Admissible rules and (almost) structural completeness for many-valued logics.

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Admissibility Theory

Given a logic L, an L-**unifier** of a formula φ is a substitution σ such that $\vdash_L \sigma\varphi$.

A rule Γ/φ is L-**admissible** in L iff every common L-unifier of Γ is also an L-unifier of φ.

Γ/φ is **passive** L-**admissible** in L iff Γ has no common L-unifier.

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Γ/φ is **passive** L-**admissible** in L iff Γ has no common L-unifier.

A logic is **structurally complete** iff every admissible rule is a derivable rule.

A logic is **almost structurally complete** iff every admissible rule is either derivable rule or a passive admissible.

- CPC is structurally complete.

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- Infinite valued Łukasiewicz logic is not structurally complete.
- n-valued Łukasiewicz logic is not structurally complete but almost structurally complete.
- Any n-contractive extension of Basic logic is almost structurally complete.

Algebraizable logics

Deductive Systems $\longleftrightarrow$ Quasivarieties

$L \longleftrightarrow \mathbb{K}$

Algebraizable logics

Deductive Systems	$\longleftrightarrow$	Quasivarieties
L	$\longleftrightarrow$	$\mathbb{K}$
$\langle Prop(X), \vdash_L \rangle$	$\longleftrightarrow$	$\langle Eq(X), \models_{\mathbb{K}} \rangle$

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$\langle Prop(X), \vdash_L \rangle$	$\longleftrightarrow$	$\langle Eq(X), \models_{\mathbb{K}} \rangle$

$\tau : Prop(X) \to \mathcal{P}(Eq(X))$ $\qquad$ $\sigma : Eq(X) \to \mathcal{P}(Prop(X))$

Algebraizable logics

$$\text{Deductive Systems} \longleftrightarrow \text{Quasivarieties}$$

$$L \longleftrightarrow \mathbb{K}$$

$$\langle Prop(X), \vdash_L \rangle \longleftrightarrow \langle Eq(X), \models_{\mathbb{K}} \rangle$$

$\tau : Prop(X) \to \mathcal{P}(Eq(X))$ | $\sigma : Eq(X) \to \mathcal{P}(Prop(X))$

$\Gamma \cup \{\varphi\} \subseteq Prop(X)$ | $\Sigma \cup \{p \approx q\} \subseteq Eq(X)$

$\Gamma \vdash_L \varphi$ iff $\tau[\Gamma] \models_{\mathbb{K}} \tau(\varphi)$ | $\Sigma \models_{\mathbb{K}} p \approx q$ iff $\sigma[\Sigma] \vdash_L \sigma(p \approx q)$

$\varphi \dashv\vdash_L \sigma(\tau(\varphi))$ | $p \approx q \;=\!\!|\!\!\models_{\mathbb{K}} \tau(\sigma(p \approx q))$

Algebraizable logics and Algebraic logic

Finitary Extensions of L $\longleftrightarrow$ Quasivarieties of $\mathbb{K}$

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Finitary Extensions of L	$\longleftrightarrow$	Quasivarieties of $\mathbb{K}$
Axiomatic Extensions	$\longleftrightarrow$	(Relative) Varieties

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Algebraizable logics and Algebraic logic

Finitary Extensions of L	$\longleftrightarrow$	Quasivarieties of $\mathbb{K}$
Axiomatic Extensions	$\longleftrightarrow$	(Relative) Varieties
(Finite) Axiomatization	$\longleftrightarrow$	(Finite) Axiomatization
Deduction Theorem	$\longleftrightarrow$	EDPCR
Local Deduction Theorem	$\longleftrightarrow$	RCEP
Interpolation Theorem	$\longleftrightarrow$	Amalgamation Property

Algebraic Admissibility Theory

Given a quasivariety $\mathbb{K}$, we say that a quasiequation

$$\alpha_1 \approx \gamma_1 \& \cdots \& \alpha_n \approx \gamma_n \Rightarrow \epsilon \approx \eta$$

is $\mathbb{K}$-**admissible** iff for every term substitution σ if $\mathbb{K} \models \sigma(\alpha_i) \approx \sigma(\gamma_i)$ for $i = 1 \div n$, then $\mathbb{K} \models \sigma(\epsilon) \approx \sigma(\eta)$.

is **passive** in $\mathbb{K}$ iff there is no term substitution σ such that $\mathbb{K} \models \sigma(\alpha_i) \approx \sigma(\gamma_i)$ for $i = 1 \div n$.

$\mathbb{K}$ is **structurally complete** iff every $\mathbb{K}$-admissible quasiequation is valid in $\mathbb{K}$.

$\mathbb{K}$ is **almost structurally complete** iff every admissible quasiequation is either valid in $\mathbb{K}$ or passive in $\mathbb{K}$.

Algebraic logic

Theorem (Rybakov 1997, Olson et al. 2008)

Let L be an algebraizable logic and $\mathbb{K}$ its quasivariety semantics, then L is (almost) structurally complete iff $\mathbb{K}$ is (almost) structurally complete.

Structural completeness and free algebras

Theorem (Bergman 1991)

Let $\mathbb{K}$ be a quasivariety, then the following properties are equivalent.

1. $\mathbb{K}$ *is structurally complete.*
2. *Each proper subquasivariety of* $\mathbb{K}$ *generates a proper subvariety of* $\mathcal{V}(\mathbb{K})$*.*
3. $\mathbb{K}$ *is the least* $\mathcal{V}(\mathbb{K})$*-quasivariety.*
4. $\mathbb{K} = \mathcal{Q}(\mathbf{Free}_{\mathbb{K}}(\omega)) = \mathcal{Q}(\mathbf{Free}_{\mathcal{V}(\mathbb{K})}(\omega))$.

