Un teorema de convergencia para modelos exponenciales con dispersión bivariados

Lila Ricci, Gabriela Boggio

Centro Marplatense de Investigaciones Matemáticas - FCEyN - UNMdP
Facultad de Ciencias Económicas y Estadística - UNR

June 4, 2019

Exponential Dispersion Models

1947 Tweedie pointed out the main properties of *EDM*s;
but his ideas remained unknown for decades.

Exponential Dispersion Models

1947 Tweedie pointed out the main properties of *EDM*s; but his ideas remained unknown for decades.

1972 Nelder and Wedderburn introduced Natural Exponential Family; they can represent the error distribution in generalized linear models (*GLM*s).

Exponential Dispersion Models

1947 Tweedie pointed out the main properties of *EDM*s; but his ideas remained unknown for decades.

1972 Nelder and Wedderburn introduced Natural Exponential Family; they can represent the error distribution in generalized linear models (*GLM*s).

1987 Jørgensen rescued Tweedie's ideas and defined an extended family of distributions named Exponential Dispersion Model. They broaden the field of *GLM*s, allowing the researchers to choose, between infinite probability distributions, the one that optimally represents their data.

Exponential Dispersion Models

1947 Tweedie pointed out the main properties of *EDM*s; but his ideas remained unknown for decades.

1972 Nelder and Wedderburn introduced Natural Exponential Family; they can represent the error distribution in generalized linear models (*GLM*s).

1987 Jørgensen rescued Tweedie's ideas and defined an extended family of distributions named Exponential Dispersion Model. They broaden the field of *GLM*s, allowing the researchers to choose, between infinite probability distributions, the one that optimally represents their data.

2012 Jørgensen and Martínez developed a unified methodology to build Multivariate Exponential Dispersion Models (*MEDM*s) with fixed known marginals and a flexible correlation structure.

Multivariate Exponential Dispersion Models

The method to obtain *MEDM*s is based on an extended convolution method.

Multivariate Exponential Dispersion Models

The method to obtain *MEDM*s is based on an extended convolution method.
It ensures a $k + k(k+1)/2$ parameters distribution for $k-$dimensional models, with marginal distributions that belong to the same family.

Multivariate Exponential Dispersion Models

The method to obtain *MEDM*s is based on an extended convolution method.

It ensures a $k + k(k+1)/2$ parameters distribution for $k-$dimensional models, with marginal distributions that belong to the same family.

These new *MEDM*s have a flexible covariance structure.

Construction of bivariate exponential dispersion models

The bivariate case is obtained as follows: let $\mathbf{Z} = (Z_1, Z_2)$ be expressed as

$$\mathbf{Z} = (U_1, U_2) + (U_1, 0) + (0, U_2)$$

where the three terms are independent with CGFs given by

$$(s_1, s_2) \to \lambda_{12}\kappa_\theta(s_1, s_2)$$
$$(s_1, s_2) \to \lambda_1\kappa_\theta(s_1, 0)$$
$$(s_1, s_2) \to \lambda_2\kappa_\theta(0, s_2)$$

with $\kappa_\theta(s_1, s_2) = \kappa(\theta_1 + s_1, \theta_2 + s_2) - \kappa(\theta_1, \theta_2)$, being κ the cumulant function of the generated bivariate natural exponential family.

The reproductive *MEDM* is defined by the scale transformation $\mathbf{Y} = \mathbf{Z}/\lambda$, the random vector $\mathbf{Y}$ has mean $\boldsymbol{\mu} = \dot{\kappa}(\boldsymbol{\theta})$ and covariance matrix

$$Cov(\mathbf{Y}) = \begin{bmatrix} \sigma_{11}\mu_1^2 & \sigma_{12}\phi\mu_1\mu_2 \\ \sigma_{12}\phi\mu_1\mu_2 & \sigma_{22}\mu_2^2 \end{bmatrix}$$

where σ_{ij} are the components of the dispersion matrix

$$\Sigma = \begin{bmatrix} \frac{1}{\lambda_{11}} & \frac{\lambda_{12}}{\lambda_{11}\lambda_{22}} \\ \frac{\lambda_{12}}{\lambda_{11}\lambda_{22}} & \frac{1}{\lambda_{22}} \end{bmatrix};$$

we will denote it by $Y \sim ED(\boldsymbol{\mu}, \Sigma)$.

Negative correlation

One slight disadvantage of the method is that only positive correlations are obtained. Recently Cuenin, Jrgensen and Kokonendji (2015) gave a variables-in-common method for constructing multivariate distributions admitting negative correlations, but it is restricted to Tweedie models.

