \infty & \text{if } d \geq 3. \end{cases}$$

$d \geq 3$:

Sopa Gaussiana no puede tener densidad puntual mayor que ρ_c.

Interlazos Gaussianos $d \geq 3$.

$W :=$ Trayectorias doblemente infinitas.

$$W := \left\{ w : \mathbb{Z} \to \mathbb{R}^d, \lim_{n \to \pm\infty} \|w(n)\| = \infty \right\}$$

P^x distribución de paseo aleatorio empezando en x con incrementos Gaussianos.

$n_A(w) :=$ Número de visitas a A:

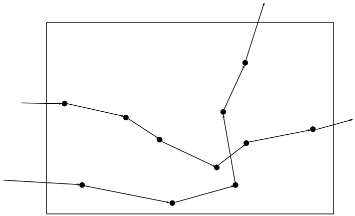
Intensidad

$$Q_A g := \int_A E^x \left[\frac{g}{n_A} \right] dx.$$

Se extiende a una medida de intensidad infinita Q.

Interlazos Gaussianos:

$$\Gamma_\rho^{\mathrm{ri}} := \text{ Proceso de Poisson en } \widetilde{W} \text{ con intensidad } \rho Q,$$
$$\mu_\rho^{\mathrm{ri}} := \text{ Distribución de } \Gamma_\rho^{\mathrm{ri}}.$$



(Sznitman 2010 Brownian interlacements)

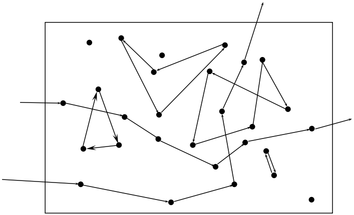
Densidad puntual de los interlazos Gaussianos es ρ.

Permutación espacial Gaussiana a densidad $\rho > 0$:

$$(\chi, \sigma)_\rho := \Gamma^{\text{ls}}_{\lambda(\rho \wedge \rho_c)} \cup \Gamma^{\text{ri}}_{(\rho - \rho_c)^+}$$

Es una superposición de realizaciones independientes de:

- Sopa de ciclos Gaussianos a densidad $\min\{\rho, \rho_c\}$
- Interlazos Gaussianos a densidad $(\rho - \rho_c)^+$.



• Sopa de ciclos Gaussiana condicionada a n puntos en Λ tiene distribución (1).

• Permutación espacial aleatoria es Gibbs (compatible) para las especificaciones inducidas por (1).

• La marginal de puntos de la sopa de ciclos Gaussiana es un proceso permanental:

para $\lambda \in (0,1]$, $\nu_\lambda^{\mathrm{ls}}$ tiene correlaciones

$$\varphi_\lambda^{\mathrm{ls}}(x_1, ..., x_n) = \mathrm{perm}\left(K_\lambda(x_i, x_j)\right)_{i,j=1}^n ,$$

• Cálculo explícito de las correlaciones de los interlazos Gaussianos.

• Las marginales puntuales coinciden con las estudiadas anteriormente por Tamura-Ito, Shirai-Takahashi y Eisenbaum

Correlaciones

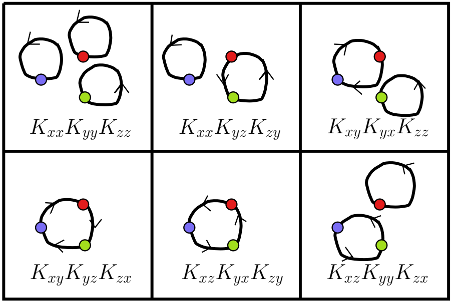


Figure 1: Gaussian loop soup 3-point correlations. A directed lace between two points means that the points belong to the same loop in the order indicated (relevant only for 3 or more points in the same loop). Point x is blue, y is red and z is green.

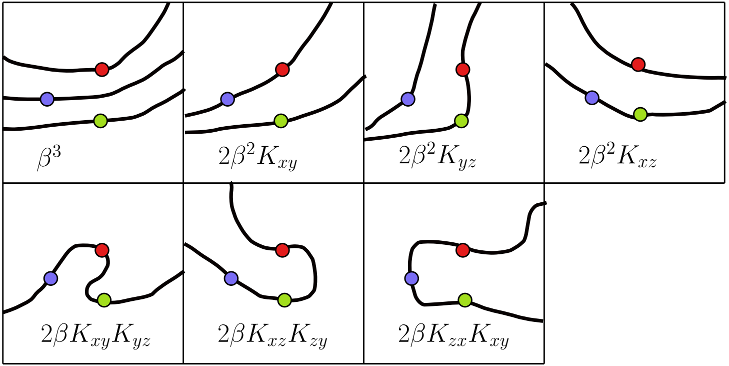


Figure 2: Gaussian random interlacements 3-point correlations. Each square represents one set of the trajectory partition. The point x is blue, y is red and z is green.