Almost Structural completeness and free algebras

Theorem (Dzik-Stronkowski 2016)

Let $\mathbb{K}$ be a quasivariety. The following are equivalent

1. $\mathbb{K}$ *is almost structurally complete.*
2. *For every* $\mathbf{A} \in \mathbb{K}$, $\mathbf{A} \times \mathbf{Free}_{\mathbb{K}}(\omega) \in \mathcal{Q}(\mathbf{Free}_{\mathbb{K}}(\omega))$.
3. *For every* $\mathbf{A} \in \mathbb{K}$, *if there is an homomorphism from* $\mathbf{A}$ *into* $\mathbf{Free}_{\mathbb{K}}(\omega)$ *then* $\mathbf{A} \in \mathcal{Q}(\mathbf{Free}_{\mathbb{K}}(\omega))$.

Almost Structural completeness and free algebras

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Let $\mathbb{K}$ be a quasivariety. The following are equivalent

1. $\mathbb{K}$ *is almost structurally complete.*
2. *For every* $\mathbf{S} \in \mathbb{K}_{SI}$, $\mathbf{S} \times \mathbf{Free}_{\mathbb{K}}(\omega) \in \mathcal{Q}(\mathbf{Free}_{\mathbb{K}}(\omega))$.
3. *For every* $\mathbf{P} \in \mathbb{K}_{FP}$, *if there is an homomorphism from* $\mathbf{A}$ *into* $\mathbf{Free}_{\mathbb{K}}(\omega)$ *then* $\mathbf{A} \in \mathcal{Q}(\mathbf{Free}_{\mathbb{K}}(\omega))$.

Almost Structural completeness and free algebras

Theorem (Dzik-Stronkowski 2016)

Let $\mathbb{K}$ be a quasivariety. The following are equivalent

1. $\mathbb{K}$ *is almost structurally complete.*
2. *There is* $\mathbf{B}$ *a subalgebra of* $\mathbf{Free}_{\mathbb{K}}(\omega)$*, such that for every* $\mathbf{S} \in \mathbb{K}_{SI}$, $\mathbf{S} \times \mathbf{B} \in \mathcal{Q}(\mathbf{Free}_{\mathbb{K}}(\omega))$.
3. *For every* $\mathbf{P} \in \mathbb{K}_{FP}$*, if there is an homomorphism from* $\mathbf{A}$ *into* $\mathbf{Free}_{\mathbb{K}}(\omega)$ *then* $\mathbf{A} \in \mathcal{ISP}(\mathbf{Free}_{\mathbb{K}}(\omega))$.

Almost Structural completeness and free algebras

Theorem (Dzik-Stronkowski 2016)

Let $\mathbb{K}$ *be a quasivariety. If* $\mathbf{B}_2$ *is a subalgebra of* $\mathbf{Free}_{\mathbb{K}}(\omega)$*, then the following are equivalent*

1. $\mathbb{K}$ *is almost structurally complete.*
2. *For every* $\mathbf{S} \in \mathbb{K}_{SI}$, $\mathbf{S} \times \mathbf{B}_2 \in \mathcal{Q}(\mathbf{Free}_{\mathbb{K}}(\omega))$.
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Goal

To algebraically study (almost) structural completeness of some algebraizable many-valued logics in order to characterize and axiomatize (all) finitary extensions.

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To study (almost) structural completeness of some varieties and quasivarieties of (many-valued) algebras in order to characterize and axiomatize (all) subquasivarieties.

- Gödel logics.
- Nilpotent minimum logics.
- Łukasiewicz logics

Gödel-Dummett Logic

Gödel-Dummett Logic (**G**) is the axiomatic extension of the Intuitionistic logic (IPC) given by the axiom

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Standard semantics:

Let $[0,1]_G = \langle \{a \in \mathbb{R} : 0 \leq a \leq 1\}; \wedge, \vee, \to, \neg, 0, 1 \rangle$. For every $a, b \in [0,1]$, $a \wedge b = min\{a,b\}$ and $a \vee b = max\{a,b\}$

$a \to b = \begin{cases} 1, & \text{if } a \leq b; \\ b & \text{otherwise.} \end{cases}$ and $\neg a := a \to 0 = \begin{cases} 1, & \text{if } a = 0; \\ 0 & \text{otherwise.} \end{cases}$

$\Gamma \models_{[0,1]_G} \varphi$ iff for every $h : Prop(x) \to [0,1]$, $h(\varphi) = 1$ whenever $h\Gamma = \{1\}$

Completeness Theorem

Theorem (Dummett 1959)

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Algebraic logic

The Gödel-Dummett logic is algebraizable with $\mathbb{G}$ the class of all Gödel-algebras as its equivalent quasivariety semantics.

Gödel-algebra

A **Gödel-algebra** is an algebra $\langle A, \wedge, \vee, \rightarrow, \neg, \bar{0}, \bar{1}\rangle$ such that

- $\langle A, \wedge, \vee, \bar{0}, \bar{1}\rangle$ is a bounded distributive lattice.
- For every $a, b \in A$, $a \rightarrow b$ is the pseudocomplent of a relative to b,
  i.e. $a \rightarrow b = max\{c \in A : a \wedge c \leq b\}$.
- $\neg a = a \rightarrow \bar{0}$.

(L) For every $a, b \in A$ $(a \rightarrow b) \vee (b \rightarrow a) = \bar{1}$.

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A Gödel algebra is a Heyting algebra satisfying (L).

Gödel-chains

We say that a Gödel-algebra is a **Gödel-chain**, provided that it is totally ordered.

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Let $\langle A, \leq, \bar{0}, \bar{1} \rangle$ be a totally ordered bounded set, if we define for every $a, b \in A$,

$$a \wedge b = min\{a, b\}, \qquad a \vee b = max\{a, b\},$$

$$a \to b = \begin{cases} \bar{1}, & \text{if } a \leq b; \\ b, & \text{otherwise.} \end{cases}, \qquad \neg a = a \to \bar{0} = \begin{cases} \bar{1}, & \text{if } a = 0; \\ 0, & \text{if } a \neq 0. \end{cases},$$

then $\mathbf{A} = \langle A, \wedge, \vee, \to, \neg, \bar{0}, \bar{1} \rangle$ is a Gödel-chain.