A particular case: bivariate Gamma

While passing from uni to multivariate distributions there is more than one direction to choose.

A particular case: bivariate Gamma

While passing from uni to multivariate distributions there is more than one direction to choose.
Kibble and Moran bivariate Gamma distribution (see Kotz, 2000 and Letac, 2007) is defined by the cumulant function

$$\kappa(\theta_1, \theta_2) = -\log\left(\theta_1\theta_2 - \rho\right), \quad \rho > 0$$

with domain

$$\Theta = \left\{\left(\theta_1, \theta_2\right), \theta_1 < 0, \theta_2 < 0, \theta_1\theta_2 - \rho > 0\right\}.$$

A particular case: bivariate Gamma

While passing from uni to multivariate distributions there is more than one direction to choose.

Kibble and Moran bivariate Gamma distribution (see Kotz, 2000 and Letac, 2007) is defined by the cumulant function

$$\kappa(\theta_1, \theta_2) = -\log\left(\theta_1\theta_2 - \rho\right), \quad \rho > 0$$

with domain

$$\Theta = \left\{\left(\theta_1, \theta_2\right), \theta_1 < 0, \theta_2 < 0, \theta_1\theta_2 - \rho > 0\right\}.$$

The moment generating function (MGF) in terms of $\boldsymbol{\mu}$ is then

$$M(\mathbf{s}; \boldsymbol{\mu}, \Lambda) = \left(1 - \mu_1\frac{s_1}{\lambda_{11}} - \mu_2\frac{s_2}{\lambda_{22}} + \mu_1\mu_2\left(1-\phi\right)\frac{s_1 s_2}{\lambda_{11}\lambda_{22}}\right)^{-\lambda_{12}} \times \left(1 - \mu_1\frac{s_1}{\lambda_{11}}\right)^{-\lambda_1}\left(1 - \mu_2\frac{s_2}{\lambda_{22}}\right)^{-\lambda_2}. \quad (1)$$

A particular case: bivariate Gamma

While passing from uni to multivariate distributions there is more than one direction to choose.

Kibble and Moran bivariate Gamma distribution (see Kotz, 2000 and Letac, 2007) is defined by the cumulant function

$$\kappa(\theta_1, \theta_2) = -\log\left(\theta_1\theta_2 - \rho\right), \quad \rho > 0$$

with domain

$$\Theta = \left\{\left(\theta_1, \theta_2\right), \theta_1 < 0, \theta_2 < 0, \theta_1\theta_2 - \rho > 0\right\}.$$

The moment generating function (MGF) in terms of $\boldsymbol{\mu}$ is then

$$M(\mathbf{s}; \boldsymbol{\mu}, \Lambda) = \left(1 - \mu_1\frac{s_1}{\lambda_{11}} - \mu_2\frac{s_2}{\lambda_{22}} + \mu_1\mu_2\left(1-\phi\right)\frac{s_1 s_2}{\lambda_{11}\lambda_{22}}\right)^{-\lambda_{12}} \times \left(1 - \mu_1\frac{s_1}{\lambda_{11}}\right)^{-\lambda_1}\left(1 - \mu_2\frac{s_2}{\lambda_{22}}\right)^{-\lambda_2}. \qquad (1)$$

Note that when $\phi = 0$, meaning independence, (1) becomes

$$M(s_1, s_2; \boldsymbol{\mu}, \Lambda) = (1 - \mu_1\frac{s_1}{\lambda_{11}})^{-\lambda_{11}}(1 - \mu_2\frac{s_2}{\lambda_{22}})^{-\lambda_{22}}. \qquad (2)$$

Extending some results

Jørgensen, Martínez and Tsao (1994) proved an important theorem, that asses Gamma convergence of some *EDM*s under weaker conditions than those required for asymptotic convergence of variance functions.

Extending some results

Jørgensen, Martínez and Tsao (1994) proved an important theorem, that asses Gamma convergence of some *EDM*s under weaker conditions than those required for asymptotic convergence of variance functions.
The above mentioned authors conjectured that previous results might be extended to those *MEDM*s that they had just defined.

Extending some results

Jørgensen, Martínez and Tsao (1994) proved an important theorem, that asses Gamma convergence of some *EDM*s under weaker conditions than those required for asymptotic convergence of variance functions.
The above mentioned authors conjectured that previous results might be extended to those *MEDM*s that they had just defined.
Our goal is to extend to bi-dimensional space the result about convergence of *EDM*s to a Gamma model, that has been proved in the univariate case.

