Stability of equilibrium and bifurcation analysis in delay differential equations

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Delay and Neutral Delay differential equations

- Delay differential equations (Ddes):
  $\dot{x} = f(x, x_\tau, \mu), \quad x_\tau = x(t - \tau), \tau > 0.$
- Neutral Ddes (NDdes): $\dot{x} = f(x, x_\tau, \dot{x}_\tau, \mu), \quad \dot{x}_\tau = \dot{x}(t - \tau),$ $\tau > 0.$

- Time Domain Approach (TDA).
- Characteristic equations.
- Equilibrium: stability and bifurcations.
- Frequency Domain Approach (FDA).
- Limit cycles: stability and bifurcations.

Characteristic equations

- **Ddes already analyzed**

$$\ddot{x} + \gamma x = f(u),$$
$$f(u) = \alpha u + \beta u^2,$$
$$u = x_\tau \text{ or } u = \dot{x}_\tau$$

Model 1: $u = x_\tau$
$P(\mathbf{s},\gamma,\alpha,\tau) = e^{\mathbf{s}}(\mathbf{s}^2 + \gamma\tau^2) - \alpha\tau^2$

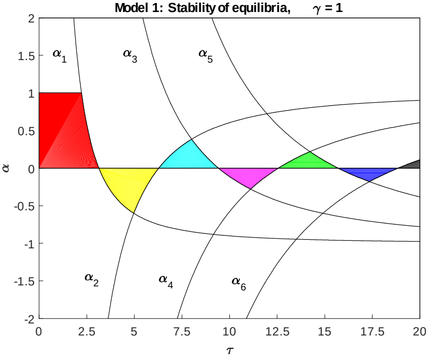
Model 2: $u = \dot{x}_\tau$
$P(\mathbf{s},\gamma,\alpha,\tau) = e^{\mathbf{s}}(\mathbf{s}^2 + \gamma\tau^2) - \alpha\tau\mathbf{s}$

Characteristic equation:
exponential polynomial
$P(\mathbf{s},\gamma,\alpha,\tau) = 0$
$\Updownarrow$
Stability analysis
(trivial equilibrium)
$\Updownarrow$
When all the roots of P
have negative real part?
Pontryagin (1955)

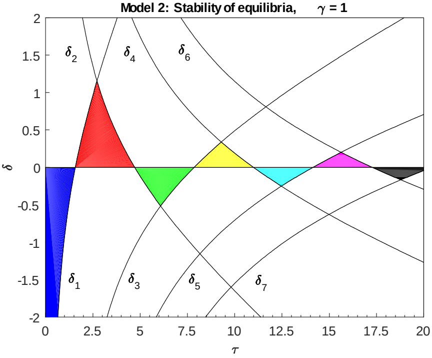
Static stability areas for Ddes

Two theorems set with Time Domain Approach (TDA)

Model 1



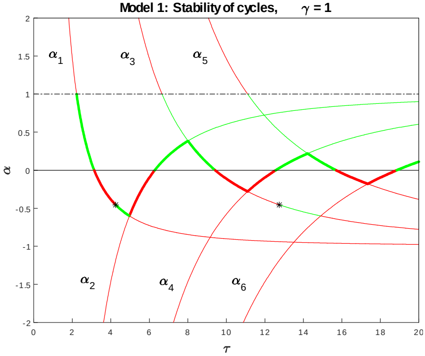
Model 2



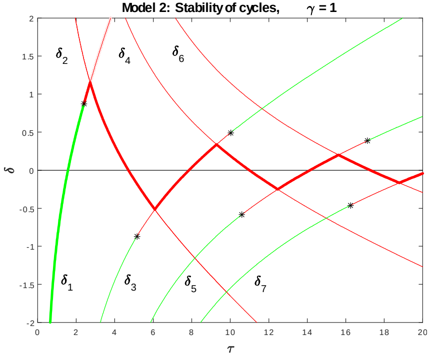
Hopf bifurcation curves and dynamic stability

Results combining Frequency and Time Domain Approaches

Model 1



Model 2



NDdes: Model A

$$\ddot{x} + \gamma x = f(u)$$
$$f(u) = \alpha u + \beta u^2,$$
$$u = \ddot{x}_\tau = \ddot{x}(t - \tau),$$
$$\gamma > 0,\ \alpha \neq 0.$$