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Every Gödel-chain is of this form.

Finite Gödel-chains

Therefore up to isomorphism for each natural number n, there is only one Gödel-chain $\mathbf{G}_n$ with exactly n elements.

$$\mathbf{G}_n = \langle \{0, 1, 2, \ldots, n-1\}, \wedge, \vee, \rightarrow, \neg, 0, n-1 \rangle.$$

Notice that $\mathbf{G}_1$ is the trivial algebra and $\mathbf{G}_2$ is the 2-element Boolean algebra.

$\mathbf{G}_n \hookrightarrow \mathbf{G}_m$ iff $n \leq m$

G-varieties

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- *A variety* $\mathbb{V}$ *of Gödel-algebras is proper subvariety of* $\mathbb{G}$ *iff* $\mathbb{V} = \mathbb{G}_n = \mathcal{V}(\mathbf{G}_n)$ *for some* $n > 0$.
- $\mathbb{G}_n$ *is axiomatizable by* $\bigvee_{i<n}((x_i \leftrightarrow x_{i+1}) \approx \overline{1}$
- $\mathbb{G}_1 \subsetneq \mathbb{G}_2 \subsetneq \mathbb{G}_3 \subsetneq \cdots \subsetneq \mathbb{G}_n \subsetneq \cdots \mathbb{G}$

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- $\mathbb{G} = \mathcal{Q}([0,1]_G) = \mathcal{Q}(\{\mathbf{G}_n : n > 1\})$.
- $\mathbb{G}_n = \mathcal{Q}(\mathbf{G}_n)$ *for every* $n > 0$.

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Theorem (Dzik-Wronski 1973)

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Let $n > 1$. *For every* $2 \leq k \leq n$, $\mathbf{G}_k$ *is embeddable into* $\mathbf{Free}_{\mathbb{G}_n}(\omega)$.

Theorem

Gödel logic is **hereditarily** *structurally complete.*

Quasivarieties of Gödel algebras

Every quasivariety of Gödel algebras is a variety.

$$L_{\mathcal{V}}(\mathbb{G}) = L_{\mathcal{Q}}(\mathbb{G})$$

Nilpotent Minimum Logic

Nilpotent Minimum Logic (**NML**) is the axiomatic extension of the Monoidal t-norm logic (MTL) given by the axioms

Inv $\neg\neg\varphi \to \varphi$

WNM $(\psi * \varphi \to \bot) \vee (\psi \wedge \varphi \to \psi * \varphi)$

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Standard Semantics: $(\models_{[0,1]_{NM}})$
$[0,1]_{NM} = \langle [0,1]; *, \to, \wedge, \vee, \neg, 0, 1 \rangle$ where for every $a, b \in [0,1]$,
$a \wedge b = min\{a,b\}$, $a \vee b = max\{a,b\}$, $\neg a = 1 - a$,

$$a * b = \begin{cases} min\{a,b\}, & \text{if } b > 1 - a; \\ 0, & \text{otherwise.} \end{cases} \quad \text{and}$$

$$a \to b = \begin{cases} 1, & \text{if } a \leq b; \\ max\{1 - a, b\} & \text{otherwise.} \end{cases}$$

Completeness Theorem

Theorem (Esteva Godo 2001, Noguera et al 2008)

$$\Sigma \vdash_{NML} \varphi \text{ iff } \Sigma \models_{[0,1]_{NM}} \varphi$$

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Algebraic logic

The Nilpotent Minimum Logic NML is algebraizable with $\mathbb{NM}$ the class of all NM-algebras as its equivalent quasivariety semantics.

NM-algebras

A **NM-algebra** is a bounded integral residuated lattice satisfying the following equations:

$$(x \to y) \vee (y \to x) \approx \bar{1} \quad \text{(L)}$$

$$\neg\neg x \approx x \quad \text{(I)}$$

$$\neg(x * y) \vee (x \wedge y \to x * y) \approx \bar{1} \quad \text{(WNM)}$$

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$$\neg(x * y) \vee (x \wedge y \to x * y) \approx \bar{1} \quad \text{(WNM)}$$

Example: $[0,1]_{NM}$ is a NM-algebra.

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Let $\langle A, \leq, \bar{0}, \bar{1} \rangle$ a totally ordered bounded set equipped with an involutive negation $\neg$,if we define for every $a, b \in A$,

$$a * b = \begin{cases} \bar{0}, & \text{if } b \leq \neg a; \\ a \wedge b, & \text{otherwise.} \end{cases} \qquad a \to b = \begin{cases} \bar{1}, & \text{if } a \leq b; \\ \neg a \vee b, & \text{otherwise.} \end{cases},$$

$$a \wedge b = min\{a, b\} \qquad a \vee b = max\{a, b\},$$

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Every NM-chain is of this form.

Finite NM-chains

Therefore up to isomorphism for each finite $n \in \mathbb{N}$, there is only one NM-chain $\mathbf{A}_n$ with exactly n elements.

$$\mathbf{A}_{2n+1} = \langle [-n, n] \cap \mathbb{Z}, *, \rightarrow, \wedge, \vee, -n, n \rangle.$$

$$\mathbf{A}_{2n} = \langle A_{2n+1} \setminus \{0\}, *, \rightarrow, \wedge, \vee, -n, n \rangle.$$

For every $n, k > 0$,

- $\mathbf{A}_{2n} \hookrightarrow \mathbf{A}_{2k+1}$ iff $\mathbf{A}_{2n} \hookrightarrow \mathbf{A}_{2k}$ iff $\mathbf{A}_{2n+1} \hookrightarrow \mathbf{A}_{2k+1}$ iff $n \leq k$.
- $\mathbf{A}_{2n+1} \not\hookrightarrow \mathbf{A}_{2k}$.

