MV-algebras, beyond algebraic logic

(Joint work with L.M.Cabrer)

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R.Cignoli about A.Monteiro

Revista Colombiana de Matemáticas XIX (1985) pp. 1-8

Actas IX Congreso Dr. Antonio A. R. Monteiro, 2007, pp.3-8

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I consider that he was mainly an algebraist, and that his interest for logic was as a source of algebraic problems, principally those capable of computable results.

Let K be any equational class of interest to you (no MV-algebras in the first part of this talk)

FIRST PROBLEM:

recognizing free generating sets in free K-algebras

Assessing four decision problems for K

Let K be an equational class of algebras and F_n the free n - generator K-algebra. For any set $t_1,\ldots,t_n$ of K-terms, all in the same variables $x_1,\ldots,x_n$, let $t'_1,\ldots,t'_n$ be their respective interpretations as elements of F_n .

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For boolean algebras and abelian groups, all four problems are easily decidable.

An old theorem by Jónsson-Tarski

THEOREM *Let K be a class of algebras such that every equation which is satisfied in all finite algebras of K is also satisfied in all algebras of K.*

Then for any algebra A in K, if A has a free generating set of n elements, then every generating set of A with n elements is a free generating set of A.

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Thus if K satisfies the hypothesis of the theorem, the above **four** decision problems are reduced to the **two** problems of deciding if $t'_1,\ldots,t'_n$ generate the free algebra, or an isomorphic copy of it

certifying that $t'_1,\ldots,t'_n$ generate the free K-algebra F_n

A **certificate** that $t'_1,\ldots,t'_n$ generate the free n-generator algebra F_n is given by K-terms $u_1,\ldots,u_n$ such that each composite term $u_i(t_1,\ldots,t_n)$ coincides with the *i*th variable x_i.

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If we can decide equality of K-terms, we can apply the Jónsson Tarski theorem and **recursively enumerate** the n-tuples $(t_1,...,t_n)$ of terms representing free generators.

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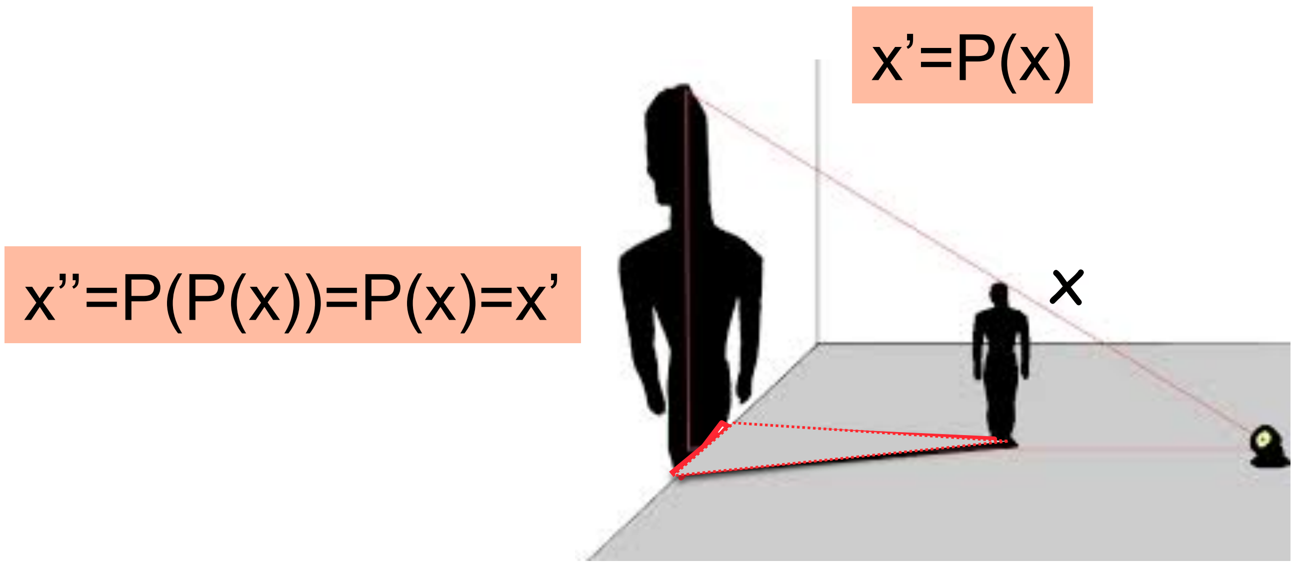
How to **certify that $t'_1,\ldots,t'_n$ do NOT generate** F_n?