Extending some results

Jørgensen, Martínez and Tsao (1994) proved an important theorem, that asses Gamma convergence of some *EDM*s under weaker conditions than those required for asymptotic convergence of variance functions.
The above mentioned authors conjectured that previous results might be extended to those *MEDM*s that they had just defined.
Our goal is to extend to bi-dimensional space the result about convergence of *EDM*s to a Gamma model, that has been proved in the univariate case.

Necessary tools

An essential tool in the study of these domains of attraction has been the theory of **regularly varying functions**. that were defined by J. Karamata (see deHaan, 1975 and deHaan and Resnick, 1987), they behave asymptotically as their Laplace transforms.

Necessary tools

An essential tool in the study of these domains of attraction has been the theory of **regularly varying functions**. that were defined by J. Karamata (see deHaan, 1975 and deHaan and Resnick, 1987), they behave asymptotically as their Laplace transforms. The next definition was given by Omey and Willekens (1989), extending the concept of regular variation to $\mathbb{R}_+^2$.

Necessary tools

An essential tool in the study of these domains of attraction has been the theory of **regularly varying functions**. that were defined by J. Karamata (see deHaan, 1975 and deHaan and Resnick, 1987), they behave asymptotically as their Laplace transforms. The next definition was given by Omey and Willekens (1989), extending the concept of regular variation to $\mathbb{R}_+^2$.

Definition

A measurable function $u : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$ *is regularly varying at infinity (zero) with indexes* $\alpha, \beta \in \mathbb{R}$ *if* $\forall x\,, y > 0$ *and* $t > 0$, *the limit*

$$\lim_{\min(t,s)\to\infty(0)} \frac{u(tx, sy)}{u(t, s)} = x^{\alpha} y^{\beta},$$

exists and is finite.

Necessary tools

An essential tool in the study of these domains of attraction has been the theory of **regularly varying functions**. that were defined by J. Karamata (see deHaan, 1975 and deHaan and Resnick, 1987), they behave asymptotically as their Laplace transforms. The next definition was given by Omey and Willekens (1989), extending the concept of regular variation to $\mathbb{R}_+^2$.

Definition

A measurable function $u : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$ *is regularly varying at infinity (zero) with indexes* $\alpha, \beta \in \mathbb{R}$ *if* $\forall x\,, y > 0$ *and* $t > 0$, *the limit*

$$\lim_{\min(t,s)\to\infty(0)} \frac{u(tx, sy)}{u(t, s)} = x^{\alpha} y^{\beta},$$

exists and is finite.

Notation: $u \in VR\left(\alpha, \beta\right)_0$;

Necessary tools

An essential tool in the study of these domains of attraction has been the theory of **regularly varying functions**. that were defined by J. Karamata (see deHaan, 1975 and deHaan and Resnick, 1987), they behave asymptotically as their Laplace transforms. The next definition was given by Omey and Willekens (1989), extending the concept of regular variation to $\mathbb{R}_+^2$.

Definition

A measurable function $u : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$ *is regularly varying at infinity (zero) with indexes* $\alpha, \beta \in \mathbb{R}$ *if* $\forall x\,, y > 0$ *and* $t > 0$*, the limit*

$$\lim_{\min(t,s)\to\infty(0)} \frac{u(tx, sy)}{u(t, s)} = x^{\alpha} y^{\beta},$$

exists and is finite.

Notation: $u \in VR\,(\alpha, \beta)_0$;
If $\alpha = \beta = 0$ the function is said to be *slowly varying at infinity (zero)*, $L \in VL_{\infty(0)}$.

Regular variation of measures

The concept of regular variation can be extended to measures as follows.

Definition

A measure ν *on* $\mathbb{R}_+^2$ *is said to vary regularly at infinity or zero with indexes* $\alpha, \beta \in \mathbb{R}$ *if the distribution function* $\bar{\nu}(x, y) = \nu\{(0, x] \times (0, y]\}$ *does.*

Bivariate Karamata theorems

Hereafter the notation "$f(x) \sim kg(x)$ when $x \to \infty$" means that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = k$.

Bivariate Karamata theorems

Hereafter the notation "$f(x) \sim kg(x)$ when $x \to \infty$" means that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = k$.
Next we extend a theorem stated by Jørgensen, Martínez and Tsao (1994), that relates regular variation of a measure with regular variation of its Laplace transform.