Negation fixpoint

Let **A** be an NM-algebra,
$a \in A$ is a **negation fixpoint** (or just **fixpoint**, for short) iff $\neg a = a$.

Let **C** *be an NM-chain. Then* $C \setminus \{c\}$ *is the universe of a subalgebra of* **C** *which we denote by* $\mathbf{C}^{-}$*.*

$$\mathbf{A_{2n}} = \mathbf{A_{2n+1}}^{-}$$

NM-varieties

Let

$$S_n(x_0, \ldots, x_n) = \bigwedge_{i<n} ((x_i \to x_{i+1}) \to x_{i+1}) \to \bigvee_{i<n+1} x_i$$

$$\nabla(x) = \neg(\neg x^2)^2 \qquad \Delta(x) = (\neg(\neg x)^2)^2$$

where x^2 is an abbreviation of $x * x$.

Lemma

Let **A** *be an NM-chain. Then we have*

1. **A** *does not have a fixpoint iff* $\nabla(x) \approx \Delta(x)$ *holds in* **A**.
2. **A** *has less than* $2n+2$ *elements if and only if* $S_n(x_0, \ldots, x_n) \approx \bar{1}$ *holds in* **A**.

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$\mathbb{NM}- = \mathbb{NM} + \nabla(x) \approx \Delta(x)$

$\mathbb{NM}- = \mathcal{V}(\{\mathbf{A}_{2n} : n > 0\})$

NM-varieties

Theorem (Gispert 03)

Every nontrivial variety of NM-algebras is of one of the following types:

1. $\mathbb{NM} = \mathcal{V}([\mathbf{0},\mathbf{1}]) = \mathcal{V}(\{\mathbf{A}_n : n > 1\})$
2. $\mathbb{NM}- = \mathcal{V}([\mathbf{0},\mathbf{1}]^-) = \mathcal{V}(\{\mathbf{A}_{2n} : n > 0\})$
3. $\mathbb{NM}_{2m+1} = \mathcal{V}(\mathbf{A}_{2m+1})$ *for some* $m > 0$
4. $\mathbb{NM}_{2n} = \mathcal{V}(\mathbf{A}_{2n})$ *for some* $n > 0$
5. $\mathbb{NM}_{2n2m+1} = \mathcal{V}(\{\mathbf{A}_{2n}, \mathbf{A}_{2m+1}\})$ *for some* $n > m > 0$
6. $\mathbb{NM}-_{2m+1} = \mathcal{V}(\{[\mathbf{0},\mathbf{1}]^-, \mathbf{A}_{2m+1}\}) = \mathcal{V}(\{\mathbf{A}_{2n} : n > 0\} \cup \{\mathbf{A}_{2m+1}\})$

NM-varieties as quasivarieties

Theorem (Noguera et al. 08)

Every nontrivial variety of NM-algebras is of one of the following types:

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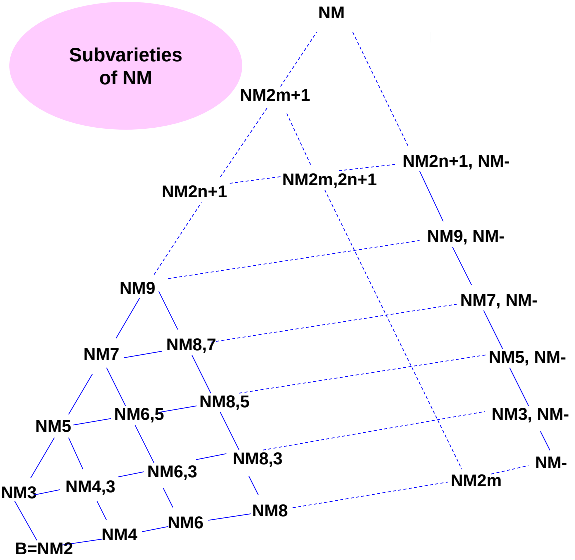
6. $\mathbb{NM}-_{2m+1} = \mathcal{Q}(\{[\mathbf{0}, \mathbf{1}]^-, \mathbf{A}_{2m+1}\}) = \mathcal{Q}(\{\mathbf{A}_{2n} : n > 0\} \cup \{\mathbf{A}_{2m+1}\})$

Lattice of NM-varieties



Structural completeness of NM

Proposition

$\mathbb{NM}$ *is not structurally complete.*

Structural completeness of NM

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Proof:

$\neg x \approx x \Rightarrow \bar{0} \approx \bar{1}$ is $\mathbb{NM}$-admissible (passive) but not valid in $\mathbb{NM}$.

Structural completeness of NM-

Theorem

$\mathbb{NM}-$ *is hereditarily structurally complete.*

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For every $n > 0$, $\mathbf{A}_{2n}$ *is embeddable into* $\mathbf{Free}_{\mathbb{NM}-}(\omega)$.

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$\mathcal{Q}(\mathbf{Free}_{\mathbb{NM}-}(\omega)) = \mathcal{Q}(\{\mathbf{A}_{2n} : n > 0\}) = \mathbb{NM}-$

For every $n > 0$, $\mathcal{Q}(\mathbf{Free}_{\mathbb{NM}_{2n}}(\omega)) = \mathcal{Q}(\mathbf{A}_{2n}) = \mathbb{NM}_{2n}$

Almost structural completeness of NM

If $\mathbb{M} \not\subseteq \mathbb{NM}-$, then

Proposition

For every $k > 1$,
$\mathbf{A}_2 \times \mathbf{A}_k$ *is embeddable into* $\mathbf{Free}_{\mathbb{M}}(\omega)$ *if and only if* $\mathbf{A}_k \in \mathbb{M}$

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Theorem

$\mathbb{M}$ *is almost structurally complete*

Almost structural completeness of NM

Theorem

$\mathbb{NM}$ *is almost structurally complete and all their subvarieties are almost structurally complete.*

Axiomatization of admissible quasiequations

Theorem

For every variety of NM-algebras the quasiequation $\neg x \approx x \Rightarrow \bar{0} \approx \bar{1}$ axiomatizes all passive admissible quasiequations.