SECOND PROBLEM:

characterizing **projective** algebras in your class K

Idempotent endomorphisms

to fix ideas, think of a projection map $P: E \rightarrow E$



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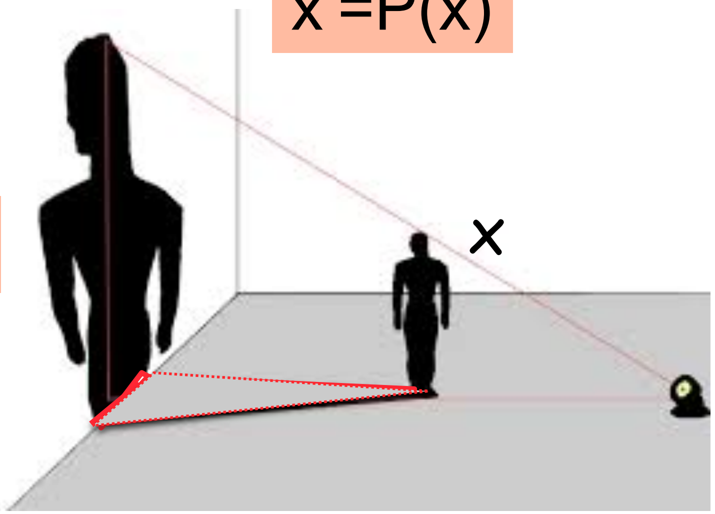
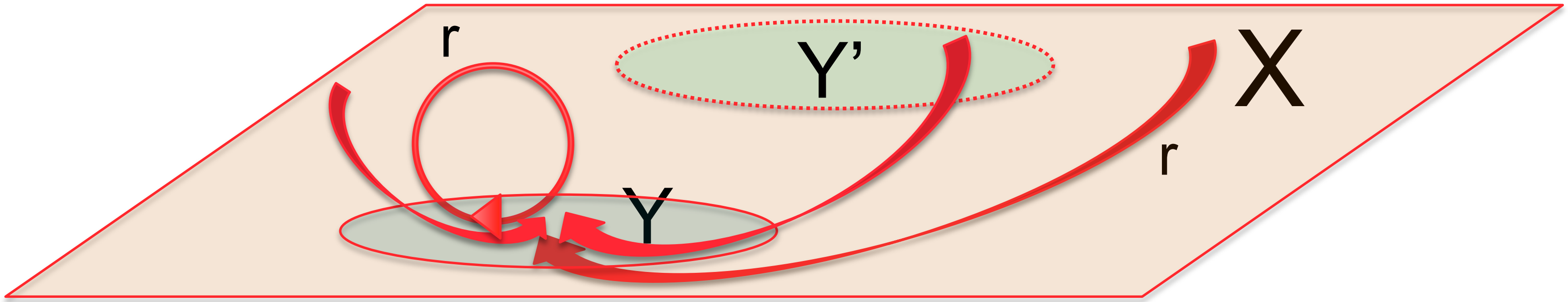