Theorem

Let ν be a measure on $\mathbb{R}_+^2$ with Laplace transform $\omega(\cdot,\cdot)$, then

$$\bar{\nu}(t,s) \sim \frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)} t^{\alpha} s^{\beta} L(t,s) \Longleftrightarrow \omega\left(\frac{1}{t},\frac{1}{s}\right) \sim t^{\alpha} s^{\beta} L(t,s)$$

when $\min(t,s) \to \infty$, $L \in VL_\infty$, α and β being non negative numbers and $\bar{\nu}$ the function given in the previous definition.

We proved that the statement on the left is equivalent to affirm that $\bar{\nu} \in VR(\alpha, \beta)_\infty$.

We proved that the statement on the left is equivalent to affirm that $\bar{\nu} \in VR(\alpha, \beta)_{\infty}$.
Now let ν be a measure of the form

$$\nu\left(dx, dy\right) = g\left(x, y\right) x^{\alpha-1} y^{\beta-1} dxdy,$$

g being analytic and zero at $(0, 0)$, then $\nu \in VR(\alpha, \beta)_0$. Theorem 1 allows us to say that the MGF of the natural exponential family generated by such a measure takes the form

$$M_{\nu}(\theta_1, \theta_2) = \left(-\theta_1\right)^{-\alpha} \left(-\theta_2\right)^{-\beta} L\left(-\theta_1, -\theta_2\right),\ \theta_1, \theta_2 < 0, \qquad (3)$$

where $L\left(-\theta_1, -\theta_2\right) \in VL_{\infty}$.

Bivariate Karamata representation

de Haan and Resnick (1987) proved an extension of Karamata representation to the multivariate regular variation case; we are interested in the particular case of bivariate slow variation.

Bivariate Karamata representation

de Haan and Resnick (1987) proved an extension of Karamata representation to the multivariate regular variation case; we are interested in the particular case of bivariate slow variation.
A slowly varying function $L : \mathbb{R}_+^2 \to \mathbb{R}$ can be represented as

$$L(x, y) = d(x, y) \exp\left\{ \int_1^{\|(x,y)\|} \frac{a(t, t)}{t} dt \right\} \tag{4}$$

where $d : \mathbb{R}_+^2 \to \mathbb{R}$ and $a : \mathbb{R}_+^2 \to \mathbb{R}$ such that
$\lim_{t\to\infty} | d(tx, ty) - d_0 |= 0$ for some $0 < d_0 < \infty$ and
$\lim_{t\to\infty} a(t, t) = 0$.

The main result

Theorem

Let $Y \sim ED(\boldsymbol{\mu}, \Sigma)$ be a bivariate EDM generated by the measure ν with support $S \subseteq (0,\infty) \times (0,\infty)$. Suppose that ν is regularly varying at zero or infinity with the same index on both variables. Given (3) and if $l(x,y)$ satisfies:

$$\lim_{c \to 0(\infty)} \frac{1}{c} \frac{\partial l\left(-\delta_1(c\boldsymbol{\mu}), -\delta_2(c\boldsymbol{\mu})\right)}{\partial \delta_i(c\boldsymbol{\mu})} = 0, \quad i = 1,2 \qquad (5)$$

for all $\boldsymbol{\mu} \in \mathbb{R}^2_+$ and $\Sigma = \begin{bmatrix} \frac{1}{\lambda_{11}} & \frac{\lambda_{12}}{\lambda_{11}\lambda_{22}} \\ \frac{\lambda_{12}}{\lambda_{11}\lambda_{22}} & \frac{1}{\lambda_{22}} \end{bmatrix}$ with $\lambda_{11} > 0$, $\lambda_{22} > 0$ and $\lambda_{12} \geq 0$, then

$$\frac{1}{c} ED(c\boldsymbol{\mu}, \Sigma) \underset{c \to 0(\infty)}{\xrightarrow{d}} \Gamma\left(\boldsymbol{\mu}, \Sigma_\alpha\right)$$

where Γ is the bivariate Gamma defined previously and

$$\Sigma_\alpha = \begin{bmatrix} \frac{1}{\alpha\lambda_{11}} & 0 \\ 0 & \frac{1}{\ } \end{bmatrix}.$$