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For every variety of NM-algebras the quasiequation $\neg x \approx x \Rightarrow \bar{0} \approx \bar{1}$ *axiomatizes all passive admissible quasiequations.*

Proof:

(Jeřábek 2010)

The rule $\neg(p \vee \neg p)^n / \bot$ *axiomatizes all passive rules for every n-contractive axiomatic extension of MTL.*

$\mathbb{NM}$ is 2 contractive ($x^2 \approx x^3$)

$$\neg p \leftrightarrow p \dashv\vdash_{NML} \neg(p \vee \neg p)^2$$

NM-quasivarieties

Proposition

Let $\mathbb{M}$ be a non trivial variety of NM-algebras and $\mathbb{K}$ be an $\mathbb{M}$-quasivariety. Then $\mathbb{K}$ is a proper $\mathbb{M}$-quasivariety iff there is $\mathbf{A}_{2n+1} \in \mathbb{M} \setminus \mathbb{K}$ for some $n > 1$.

NM-quasivarieties

Theorem

Let $\mathbb{M}$ *be a non trivial NM-variety. If* $\mathbb{K}$ *is proper* $\mathbb{M}$*-quasivariety and* $k = \max\{n \in \mathbb{N} : \mathbf{A}_{2n+1} \in \mathbb{K}\}$*, then*

$$\mathbb{K} = \mathcal{Q}(\{\mathbf{A}_{2n} : \mathbf{A}_{2n} \in \mathbb{M}\} \cup \{\mathbf{A}_2 \times \mathbf{A}_{2m+1} : \mathbf{A}_{2m+1} \in \mathbb{M}\} \cup \{\mathbf{A}_{2k+1}\})$$

Moreover, $\mathbb{K}$ *is axiomatized relative to* $\mathbb{M}$ *by the quasiequation*

$$x \approx \neg x \Rightarrow S_k(x_0, \ldots, x_k) \approx \bar{1} \textit{ if } k > 0$$

or

$$x \approx \neg x \Rightarrow \bar{0} \approx \bar{1} \textit{ if } k = 0.$$

Quasivarieties of $\mathbb{NM}$

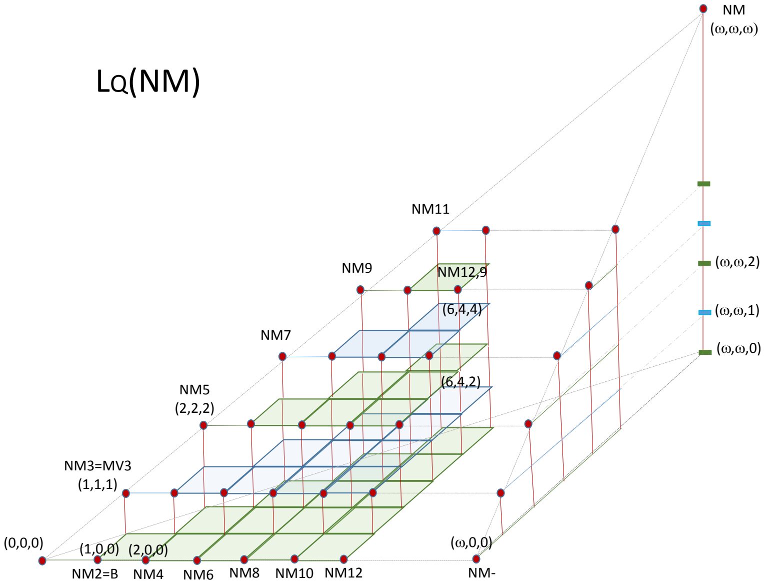
Theorem

$$L_Q(\mathbb{NM}) \cong \langle \{(n, m, k) \in \left(\omega^+\right)^3 : n \geq m \geq k\}, \leq^3 \rangle$$

where
$(n_1, m_1, k_1) \leq^3 (n_2, m_2, k_2)$ *iff* $n_1 \leq n_2$, $m_1 \leq m_2$ *and* $k_1 \leq k_2$

Quasivarieties of $\mathbb{NM}$

LQ(NM)



Łukasiewicz logics

The **Infinite valued Łukasiewicz Calculus**, $Ł_\infty$

Axioms:

Ł1. $\varphi \to (\psi \to \varphi)$

Ł2. $(\varphi \to \psi) \to ((\psi \to \nu) \to (\varphi \to \nu))$

Ł3. $((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$

Ł4. $(\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$

Rules:

Modus Ponens. $\{\varphi, \varphi \to \psi\}/\psi$.

Original logic semantics

$$[0,1]_{Ł} = \langle \{a \in \mathbb{R} : 0 \leq a \leq 1\}; \rightarrow, \neg \rangle$$

For all $a, b \in [0,1]$,

$$a \rightarrow b = \begin{cases} 1, & \text{if } a \leq b; \\ 1 - a + b, & \text{otherwise.} \end{cases}, \qquad \neg a = 1 - a.$$

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For all $a, b \in [0,1]$,

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Let $\Gamma \cup \{\varphi\} \subseteq Prop(X)$, then

$\Gamma \models_{[0,1]_{Ł}} \varphi$ iff
for every $h : Prop(x) \rightarrow [0,1]$, $h(\varphi) = 1$ whenever $h\Gamma = \{1\}$

Completeness Theorems

Weak Completeness Theorem

Theorem (Rose-Rosser 1958, Chang 1959)

$$\vdash_{Ł_\infty} \varphi \textit{ iff } \models_{[0,1]_Ł} \varphi$$

Completeness Theorems

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Strong Finite Completeness Theorem

Theorem (Hay 1963)

$$\varphi_1, \ldots, \varphi_n \vdash_{Ł_\infty} \varphi \textit{ iff } \varphi_1, \ldots, \varphi_n \models_{[0,1]_Ł} \varphi$$

Algebraic logic

The infinite valued Łukasiewicz calculus $Ł_\infty$ is algebraizable with $\mathbb{MV}$ the class of all MV-algebras as its equivalent quasivariety semantics.