a projection map is **idempotent**, P(P)=P.
P acts identically over its range

Idempotent endomorphisms ≈ retractions



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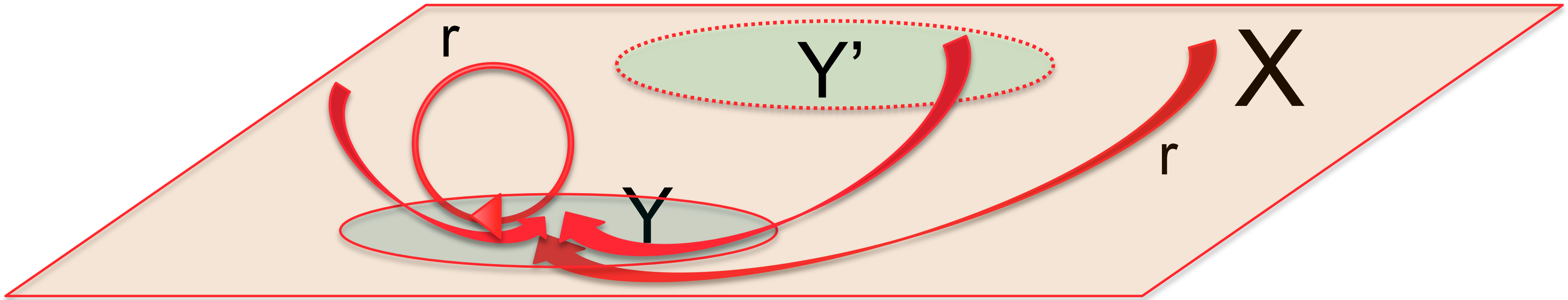

We all know what a retraction *r* : X→Y is. The map r acts identically on its range, $r^2 = r$, like a projection.

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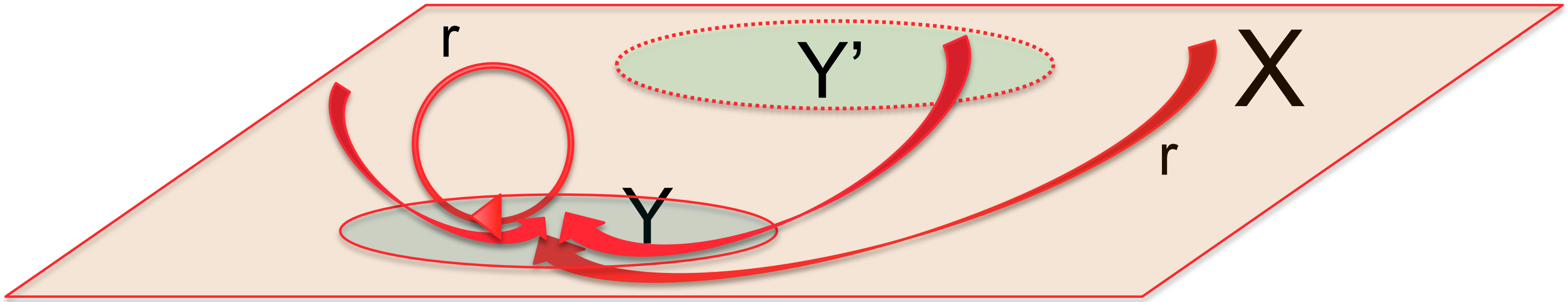

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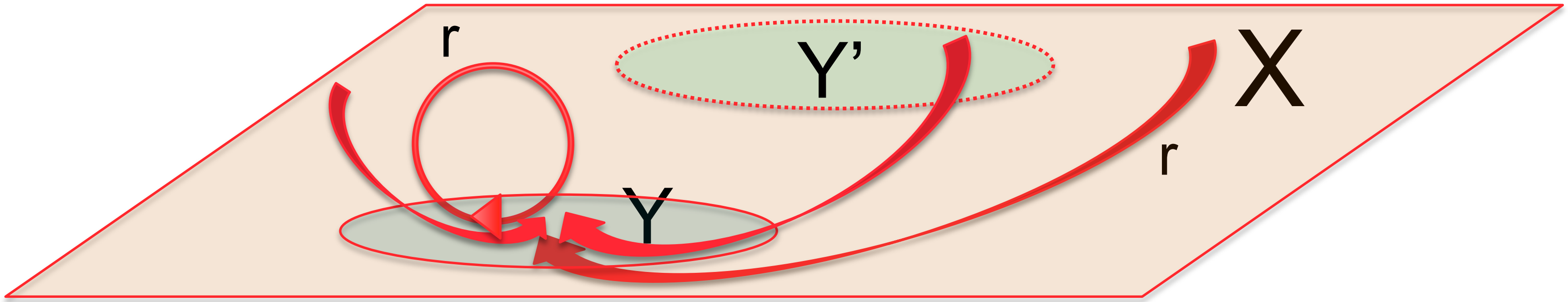

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What about the behavior of r over the domain X\Y ?