Proof

Let $\mathbf{Z} = (Z_1, Z_2)^T \sim ED^*(\boldsymbol{\theta}, \Lambda)$ be the bivariate additive *EDM* generated by ν, constructed as described above with *MGF*

$$M^*(\mathbf{s}; \boldsymbol{\theta}, \Lambda) = \frac{\left[e^{\kappa(\theta_1+s_1,\theta_2)}\right]^{\lambda_1} \left[e^{\kappa(\theta_1,\theta_2+s_2)}\right]^{\lambda_2} \left[e^{\kappa(\theta_1+s_1,\theta_2+s_2)}\right]^{\lambda_{12}}}{\left[e^{\kappa(\theta_1,\theta_2)}\right]^{\lambda_1+\lambda_2+\lambda_{12}}}.$$

Replacing by κ we obtain

$$\begin{aligned} M^*(\mathbf{s}; \boldsymbol{\theta}, \Lambda) = & \left(1 + \frac{s_1}{\theta_1}\right)^{-\alpha\lambda_{11}} \left(1 + \frac{s_2}{\theta_2}\right)^{-\alpha\lambda_{22}} \\ & \times \left[\frac{L\left(-\theta_1 - s_1, -\theta_2\right)}{L\left(-\theta_1, -\theta_2\right)}\right]^{\lambda_1} \left[\frac{L\left(-\theta_1, -\theta_2 - s_2\right)}{L\left(-\theta_1, -\theta_2\right)}\right]^{\lambda_2} \\ & \times \left[\frac{L\left(-\theta_1 - s_1, -\theta_2 - s_2\right)}{L\left(-\theta_1, -\theta_2\right)}\right]^{\lambda_{12}} \end{aligned}$$

Given the scale transformation $\mathbf{Y} = (Y_1, Y_2)^T = \left(\frac{Z_1}{\lambda_{11}}, \frac{Z_2}{\lambda_{22}}\right)^T$ and MGFs properties, the MGF for the perturbed reproductive model $\frac{1}{c}ED(c\boldsymbol{\mu}, \Sigma)$ results, for $\mu_i > 0$ fix, $i = 1, 2$ and c small enough to ensure that $c\boldsymbol{\mu} \in \Omega$

$$
\begin{aligned}
M\left(\frac{1}{c}\mathbf{s}; \boldsymbol{\tau}^{-1}(c\boldsymbol{\mu}), \Lambda\right) &= \left(1 + \frac{s_1}{\lambda_{11} c\delta_1(c\boldsymbol{\mu})}\right)^{-\alpha\lambda_{11}} \\
&\times \left(1 + \frac{s_2}{\lambda_{22} c\delta_2(c\boldsymbol{\mu})}\right)^{-\alpha\lambda_{22}} \\
&\times \left[\frac{L\left(-\delta_1(c\boldsymbol{\mu}) - \frac{s_1}{\lambda_{11}c}, -\delta_2(c\boldsymbol{\mu})\right)}{L(-\delta_1(c\boldsymbol{\mu}), -\delta_2(c\boldsymbol{\mu}))}\right]^{\lambda_1} \\
&\times \left[\frac{L\left(-\delta_1(c\boldsymbol{\mu}), -\delta_2(c\boldsymbol{\mu}) - \frac{s_2}{\lambda_{22}c}\right)}{L(-\delta_1(c\boldsymbol{\mu}), -\delta_2(c\boldsymbol{\mu}))}\right]^{\lambda_2} \\
&\times \left[\frac{L\left(-\delta_1(c\boldsymbol{\mu}) - \frac{s_1}{\lambda_{11}c}, -\delta_2(c\boldsymbol{\mu}) - \frac{s_2}{\lambda_{22}c}\right)}{L(-\delta_1(c\boldsymbol{\mu}), -\delta_2(c\boldsymbol{\mu}))}\right]^{\lambda_{12}},
\end{aligned}
$$

Let us denote by $h_i(\mathbf{s}; c, \boldsymbol{\mu}, \Lambda)$ the expressions with exponent λ_i $(i = 1, 2)$ and by $h_{12}(\mathbf{s}; c, \boldsymbol{\mu}, \Lambda)$ the one with exponent λ_{12} so the *MGF* can be written as follows:

$$
\begin{aligned}
M\left(\frac{1}{c}\mathbf{s}; \boldsymbol{\tau}^{-1}(c\boldsymbol{\mu}), \Lambda\right) &= \left(1 + \frac{s_1}{\lambda_{11} c \delta_1(c\boldsymbol{\mu})}\right)^{-\alpha\lambda_{11}} \\
&\times \left(1 + \frac{s_2}{\lambda_{22} c \delta_2(c\boldsymbol{\mu})}\right)^{-\alpha\lambda_{22}} \\
&\times h_1^{\lambda_1}(\mathbf{s}; c, \boldsymbol{\mu}, \Lambda) \\
&\times h_2^{\lambda_2}(\mathbf{s}; c, \boldsymbol{\mu}, \Lambda) \\
&\times h_{12}^{\lambda_{12}}(\mathbf{s}; c, \boldsymbol{\mu}, \Lambda). \qquad (6)
\end{aligned}
$$