Current problems.

Thermodynamic limit of canonical measure.

Gaussian random permutation in a box Λ converges to the infinite volume GRP constructed here.

Extensions: Poisson process on $\mathbb{Z}^d$

Other interactions besides Gaussian.

Quantum case, when the Brownian trajectories from x to y interact. (BCMP 2005 treated the mean field case).

Gracias!

Conditioned Gaussian loop soup is Canonical permutation

$$\mathcal{H}_{\Lambda,N} := \{\Gamma : \textstyle\sum_{\gamma\in\Gamma}|\gamma|\mathbf{1}_{\{\gamma\}\subset\Lambda} = N\}$$

Recall:

$$\mu_\lambda^{\mathrm{ls}} = \text{ loop soup;}$$

$$G_{\Lambda,N}g = \frac{1}{Z_{\Lambda,N}} \sum_{\sigma\in\mathcal{S}_N} \int_{\Lambda^N} g(x,\sigma)e^{-H(x,\sigma)}dx \quad \text{canonical}$$

Proposition 1. *For g depending on the cycles totally included in Λ:*

$$\mu_\lambda^{\mathrm{ls}}(g|\mathcal{H}_{\Lambda,N}) = G_{\Lambda,N}g.$$

Proof. Density of loop soup is the product of intensity of its cycles. Same as density given by the permutation. □

Specifications Λ compact, (χ, σ) and (ζ, κ) spatial permutations.

Λ-compatibility: $(\chi, \sigma) \sim_\Lambda (\zeta, \kappa)$ if

$$\begin{aligned} \chi \setminus \Lambda &= \zeta \setminus \Lambda, \\ \sigma(x) &= \kappa(x) \quad \text{if } \{x, \kappa(x)\} \subset \zeta \setminus \Lambda, \\ \sigma^{-1}(\chi \cap \Lambda) \setminus \Lambda &= \kappa^{-1}(\zeta \cap \Lambda) \setminus \Lambda, \\ \sigma(\chi \cap \Lambda) \setminus \Lambda &= \kappa(\zeta \cap \Lambda) \setminus \Lambda. \end{aligned}$$

Conditioned Hamiltonian:

$$H_\Lambda((\chi, \sigma)|(\zeta, \kappa)) := \sum_{x \in [\chi \cap \Lambda] \cup [\kappa^{-1}(\zeta \cap \Lambda) \setminus \Lambda]} \|x - \sigma(x)\|^2.$$

Specification: measure $G_{\Lambda,\lambda}(\cdot|(\zeta, \kappa))$ with density

$$f_{\Lambda,\lambda}((\chi, \sigma)|(\zeta, \kappa)) := \frac{1}{Z_{\Lambda,\lambda}(\zeta, \kappa)} \lambda^{|\chi \cap \Lambda|} \exp\big(- \alpha H_\Lambda((\chi, \sigma)|(\zeta, \kappa))\big),$$

Gaussian random permutation on $\mathbb{R}^d$ is Gibbs

Theorem 2 (AFY). *For $d \geq 3$ and $\lambda \leq 1$ the loop soup measure*

$$\mu_\lambda^{\text{ls}} \textit{ is Gibbs for the specifications } (G_{\Lambda,\lambda} : \Lambda \textit{ compact}):$$

$$\mu_\lambda^{\text{ls}} g = \int d\mu_\lambda^{\text{ls}}(\zeta, \kappa)\, G_{\Lambda,\lambda}(g|(\zeta, \kappa)) \qquad \text{DLR}$$

For all $\rho \geq \rho_c$ the measure

$$\mu_1^{\text{ls}} * \mu_{\rho-\rho_c}^{\text{ri}} \textit{ is Gibbs for the specifications } (G_{\Lambda,1} : \Lambda \textit{ compact}). \tag{2}$$