In the picture we have a region Y'≠Y where r **acts bijectively** onto Y. The number of such regions might yield new invariants for projective algebras.

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if in particular, A is n-generated then we can assume A to be **equal to** the image of an idempotent endomorphism ∂ of the free n-generator K-algebra F_n.

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Problem b. Can A be the image of **infinitely many** idempotent endomorphisms of F_n ?

Problem c. For each i=1,2,... does there exist algebras A_i such that the number of idempotent endomorphisms of F_n onto A_i is finite and > i ?

THIRD PROBLEM:

which algebras in your class K are hopfian ?

hopficity: a finiteness property

DEFINITION An algebra H is ***hopfian*** if every homomorphism of H onto H is an automorphism.

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THEOREM *Every finitely generated residually finite algebra is hopfian.*

we will study these problems from the topological and algorithmic viewpoint

in the spirit of Monteiro's study of extensions of the Stone theorem, connecting algebra and topology

we start from the recognition problem for combinatorial manifolds

the recognition problem for manifolds

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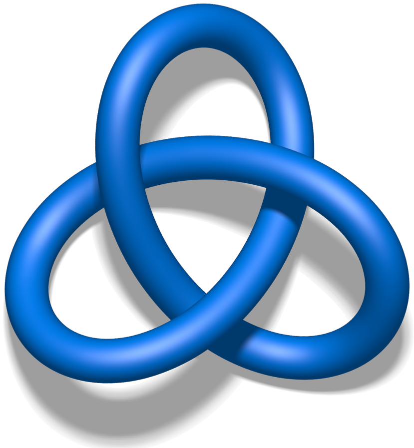
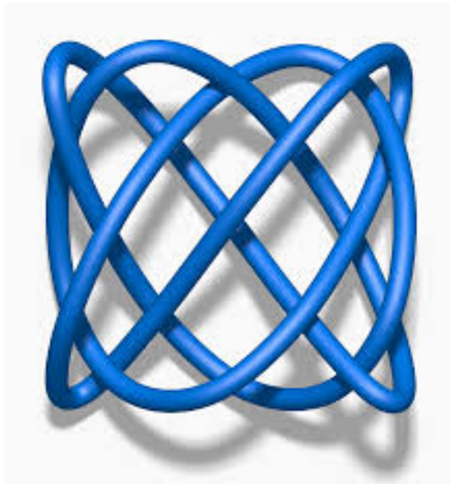
EXAMPLE: are these two manifolds homeomorphic?





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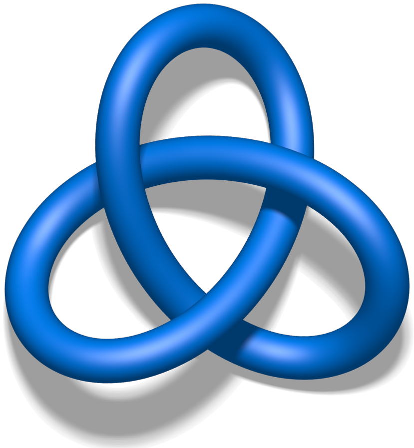

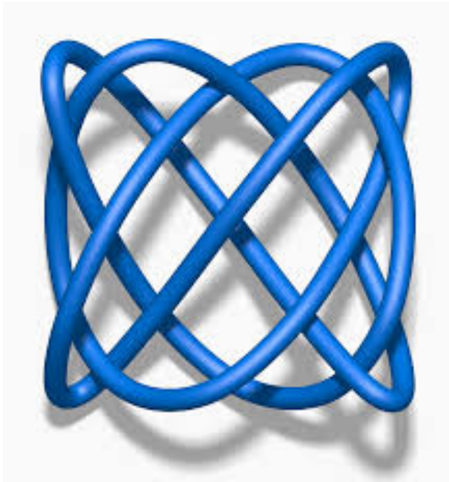

for this problem to make computational sense in general, the two combinatorial manifolds must be (triangulated and) presented to the computer as strings of symbols.