The following equalities will be proved:

$$\lim_{c \to 0} c\delta_i(c\boldsymbol{\mu}) = -\frac{\alpha}{\mu_i}, \quad i = 1, 2, \qquad (7a)$$

$$\lim_{c \to 0} h_i(\mathbf{s}; c, \boldsymbol{\mu}, \Lambda) = 1, \quad i = 1, 2, \qquad (7b)$$

$$\lim_{c \to 0} h_{12}(\mathbf{s}; c, \boldsymbol{\mu}, \Lambda) = 1, \qquad (7c)$$

To prove (7a) note that given condition (5), we have that

$$c\delta_i(c\boldsymbol{\mu}) = \frac{-\alpha}{\mu_i + \frac{1}{c}\frac{\partial l(-\boldsymbol{\tau}^{-1}(c\boldsymbol{\mu}))}{\partial \delta_i(c\boldsymbol{\mu})}} = -\frac{\alpha}{\mu_i}.$$

Before proving $(7b)$ and in order to simplify the notation we define $\widetilde{\boldsymbol{\theta}}^T = \left(\tilde{\theta}_1, \tilde{\theta}_2\right)^T = (\delta_1(c\boldsymbol{\mu}), \delta_2(c\boldsymbol{\mu}))^T$, it can be proved that $\delta_i(c\boldsymbol{\mu}),\ \ i = 1, 2$ are strictly increasing functions.

Now, to obtain $\lim_{c\to 0} h_1(\mathbf{s}; c, \boldsymbol{\mu}, \Lambda)$, we apply (4), in such a way that $h_1(\mathbf{s}; c, \boldsymbol{\mu}, \Lambda)$, with $\boldsymbol{\tau}^{-1}(c\boldsymbol{\mu})$ in terms of $\widetilde{\boldsymbol{\theta}}$ can be expressed as

$$h_1\left(\mathbf{s}; c, \widetilde{\boldsymbol{\theta}}, \Lambda\right) = \frac{L\left(-\widetilde{\theta}_1(1+z_1), -\tilde{\theta}_2\right)}{L\left(-\tilde{\theta}_1, -\tilde{\theta}_2\right)}$$

$$= \frac{d\left(-\tilde{\theta}_1(1+z_1), -\tilde{\theta}_2\right)}{d\left(-\tilde{\theta}_1, -\tilde{\theta}_2\right)} \exp\left\{\int_m^n \frac{a(t,t)}{t} dt\right\} \qquad (8)$$

where $m = \left\| \left(-\tilde{\theta}_1, -\tilde{\theta}_2 \right) \right\|$, $n = \left\| \left(-\tilde{\theta}_1 \left\{ 1 + z_1 \right\}, -\tilde{\theta}_2 \right) \right\|$, being $\| \cdot \|$ any norm and $z_1 = \frac{s_1}{\lambda_1 c \tilde{\theta}_1}$.

Taking into account that $\lim_{c \to 0} (1 + z_1) = 1 - \frac{s_1 \mu_1}{\lambda_1 \alpha} > 0$ because $s_1 < 0$ and given the conditions required for bivariate Karamata representation, it can be deduced that

$$\lim_{c \to 0} \frac{d\left(-\tilde{\theta}_1 (1 + z_1), -\tilde{\theta}_2\right)}{d\left(-\tilde{\theta}_1, -\tilde{\theta}_2\right)} = \frac{d_0}{d_0} = 1. \tag{9}$$

We also have that

$$a(t, t) \leq \sup_{m \leq t \leq n} a(t, t) \text{ and that } \frac{1}{t} \leq \frac{1}{m},$$

and then

$$\int_m^n \frac{a(t, t)}{t} dt \leq \sup_{m \leq t \leq n} a(t, t) \frac{1}{m} (n - m).$$