Corollary 3. *Point and permutation marginals can be computed explicitely.*

Point marginal of Gaussian loop soup is permanental

$$\nu_\lambda^{\mathrm{ls}} := \text{Point marginal of Gaussian loop soup } \mu_\lambda^{\mathrm{ls}}$$

Recall fugacity $\lambda \in (0,1]$ and

$$K_\lambda(x,y) := \sum_{k\geq 1} \lambda^k \left(\frac{\alpha}{\pi k}\right)^{d/2} e^{-\frac{\alpha}{k}\|x-y\|^2}, \quad x,y \in \mathbb{R}^d$$

For $\lambda \in (0,1]$, $\nu_\lambda^{\mathrm{ls}}$ has correlations

$$\varphi_\lambda^{\mathrm{ls}}(x_1,...,x_n) = \mathrm{perm}\left(K_\lambda(x_i,x_j)\right)_{i,j=1}^n ,$$

(Permanental point process).

AFY using the density of the Poisson process $\nu_\lambda^{\mathrm{ls}}$.

Point marginal of Gaussian interlacements

$$\nu_\rho^{\mathrm{ri}} := \text{Point marginal of Gaussian interlacements } \mu_\rho^{\mathrm{ri}}$$

Correlations:

$$\varphi_\rho^{\mathrm{ri}}(x_1, \ldots, x_n) = \sum_{P \in \mathcal{P}_n} \prod_{I \in P} \sum_{\sigma \in \mathcal{S}_I} V_\rho(x_{\sigma(i_1)}, \ldots, x_{\sigma(i_{|I|})}). \tag{3}$$

$\mathcal{P}_n :=$ partitions of $\{1, \ldots, n\}$ with nonempty sets,
$\mathcal{S}_I :=$ permutations of set I,
$(i_1, \ldots, i_{|I|})$ arbitrary order of I and

$$V_\rho(x_1, \ldots, x_\ell) := \rho K(x_1, x_2) \ldots K(x_{\ell-1}, x_\ell)$$

AFY using the density of the Poisson process ν_ρ^{ri}.

Thermodynamic limit of Point marginal

Theorem (Shirai-Takahashi, Tamura-Ito). *Fix density* $\rho > 0$.

$G^{\text{point}}_{\Lambda,|\Lambda|\rho} :=$ *law of* X*-marginal with* $|\Lambda|\rho$ *points.*

$$G^{\text{point}}_{\Lambda,|\Lambda|\rho} \Rightarrow \nu^{\text{TI}}_{\rho} \text{ as } \Lambda \nearrow \mathbb{R}^d$$

Subcritical case. $\rho \leq \rho_c$ or $d \leq 2$ Fichtner 1991; Tamura-Ito 2006.

$\nu^{\text{TI}}_{\rho} = \nu^{\text{ls}}_{\lambda(\rho)}$, the point marginal of loop soup at intensity $\lambda(\rho)$.

Supercritical case. $\rho > \rho_c$ and $d \geq 3$ Tamura-Ito 2007.

$$\nu^{\text{point}}_{\rho} = \nu^{\text{TI}}_{\rho_c} * \nu^{\infty}_{\rho-\rho_c}. \tag{4}$$

$\nu^{\infty}_{\rho} = \nu^{\text{ri}}_{\rho}$, the point marginal of Gaussian interlacements at ρ.

Partial "Thermodynamic limit" of permutation marginal

$G^{\text{permut}}_{\Lambda,|\Lambda|\rho} :=$ σ-marginal of $G_{\Lambda,|\Lambda|\rho}$

$$G^{\text{permut}}_{\Lambda,\rho} \Rightarrow \nu^{\text{permut}}_{\rho} \text{ for } \textit{cycle-size distribution}.$$

Macroscopic cycles: cycles with size bigger than $\varepsilon|\Lambda|$.

Subcritical case. $\rho \leq \rho_c$ or $d = 1, 2$

The expected fraction of points in macroscopic cycles is zero. BU 2011

Supercritical case. $\rho > \rho_c$ or $d \geq 3$

(a) expected fraction of points in macroscopic cycles is $\frac{\rho-\rho_c}{\rho}$.

(b) Rescaled macroscopic cycles have random lenght:
Benfatto, Cassandro, Merola Presutti 2005.

Poisson-Dirichlet distribution (as uniform permut): Betz-Ueltschi 2011.

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