success: surface classification

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*Thus it is decidable whether two surfaces are homeomorphic, once they are **presented** as **rational polyhedra**.*

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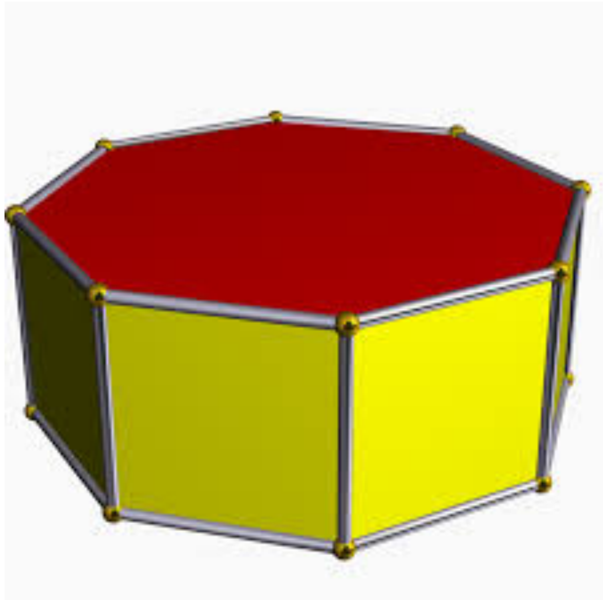

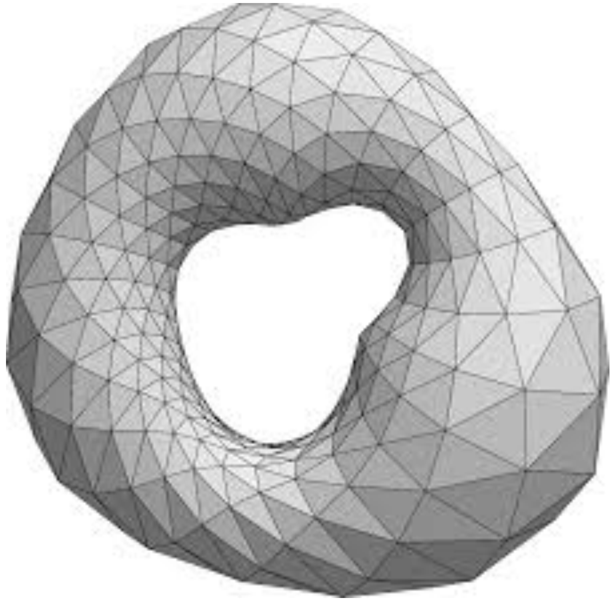

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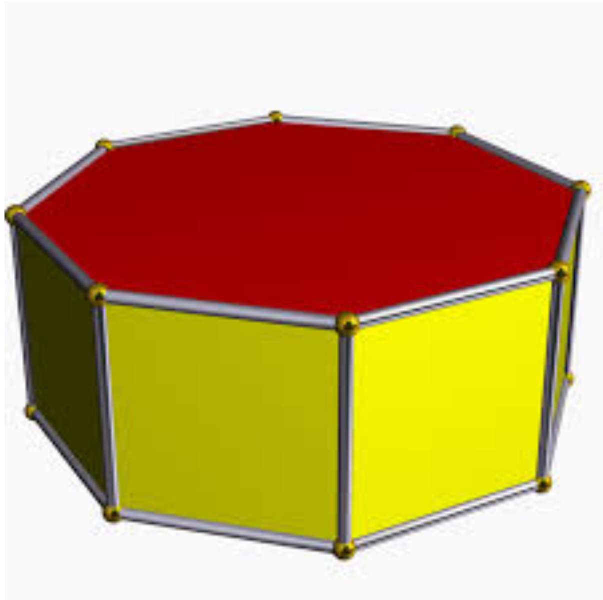