For type 1 norm defined as $\|(x, y)\|_1 = |x| + |y|$ we have that

$$\begin{aligned} n - m &= \mid \tilde{\theta}_1 \mid\mid 1 + z_1 \mid + \mid \tilde{\theta}_2 \mid - \mid \tilde{\theta}_1 \mid - \mid \tilde{\theta}_2 \mid \\ &= \mid \tilde{\theta}_1 \mid (\mid 1 + z_1 \mid -1) \,, \end{aligned}$$

and

$$\int_m^n \frac{a\left(t,t\right)}{t} dt \leq \sup_{m \leq t \leq n} a(t,t) \frac{1}{\mid \tilde{\theta}_1 \mid + \mid \tilde{\theta}_2 \mid} \mid \tilde{\theta}_1 \mid (\mid 1 + z_1 \mid -1) \,.$$

Now, given that $\frac{1}{|\tilde{\theta}_1|} \geq \frac{1}{|\tilde{\theta}_1| + |\tilde{\theta}_2|}$

$$\lim_{c \to 0} \int_m^n \frac{a\left(t,t\right)}{t} dt \leq \lim_{c \to 0} \sup_{m \leq t \leq n} a(t,t) \frac{1}{\mid \tilde{\theta}_1 \mid} \mid \tilde{\theta}_1 \mid (\mid 1 + z_1 \mid -1) = 0,$$

hence

$$\lim_{c \to 0} \exp \left\{ \int_m^n \frac{a\left(t,t\right)}{t} dt \right\} = 1. \tag{10}$$

Putting together both results (9) and (10) and replacing in (8):

$$\lim_{c \to 0} h_1 (\mathbf{s}; c, \boldsymbol{\mu}, \Lambda) = 1.$$

Limits for h_2 and h_{12} when $c \to 0$ can be obtained in a similar way, then taking limits in (6) we have that

$$\lim_{c \to 0} M \left\{ \frac{1}{c}\mathbf{s}; \boldsymbol{\tau}^{-1} (c\boldsymbol{\mu}), \Lambda \right\} = \left(1 - \mu_1 \frac{s_1}{\alpha\lambda_{11}}\right)^{-\alpha\lambda_{11}} \left(1 - \mu_2 \frac{s_2}{\alpha\lambda_{22}}\right)^{-\alpha\lambda_{22}},$$

and this is the expression for the *MGF* of the bivariate dispersion model $\Gamma(\boldsymbol{\mu}, \Sigma_\alpha)$ for independent variables, as was proved in (2). The matrix Σ_α takes the following form:

$$\Sigma_\alpha = \begin{bmatrix} \frac{1}{\alpha\lambda_{11}} & 0 \\ 0 & \frac{1}{\alpha\lambda_{22}} \end{bmatrix}.$$

□

Example

Let the bivariate *EDM*, $ED(\boldsymbol{\mu}, \Sigma)$ be generated by the following measure, that is an extension of the measure presented by Letac (1992):

$$\nu\left(dy_1, dy_2\right) = \left(e^{2y_1} - 1\right)\left(e^{2y_2} - 1\right) dy_1 dy_2, \quad \left(y_1, y_2\right) \in \mathbb{R}_+^2. \quad (11)$$

The *CGF* is:

$$\begin{aligned}\kappa(\theta_1, \theta) &= \log \int_0^\infty \int_0^\infty e^{y_1\theta_1 + y_2\theta_2} \nu\left(dy_1, dy_2\right) \\ &= \log \frac{2}{\theta_1^2 + 2\theta_1} + \log \frac{2}{\theta_2^2 + 2\theta_2}.\end{aligned}$$

In order to analyse if ν varies regularly we obtain the distribution function $\bar{\nu}(y_1, y_2) = \nu\left\{(0, y_1] \times (0, y_2]\right\}$:

$$\begin{aligned}\bar{\nu}(y_1, y_2) &= \int_0^{y_1} \left(e^{2u} - 1\right) du \int_0^{y_2} \left(e^{2s} - 1\right) ds \\ &= \left[\frac{e^{2y_1}}{2} - y_1 - \frac{1}{2}\right]\left[\frac{e^{2y_2}}{2} - y_2 - \frac{1}{2}\right]\end{aligned}$$

