Syntactical Reasons

Miguel Campercholi

Facultad de Matemática, Astronomía y Física
Universidad Nacional de Córdoba
Argentina

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Certificates

Gödel's Completeness Theorem

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Theorem (Gödel 1929)

Let Σ be a set of first-order axioms. If φ is a first-order property true in every structure satisfying Σ, then there is a first-order proof of φ from Σ.

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Theorem (Malcev 1954)

A variety $\mathscr{V}$ is CP *iff there is a term $p(x,y,z)$ such that $\mathscr{V}$ satisfies $p(x,y,y) \approx x$ and $p(x,x,y) \approx y$.*

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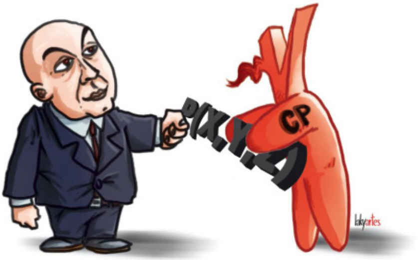

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Theorem (Pixley 1963)

A variety $\mathscr{V}$ is arithmetical iff there is a term $m(x, y, z)$ such that $\mathscr{V}$ satisfies $m(x, y, x) \approx m(x, y, y) \approx m(y, y, x) \approx x$.

Certificates Are Useful

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Let $\mathbf{A}$ be a finite algebraic structure. There is an algorithm that checks whether $\mathbb{V}(\mathbf{A})$ is CP.

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Proof.

Check if there is a function $f \in \{t^{\mathbf{A}}(x,y,z) : t \text{ is a ternary term}\}$ satisfying $f(x,y,y) \approx x$ and $f(x,x,y) \approx y$.

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Note the same is true for CD and arithmeticity.

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Finite bases

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Smart use of the Jónsson terms. $\square$

Three Theorems Cooked in Córdoba

Filtral Quasivarieties

Discriminator Varieties

- The (quaternary) discriminator on a set A is the function

$$n^A(a,b,c,d) = \begin{cases} c & \text{if } a = b, \\ d & \text{if } a \neq b. \end{cases}$$

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- A variety $\mathscr{V}$ is a discriminator variety if there are a class $\mathscr{K}$ and a term $t(x,y,z,w)$ such that:
  - $\mathscr{K}$ generates $\mathscr{V}$
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- The variety of Boolean algebras is a discriminator variety:
  - $\mathscr{K} = \{\mathbf{2}\}$
  - $t = ((x \leftrightarrow y) \wedge z) \vee ((x \leftrightarrow y)^c \wedge w)$.

Discriminator Varieties and Filtral Quasivarieties

Theorem (Folklore)

For a variety $\mathscr{V}$ t.f.a.e.:

- *$\mathscr{V}$ is artihmetical, semisimple and $\mathscr{V}_{FSI}$ is a universal class.*

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  - $\mathscr{K}$ generates $\mathscr{Q}$,
  - φ defines n^A in $\mathbf{A}$ for all $\mathbf{A} \in \mathscr{K}$.

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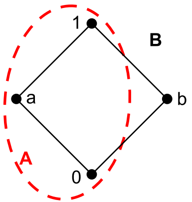
Example

Let $\mathscr{D}$ be the variety of bounded distributive lattices.



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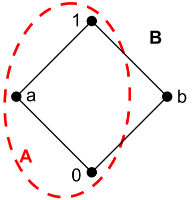

Observe that $b \in \mathrm{dom}_{\mathscr{D}}^{\mathbf{B}}\mathbf{A}$.

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Now, $\mathbf{B} \models \varphi(a,b)$, so $\mathbf{C} \models \varphi(ha,hb) \ \& \ \varphi(h'a,h'b)$. Since $ha = h'a$, we have $hb = h'b$.

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Theorem (Campercholi 2016)

Let $\mathscr{K}$ be closed under ultraproducts, and let $\mathbf{A} \leq \mathbf{B}$. Then $b \in \mathrm{dom}^{\mathbf{B}}_{\mathscr{K}} \mathbf{A}$ iff there are a pp formula $\varphi(\bar{x}, y)$ and $\bar{a} \in A^n$ such that:

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- $[\varphi]^{\mathbf{B}}(\bar{a}) = b$.