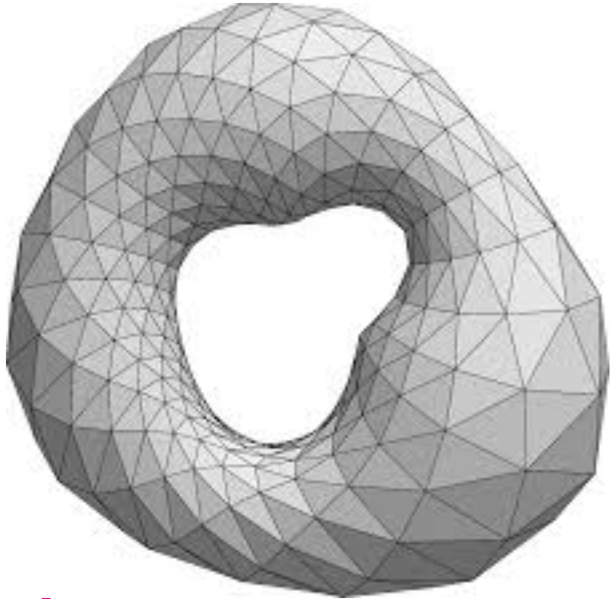

P need not be convex, nor connected

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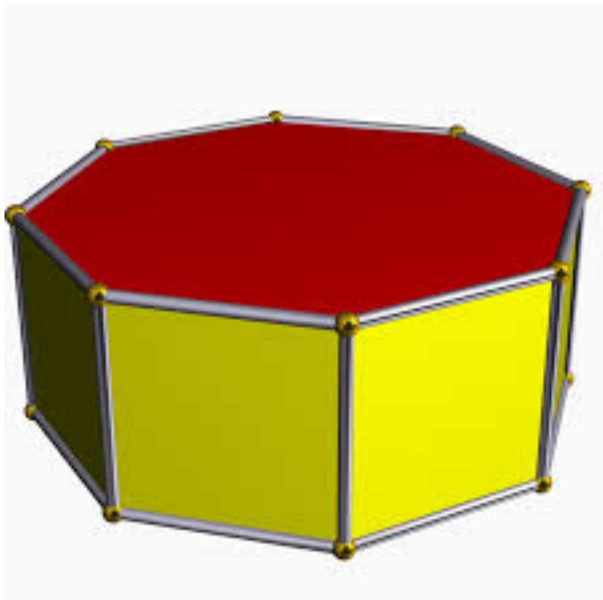

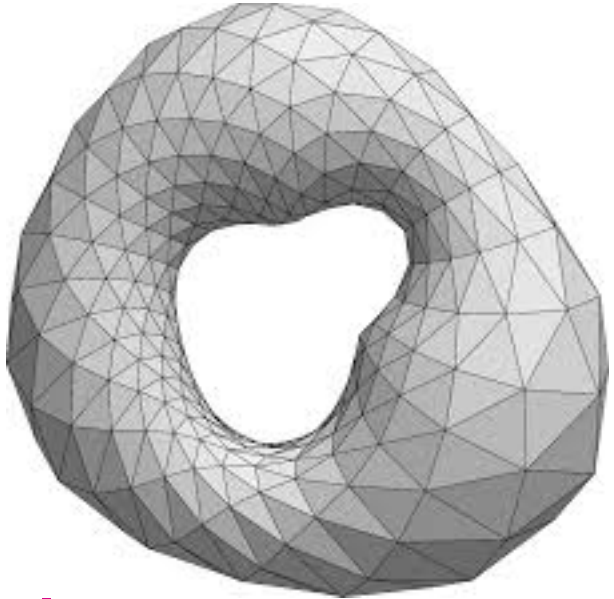

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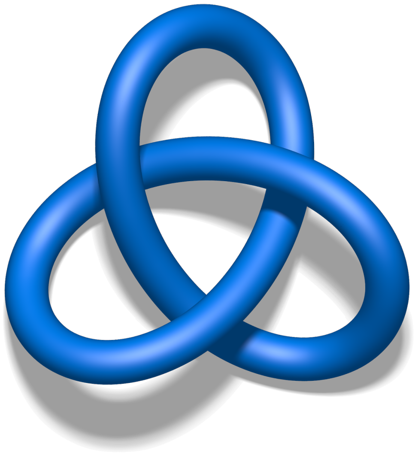
P is said to be **rational** if the vertices of each simplex S_i can be assumed to have rational coordinates