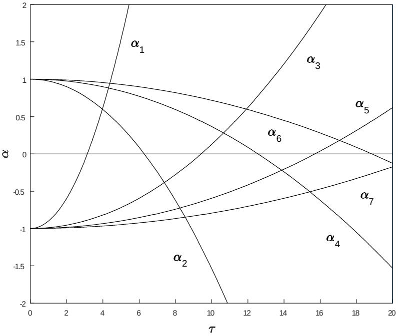
Ddes analyzed as Models 1 and 2

$$\ddot{x} + \gamma x = f(u),$$

$$f(u) = \alpha u + \beta u^2,$$
$$u = x_\tau \text{ or } u = \dot{x}_\tau$$

Hopf bifurcation curves

$$\alpha_k = (-1)^k(1 - \tau^2/(k\pi)^2)$$



Static stability theorem for Model A

Theorem

$P(\mathbf{s}) = e^{\mathbf{s}}\mathbf{s}^2 + e^{\mathbf{s}}\gamma\tau^2 - \alpha\mathbf{s}^2 = 0$, $\gamma, \tau > 0$, $\alpha \neq 0$. Let $\gamma = 1$ and $r_k = y_k^{-1} = (k\pi)^{-1}$, $k \in \mathbb{N}$ and other restrictive conditions on the parameters. Then, all the roots of P lie on the left half plane iff these conditions are fulfilled:

I) For $0 < \tau < y_1 \Rightarrow \alpha < 0$ and $\alpha > -1 + r_1^2\tau^2$.

II) For $y_k < \tau < y_{k+1}$, $k \in \mathbb{N}$, it is required:

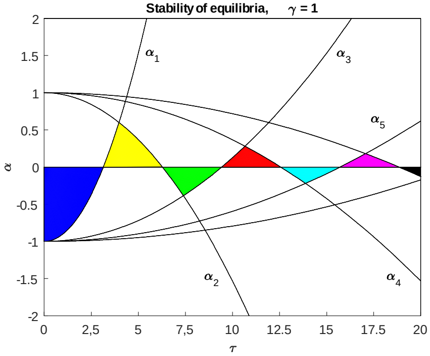
*a) If k is odd $\Rightarrow \alpha > 0$ **,** $\alpha < -1 + r_k^2\tau^2$ and $\alpha < 1 - r_{k+1}^2\tau^2$.*

*b) If k is even $\Rightarrow \alpha < 0$ **,** $\alpha > 1 - r_k^2\tau^2$ and $\alpha > -1 + r_{k+1}^2\tau^2$.*

Corollary

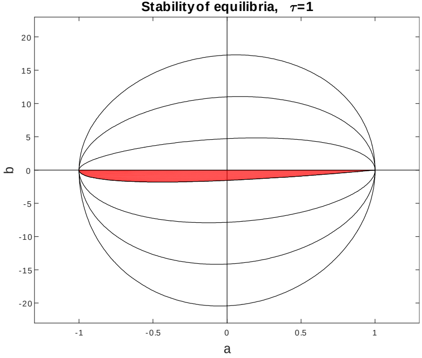
Let $\ddot{x} + \gamma x = \alpha\ddot{x}_\tau + \beta\ddot{x}_\tau^2$. Then $x = 0$ is asymptotically stable under the parameter conditions set in the Theorem.

Static stability areas for Model A



Zhang & Stépán (2018)

$$\dot{x} = \color{red}{a} v + \color{red}{b} \dot{v} + \color{red}{c} \dot{v}^3$$
$$\dot{x} = \frac{dx}{dt}, \quad v = x(t-1)$$
$$a, b, c \in R$$



Hopf curves

$$\begin{cases} a = \cos y \\ b = -y \sin y \end{cases}, \quad y \geq 0$$

Stability condition

$$0 > b > -\sqrt{1-a^2} \arccos a$$

Static stability theorem for Model B

Theorem

$P(s) = e^s\mathbf{s} - a\mathbf{s} - b\tau = 0, \tau > 0,\ a, b \in R$. *Let* $\tau = 1,\ |a| < 1,\ y_0 = \arccos a$. *Then, all the roots of P lie on the left half plane iff holds* $0 > b > -\sqrt{1-a^2}\,y_0$.

Corollary

Let $\dot{x} = ax + b\dot{v} + c\dot{v}^3$. *Then* $x = 0$ *is asymptotically stable under the parameter conditions set in the Theorem.*

Feedback control theory for Ddes and NDdes

- **Extended** Graphical Hopf Bifurcation theorem (Mees, Chua and Allwright, 1979b, 1979a).
  1. Local existence of a branch of periodic solutions.
  2. Approximation of amplitude θ and frequency ω of each periodic solution.
  3. Stability of each periodic solution.
  4. Stability along the Hopf bifurcations curves.
- Outcomes about Hopf degeneracies (up to now with NDdes)
  1. Determination of fold of cycles bifurcations.