MV-algebras

An **MV-algebra** is an algebra $\langle A, \oplus, \neg, 0\rangle$ satisfying the following equations:

MV1 $(x \oplus y) \oplus z \approx x \oplus (y \oplus z)$

MV2 $x \oplus y \approx y \oplus x$

MV3 $x \oplus 0 \approx x$

MV4 $\neg(\neg x) \approx x$

MV5 $x \oplus \neg 0 \approx \neg 0$

MV6 $\neg(\neg x \oplus y) \oplus y \approx \neg(\neg y \oplus x) \oplus x.$

- $1 =_{def} \neg 0$.
- $x \to y =_{def} \neg x \oplus y$.
- $x \vee y =_{def} (x \to y) \to y$.
- $x \wedge y =_{def} \neg(\neg x \vee \neg y)$.
- $x \odot y =_{def} \neg(\neg x \oplus \neg y)$.

For any MV-algebra **A**, $a \leq b$ iff $a \to b = 1$ endows **A** with a distributive lattice-order $\langle A, \vee, \wedge \rangle$, called the *natural order* of A.

An MV-algebra whose natural order is total is said to be an **MV-chain**.

MV-chains and totally ordered abelian groups

Let $\langle G, , +, -, 0, \leq \rangle$ be a totally ordered abelian group and an element $0 < u \in G$, if we define $\Gamma(G, u) = \langle [0, u], \oplus, \neg, 0 \rangle$ by

$$[0, u] = \{a \in G \mid 0 \leq a \leq u\}, \quad a \oplus b = u \wedge (a + b), \quad \neg a = u - a,$$

then $\langle [0, u], \oplus, \neg, 0 \rangle$ is an MV-chain.

Moreover every MV-chain is of this form.

Examples

- $[0,1]_{Ł} = \Gamma(\mathbb{R}, 1)$,
- $[0,1]_{Ł} \cap \mathbb{Q} = \Gamma(\mathbb{Q}, 1)$,

For every $0 < n < \omega$

- $\mathrm{L}_n = \Gamma(\mathbb{Z}, n) = \langle \{0, 1, \ldots, n\}, \oplus, \neg, 0 \rangle$,
- $\mathrm{L}_n^{\omega} = \Gamma(\mathbb{Z} \times_{lex} \mathbb{Z}, (n, 0)) = \langle \{(k, i) : (0,0) \leq (k, i) \leq (n, 0)\}, \oplus, \neg, 0 \rangle$.
- $\mathrm{L}_n^{s} = \Gamma(\mathbb{Z} \times_{lex} \mathbb{Z}, (n, s)) = \langle \{(k, i) : (0,0) \leq (k, i) \leq (n, s)\}, \oplus, \neg, 0 \rangle$, where $0 \leq s < n$.
- $\mathrm{S_n} = \Gamma(T, n)$ where T is a totally ordered dense subgroup of $\mathbb{R}$ such that $T \cap \mathbb{Q} = \mathbb{Z}$.

Finite MV-chains

For every $0 < n < \omega$, *every* $n + 1$ *element MV-chain is isomorphic to* $\mathrm{L}_n = \Gamma(\mathbb{Z}, n) = \langle\{0, 1, \ldots, n\}, \oplus, \neg, 0\rangle$

Let $0 < n, k < \omega$. $\mathrm{L}_n \hookrightarrow \mathrm{L}_k$ *if and only if* $n|k$.

MV-varieties

The class $\mathbb{MV}$ of all MV-algebras is a (not locally finite) variety.

(Chang's completeness thorem)

$\mathbb{MV} = \mathcal{V}([0,1]) = \mathcal{V}(\{\mathrm{L}_n : n > 0\})$.

For every $n > 0$, $\mathbb{MV}_n = \mathcal{V}(\mathrm{L}_n)$ is a locally finite variety. $\mathbb{MV}_n$ is the equivalent quasivariety semantics of $Ł_{n+1}$ the $n+1$-valued Łukasiewicz logic.

Moreover if $\mathbb{V}$ is a variety of MV-algebras, $\mathbb{V}$ is locally finite iff $\mathbb{V} \subseteq \mathbb{MV}_n$ for some $n > 0$

MV-varieties

Theorem (Komori, 1981)

$\mathbb{V}$ is a proper subvariety of $\mathbb{MV}$ iff there exist two finite sets I and J (in a reduced form) of integers ≥ 1, not both empty, such that

$$\mathbb{V} = \mathcal{V}_{I,J} := \mathcal{V}(\{\mathrm{L}_m \mid m \in I\} \cup \{\mathrm{L}_n^{\omega} \mid n \in J\}).$$

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- *Every proper subvariety of $\mathbb{MV}$ is finitely axiomatizable.*
- *The lattice of all varieties of MV-algebras is a Pseudo-Boolean algebra.*

MV-varieties as quasivarieties

$$\mathbb{MV} = \mathcal{Q}([0,1] \cap \mathbb{Q}) = \mathcal{Q}([0,1]) = \mathcal{Q}(\{\mathrm{L}_n : n > 0\}).$$

$$\mathcal{V}_{I,J} := \mathcal{Q}(\{\mathrm{L}_m \mid m \in I\} \cup \{\mathrm{L}_n^\omega \mid n \in J\}).$$