combinatorial manifold recognition



for the presentation of X as a finite string of symbols, X is **triangulated** by a finite simplicial complex Δ such that all simplexes in Δ have **rational vertices**

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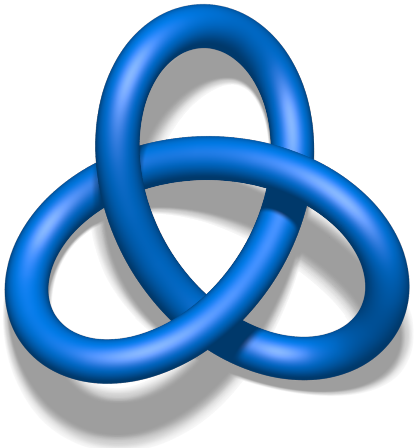

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pairs of piecewise linear (PL) homeomorphic polyhedra can be certified by combinatorially isomorphic rational triangulations

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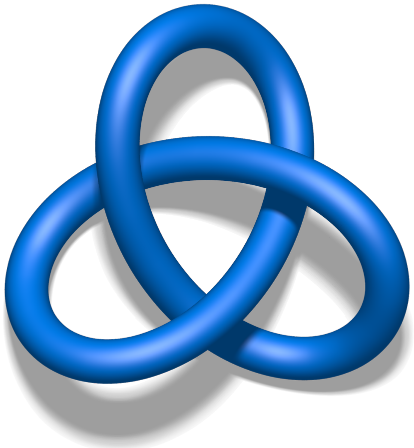

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pairs of piecewise linear (PL) homeomorphic polyhedra can be certified by combinatorially isomorphic rational triangulations

the problem is: how to **certify pairs of rational polyhedra which are not rationally** PL-homeomorphic?

A FUNDAMENTAL THEOREM

THEOREM (Markov, 1958)

Manifolds are not recognizable by Turing machines.

Thus there is no effective procedure to attach numerical invariants to rational polyhedra P, Q, so that P and Q are rationally PL-homeomorphic precisely when they have the same invariants.

a corollary of Baker-Beynon duality

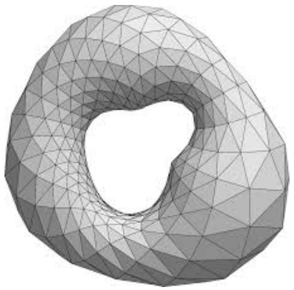
THEOREM (Baker-Beynon duality) *The category of rational polyhedra in euclidean space with arrows consisting of* ***rational*** *piecewise linear maps is dually equivalent to the category of finitely presented lattice-ordered abelian groups (l-groups).*

COROLLARY OF MARKOV THEOREM + DUALITY (Glass-Madden)

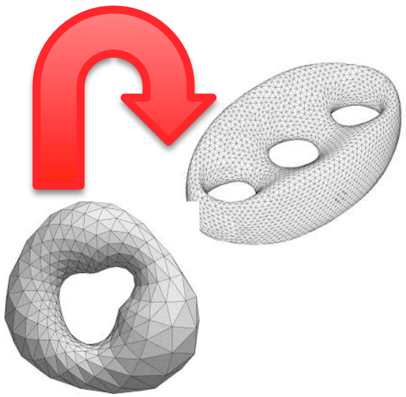
The isomorphism problem for finitely presented l-groups is undecidable.