Taking limits:

$$\lim_{t\to 0}\frac{\bar{\nu}(ty_1,ty_2)}{\bar{\nu}(t,t)} = \lim_{t\to 0}\frac{e^{2ty_1}-2ty_1-1}{e^{2t}-2t-1}\frac{e^{2ty_2}-2ty_2-1}{e^{2t}-2t-1}$$
$$= (y_1y_2)^2\,,$$

then $\bar{\nu}\in VR(2,2)_0$ and by Definition 2 the measure ν varies regularly. Given (3) we can affirm that the MGF is

$$M_\nu(\theta_1,\theta_2) = e^{\kappa(\boldsymbol{\theta})}$$
$$= \frac{2}{\theta_1^2+2\theta_1}\frac{2}{\theta_2^2+2\theta_2}$$
$$= (-\theta_1)^{-2}(-\theta_2)^{-2}L(-\theta_1,-\theta_2)\,,$$

where $L\in VL_\infty$ is

$$L(-\theta_1,-\theta_2) = \frac{4\theta_1\theta_2}{(\theta_1+2)(\theta_2+2)}.$$

Let us analyse conditions (5):

$$\frac{\partial I\left(-\theta_1,-\theta_2\right)}{\partial \theta_i}=\frac{1}{\theta_i}-\frac{1}{\theta_i+2}=\frac{2}{\theta_i\left(\theta_i+2\right)}\ i=1,2.$$

and taking into account that

$$\frac{\partial^2 \kappa}{\partial \theta_i \partial \theta_i}=\frac{2}{\theta_i^2+2\theta_i}+\frac{4\left(\theta_i+1\right)^2}{\left(\theta_i+2\right)^2}\ i=1,2$$

these second derivatives can be expressed in terms of mean values:

$$\ddot{\kappa}_{\theta_i\theta_i}\left(\theta_1,\theta_2\right)=V_i\left(\mu_i\right)=\mu_i^2+1-\sqrt{\mu_i^2+1}\ i=1,2,$$

giving

$$\frac{\partial I\left(-\theta_1,-\theta_2\right)}{\partial \theta_i}=\sqrt{\mu_i^2+1}-1\ i=1,2.$$

Then, taking limits and applying L' Hôpital:

$$\lim_{c\to 0}\frac{1}{c}\frac{\partial l\left(-\tilde{\theta}_1,-\tilde{\theta}_2\right)}{\partial\tilde{\theta}_i}=\lim_{c\to 0}\frac{\sqrt{c^2\mu_i^2+1}-1}{c}$$
$$=\lim_{c\to 0}\frac{1}{2}\left(c^2\mu_i^2+1\right)^{-\frac{1}{2}}2c\mu_i^2=0,$$

for $i=1,2$, so conditions (5) are satisfied. Now, given (4) and according with Theorem 2, *EDM*s generated by (11) satisfy that when $c\to 0$:

$$\frac{1}{c}ED(c\boldsymbol{\mu},\Sigma)\xrightarrow{d}\Gamma\left(\boldsymbol{\mu},\Sigma_\alpha\right)$$

Γ being the bivariate Gamma distribution for independent variables with $\Sigma_\alpha=\begin{bmatrix}\frac{1}{2\lambda_{11}} & 0\\ 0 & \frac{1}{2\lambda_{22}}\end{bmatrix}$.

Conclusions and discussion

- Fisher (1953) pointed out the importance of describing data in their natural *habitat*.

Conclusions and discussion

- Fisher (1953) pointed out the importance of describing data in their natural *habitat*.
- Gaussian distribution is particularly relevant for small dispersion data. EDMs are appropiate to modelize large dispersion data.

Conclusions and discussion

- Fisher (1953) pointed out the importance of describing data in their natural *habitat*.
- Gaussian distribution is particularly relevant for small dispersion data. EDMs are appropiate to modelize large dispersion data.
- Jørgensen and Martínez conjectured that results proved by them in the univariate case, could be proved for those *MEDM*s they defined in 2013.

Conclusions and discussion

- Fisher (1953) pointed out the importance of describing data in their natural *habitat*.
- Gaussian distribution is particularly relevant for small dispersion data. EDMs are appropiate to modelize large dispersion data.
- Jørgensen and Martínez conjectured that results proved by them in the univariate case, could be proved for those *MEDM*s they defined in 2013.
- Contributions made by J. Karamata extended to $\mathbb{R}^k$ some theorems for regular variation functions establishing that they behave asymptotically as their Laplace transforms.

- We proved that bivariate *EDM*s generated by regularly varying measures, tend to a bivariate independent Gamma distribution when the mean parameter goes to some extreme in the parameter domain, imposing no conditions on the asymptotic behaviour of the variance function.

- We proved that bivariate *EDM*s generated by regularly varying measures, tend to a bivariate independent Gamma distribution when the mean parameter goes to some extreme in the parameter domain, imposing no conditions on the asymptotic behaviour of the variance function.
- Hitz and Evans (2016) developed an extension of Karamata theorem to multivariate regular variation functions, their results open a new line of research of convergence properties of dispersion models for extremes (Jørgensen, 2010).