Structural completeness of Łukasiewicz logics

Theorem (Pogorzelski, Torzak, Wojtylak 1970's, Dzik 2008, Jerabek 2010)

- $Ł_\infty$ ($\mathbb{MV}$) *is not structurally complete.*
- $Ł_\infty$ ($\mathbb{MV}$) *is not almost structurally complete.*
- $Ł_{n+1}$ ($\mathbb{MV}_n$) *is not structurally complete.*
- $Ł_{n+1}$ ($\mathbb{MV}_n$) *is hereditarily almost structurally complete.*

Structural completeness of Łukasiewicz logics

Theorem

- $\mathcal{V}_{\emptyset,\{1\}} = \mathcal{V}(\mathrm{L}_1^\omega)$ *is structurally complete.*
- $\mathcal{V}_{\emptyset,\{1\}}$ *and* $\mathbb{B}$ *are the only structurally complete varieties of MV-algebras.*
- $\mathbb{V}$ *is almost structurally complete iff* $\mathbb{V}$ *is locally finite or* $\mathbb{V} = \mathcal{V}_{I,\{1\}}$ *for some reduced pair* $(I, \{1\})$.

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For every reduced pair (I, J),
$\mathcal{Q}(\mathbf{Free}_{\mathcal{V}_{I,J}}) = \mathcal{Q}(\{\mathrm{L}_1 \times \mathrm{L}_n : n \in I\} \cup \{\mathrm{L}_1 \times \mathrm{L}_m^1 : m \in J\})$.

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$\mathcal{V}_{I,\{1\}} = \mathcal{Q}(\{\mathrm{L}_n : n \in I\} \cup \{\mathrm{L}_1^1\})$.

MV-quasivarieties

Theorem (Adams-Dziobiak)

The class $\mathbb{MV}$ *is* Q-universal, *in the sense that, for every quasivariety* $\mathbb{K}$ *of algebras of finite type (not necessarily MV-algebras), the lattice of all quasivarieties of* $\mathbb{K}$ *is the homomorphic image of a sublattice of the lattice of all quasivarieties of* $\mathbb{MV}$*.*

$$L_Q(\mathbb{K}) \in \mathcal{HS}(L_Q(\mathbb{MV}))$$

Locally finite MV-quasivarieties

In the case of MV-algebras:

Locally finite MV-quasivarieties

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The following conditions are equivalent:

- $\mathbb{K}$ is a locally finite quasivariety.
- $\mathbb{K} \subseteq \mathbb{MV}_n$ for some $n \in \mathbb{N}$.
- $\mathbb{K}$ is subquasivariety contained in a discriminator variety.

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Vaggione et al: "The subquasivariety lattice of a discriminator variety"

Locally finite MV-quasivarieties

Every locally finite quasivariety of MV-algebras is generated by a finite set of critical algebras.

A **critical** algebra is a finite algebra not belonging to the quasivariety generated by all its proper subalgebras. From the characterization of critical MV-algebras (Gispert-Torrens)

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Let $\mathbb{V}$ be a locally finite MV-variety. Then

- *$L_Q(\mathbb{V})$ is finite*
- *Every member of $L_Q(\mathbb{V})$ is finitely based.*

Moreover for any $\mathbb{K} \in L_Q(\mathbb{V})$,
$\mathbb{K}$ is a variety iff $\mathbb{K}$ is generated by MV-chains.

Quasivarieties generated by MV-chains

Theorem

Two MV-chains generate the same quasivariety iff they generate the same variety and they contain the same rational elements.

Given $\mathbf{A} = \Gamma(G, b)$, a is a **rational element** of $\mathbf{A}$ iff there exist $m, n \in \omega$, $0 \leq n \leq m \neq 0$ and $c \in G$ such that $b = mc$ and $a = nc$. In that case, we say that $a = \frac{n}{m}$.

Quasivarieties generated by MV-chains

Theorem

$\mathbb{K}$ is a quasivariety generated by MV-chains if and only if there are α, γ, κ subsets of positive integers, not all of them empty, and for every $i \in \gamma$, a nonempty subset $\gamma(i) \subseteq Div(i)$ such that

$$\mathbb{K} = \mathcal{Q}(\{\mathrm{L}_n : n \in \alpha\} \cup \{\mathrm{L}_i^{d_i} : i \in \gamma,\ \ d_i \in \gamma(i)\} \cup \{\mathrm{S}_k : k \in \kappa\}).$$

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- *Every quasivariety generated by MV-chains contained in a proper subvariety of $\mathbb{MV}$ is finitely axiomatizable.*
- *The lattice of all quasivarieties generated by MV-chains is a bounded distributive lattice*

From the characterization of quasivarieties generated by MV-chains it can be deduced:

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- $\mathcal{Q}(\mathrm{L}_n)$ is the least $\mathcal{V}(\mathrm{L}_n)$-quasivariety generated by chains.
- For every reduced pair (I, J), $\mathcal{Q}(\{\mathrm{L}_n \ : \ n \in I\} \cup \{\mathrm{L}_m^1 \ : \ m \in J\})$ is the least $\mathcal{V}_{I,J}$-quasivariety generated by chains.

Structurally complete quasivarieties and least $\mathbb{V}$-quasivarieties.

Theorem

For every reduced pair (I, J)*,*
$\mathcal{Q}(\{\mathrm{L}_1 \times \mathrm{L}_n : n \in I\} \cup \{\mathrm{L}_1 \times \mathrm{L}_m^1 : m \in J\}) = \mathcal{Q}(\mathbf{Free}_{\mathcal{V}_{I,J}})$ *and therefore it is the least* $\mathcal{V}_{I,J}$*-quasivariety.*

(Almost) structural completeness again

For every reduced pair (I, J),

- $\mathcal{Q}(\{\mathrm{L}_1 \times \mathrm{L}_n : n \in I\} \cup \{\mathrm{L}_1 \times \mathrm{L}_m^1 : m \in J\})$ is the least $\mathcal{V}_{I,J}$-quasivariety.
- $\mathcal{Q}(\{\mathrm{L}_n : n \in I\} \cup \{L_m^1 : m \in J\})$ is the least $\mathcal{V}_{I,J}$-quasivariety generated by chains.