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Theorem (Blok & Hoogland, 2006)

Let L be a semantically algebraizable general logic and let $\mathscr{K}$ be its algebraic counterpart. T.f.a.e.:

- *L has the Beth Property.*
- *Every $\mathscr{K}$-epimorphism is surjective.*

An Algorithm Deciding Surjectivity of Epimorphisms

A term $M(x, y, z)$ is majority-term for the class $\mathscr{K}$ if $\mathscr{K}$ satisfies

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Theorem

Let $\mathscr{F}$ be a finite set of finite algebras with a majority-term. It is decidable whether the (quasi)variety generated by $\mathscr{F}$ has surjective epimorphisms.

Factor Congruences

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Every decomposition of $\mathbf{A} \overset{\gamma}{\cong} \mathbf{A}_1 \times \mathbf{A}_2$ produces a pair of congruences of $\mathbf{A}$

$$\theta_1 = \{\langle a, b\rangle : \pi_1\gamma(a) = \pi_1\gamma(b)\}$$
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$$\text{CD} \Rightarrow \text{FHP} \Rightarrow \text{BFC}.$$

Central Elements

Encoding factor congruences

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E.g., e is central in a bounded lattice $\mathbf{L}$ iff e is complemented in $\mathbf{L}$.

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E.g., e is central in a bounded lattice $\mathbf{L}$ iff e is complemented in $\mathbf{L}$.
Does this always work?

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Definable Factor Congruences

Let $\mathscr{V}$ be a $0,1$-variety.
Say that $\mathscr{V}$ has DFC if there is a formula $\Phi(x,y,z)$ such that for all $\mathbf{A},\mathbf{B} \in \mathscr{V}$

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We can use first-order language to "talk" about the coordinates of elements in direct products.

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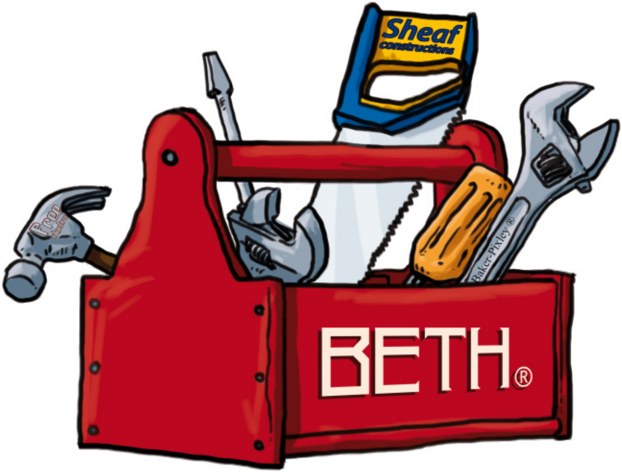
(3)$\Rightarrow$(1) Easy. $\square$

The Toolbox



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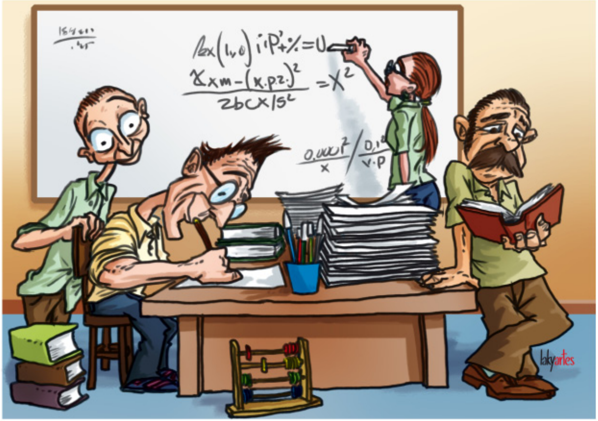
- Sheaf constructions such as Boolean products and global products.

In The Works



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- Do not vote for Macri.


Marvin Berry
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STARLIGHTERS
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