Model B

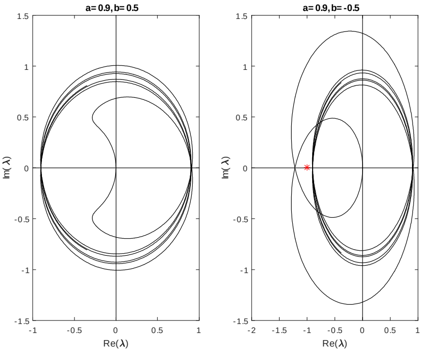
Frequency Domain Approach (FDA)

$$\dot{x} = av + b\dot{v} + c\dot{v}^3$$
$$\dot{x} = \frac{dx}{dt}, \quad v = x(t-1)$$
$$a, b, c \in R$$

$$\Updownarrow$$

$$(y = -\dot{x}(t-1))$$
$$\mathcal{L}(-y) = G^*(s)\mathcal{L}(g(y)),$$
$$G^*(s) = \frac{se^{-s}}{s - be^{-s}},$$
$$g(y) = -ay - cy^3,$$
$$-\hat{y} = G^*(0)g(\hat{y}) \Longrightarrow \hat{y} = 0$$

$$\lambda(s) = G^*(s) \left. \frac{dg}{dy} \right|_{\hat{y}=0} = \frac{-ase^{-s}}{s - be^{-s}}$$



Model B

Stability of cycles, Hopf degeneracy condition and cycles approximations

Curvature coefficient: σ_1

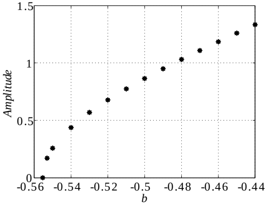
$$\operatorname{sgn}\sigma_1 = \begin{cases} -1 \Rightarrow \text{HB stable} \\ \ \ 1 \Rightarrow \text{HB unstable} \end{cases}$$

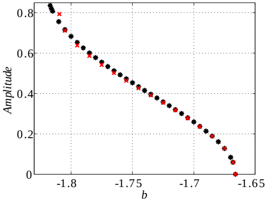
Hopf degeneracy condition

$$\begin{cases} \operatorname{sgn}\sigma_1 = -\operatorname{sgn}(cb(1-a^2-ab)) \\ b = -\arccos a\sqrt{1-a^2}, \end{cases}$$

Approximations of ω and θ

$$\begin{aligned} b &= -\omega\sin\omega \\ \theta &= \frac{2}{\sqrt{3c}}\sqrt{\cos\omega - a} \end{aligned}$$





Conclusions and future work

Conclusions

1. Static and dynamic analyses of Ndes have been performed .
2. Static analysis: TDA and studying the roots of characteristic equations.
3. Dynamic analysis: FDM and setting Graphical Hopf bifurcation Theorem (GHT) outcomes.

Future work

1. Continue exploring NDdes.
2. Get higher order approximations of periodic solutions via FDA and GHT.
3. Understand the dynamics close to diverse Hopf degeneracies.
4. Contrast results with other analytical and numerical techniques.

References

Bellman, R. and Cooke, K. [1963] Differential-Difference Equations, Academic Press, New York.

Bellman, R. and Danskin, J. Jr.[1954] A Survey of the Matematical Theory of Time-Lag, Retarded Control and Hereditary Processes, Report 256, The Rand Corporation, California.

Mees, A. I. and Allwright, D. J. [1979a]. "Using characteristic loci in the Hopf bifurcation", Proceedings Instn. Electrical Engrs., 126:628-632.

Mees, A.I. and Chua, L. [1979b] "The Hopf bifurcation theorem and its applications to nonlinear oscillations in circuits and systems", IEEE Transactions on Circuits and Systems, 26(4), 235-254.

Pontryagin, L. S. [1955] "On the zeros of some elementary trascendental functions", American Mathematical Society Tanslations 2(1), 95-110.

Stépán, G. [1989] Retarded Dynamical Systems: Stability and Characteristic Functions, Pitman Research Notes in Mathematics Series, Vol. 210, Longman, UK.

Zhang, L. and Stépán, G. [2018] "Hopf bifurcation analysis of scalar implicit neutral delay differential equation", Electronic J. of Qualitative Theory of Diff. Eqns., 62, pp. 1-9.

Thank you for your attention!