(Almost) structural completeness again

For every reduced pair (I, J),

- $\mathcal{Q}(\{\mathrm{L}_1 \times \mathrm{L}_n : n \in I\} \cup \{\mathrm{L}_1 \times \mathrm{L}^1_m : m \in J\})$ is the least $\mathcal{V}_{I,J}$-quasivariety.
- $\mathcal{Q}(\{\mathrm{L}_n : n \in I\} \cup \{L^1_m : m \in J\})$ is the least $\mathcal{V}_{I,J}$-quasivariety generated by chains.

Thus,

Theorem

Let (I, J) be a reduced pair. Then $\mathcal{Q}(\{\mathrm{L}_n : n \in I\} \cup \{L^1_m : m \in J\})$ is almost structurally complete.

Axiomatization of admissible rules.

$\mathbb{MV}$-admissible quasiequations.

(Jeřábek)

- *An infinite basis of non passive admissible rules in order to axiomatize all admissible $Ł_\infty$-rules. Infinite axiomatization of* $\mathbb{MV}$*-admissible quasiequations.*
- $\mathbb{MV}$*-admissible quasiequations are not finitely axiomatizable.*
- *Let* $\mathbb{V}$ *be a variety of MV-algebras. Then* $\{(x \vee \neg x)^n \approx 0 \Rightarrow 0 \approx 1 \ : n \in \omega\}$ *is a basis for passive* $\mathbb{V}$*-admissible quasiequations.*

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$\mathcal{Q}(\mathbf{Free}_{\mathbb{MV}}) = \mathcal{Q}(\mathcal{M}([0,1]))$ *is the only almost structurally complete* $\mathbb{MV}$*-quasivariety*

Axiomatization of admissible rules.

Admissible quasiequations in locally finite MV-varieties

- *Let* $\mathbb{V}$ *be an MV-variety. Then* $\mathbb{V}$ *is locally finite iff* $\mathbb{V}$ *is n-contractive for some* $n \in \omega$*.*
- *Every locally finite MV-variety is almost structurally complete. (Dzik)*
- $(x \vee \neg x)^n \approx 0 \Rightarrow 0 \approx 1$ *is a basis of passive admissible quasiequations for every n-contractive subvariety of* $\mathbb{MV}$*. (Jeřábek)*

Basis for admissible quasiequations for proper subvarieties of $\mathbb{MV}$

Let $\mathcal{V}_{I,J}$ be a proper subvariety of $\mathbb{MV}$.

- $Q^1_{I,J} := \mathcal{Q}(\{\mathrm{L}_n : n \in I\} \cup \{\mathrm{L}^1_m : m \in J\})$ is almost structurally complete.

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- $Q^1_{I,J} := \mathcal{Q}(\{\text{Ł}_n : n \in I\} \cup \{\text{Ł}^1_m : m \in J\})$ is almost structurally complete.
- $Q^1_{I,J}$ is a $\mathcal{V}_{I,J}$-quasivariety ($\mathcal{V}(Q^1_{I,J}) = \mathcal{V}_{I,J}$)

Basis for admissible quasiequations for proper subvarieties of $\mathbb{MV}$

Let $\mathcal{V}_{I,J}$ be a proper subvariety of $\mathbb{MV}$.

- $Q^1_{I,J} := \mathcal{Q}(\{\mathrm{L}_n : n \in I\} \cup \{\mathrm{L}^1_m : m \in J\})$ is almost structurally complete.
- $Q^1_{I,J}$ is a $\mathcal{V}_{I,J}$-quasivariety $(\mathcal{V}(Q^1_{I,J}) = \mathcal{V}_{I,J})$
- $\mathcal{Q}(\{\mathrm{L}_n : n \in I\} \cup \{\mathrm{L}^1_m : m \in J\})$ is finitely axiomatizable.

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Theorem

All $\mathcal{V}_{I,J}$-admissible quasiequatons are finitely axiomatizable.

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Theorem

Let (I, J) be a reduced pair, then a base for admissible quasiequations of $\mathcal{V}_{I,J}$ is given by

- $\Delta'(Q_m) := [(\neg x)^{m-1} \leftrightarrow x] \vee y \approx 1 \Rightarrow y \approx 1$ *for every* $m \in Div(J) \setminus Div(I)$ *minimal with respect the divisibility.*
- $\Delta'(U_k) := [(\neg x)^{k-1} \leftrightarrow x] \vee y \approx 1 \Rightarrow \alpha_{I_k,\emptyset}(z) \vee y \approx 1$ *for every* $1 < k \in Div(I)$*, where* $I_k = \{n \in I : k|n\}$.
- $CC_n^1 := (\varphi \vee \neg\varphi)^n \approx 0 \Rightarrow 0 \approx 1$ *where* $n = \max\{I \cup \{\max J + 1\}\}$.

Conclusions

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Conclusions

- Results on admissibilty theory allow to characterize and axiomatize the lattice of subquasivarieties (finitary extensions).
- Results on certain quasivarieties (locally finite, generated by chains) allow to obtain axiomatization of admissible quasiequations.
- There is a relation among least V-quasivarieties generated by chains and (almost) structural completeness

Future Work

- Similar algebraic approach to admissible rules for other many-valued logics: BL, MTL, FL...

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- Study the relation among almost structural completeness and least V-quasivarieties generated by (finite) subdirectly irreducible algebras.
- Multiple conclusion admissible rules and universal classes.

THANK YOU FOR YOUR ATTENTION