other possible categories of rational polyhedra can be considered, proceeding as in the transition from a topological space X to the algebra C(X) of continuous real, or complex functions defined over X and, more importantly for our purposes here, also in the spirit of Stone's and Monteiro's analysis of the map X→L(X) from topological spaces X to suitable lattices L(X) associated to X

natural desiderata for a polyhedral category



OBJECTS: our polyhedra must be equipped with rational triangulations, so that we can make computations

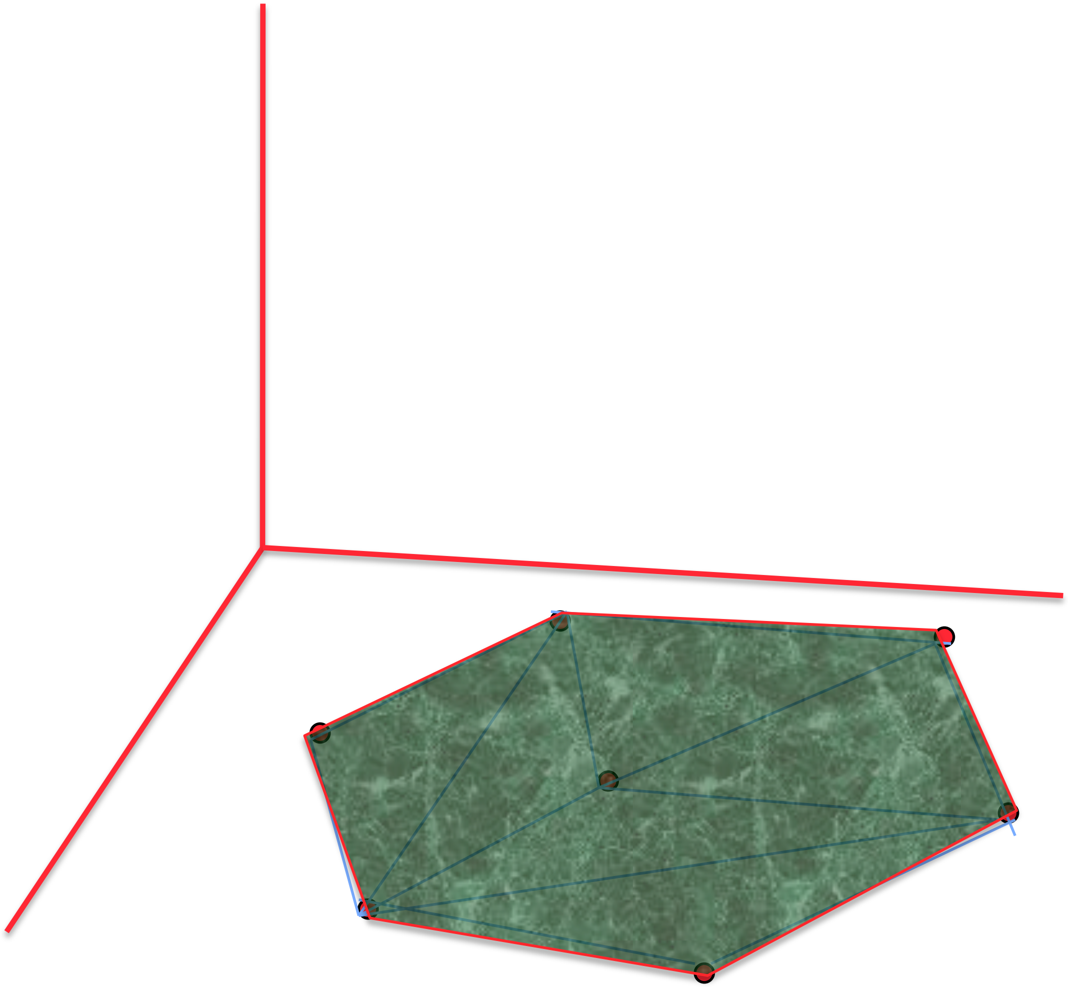


ARROWS: morphisms should be continuous functions whose graph is also a rational polyhedron, so that a morphism preserves both polyhedrality and rationality

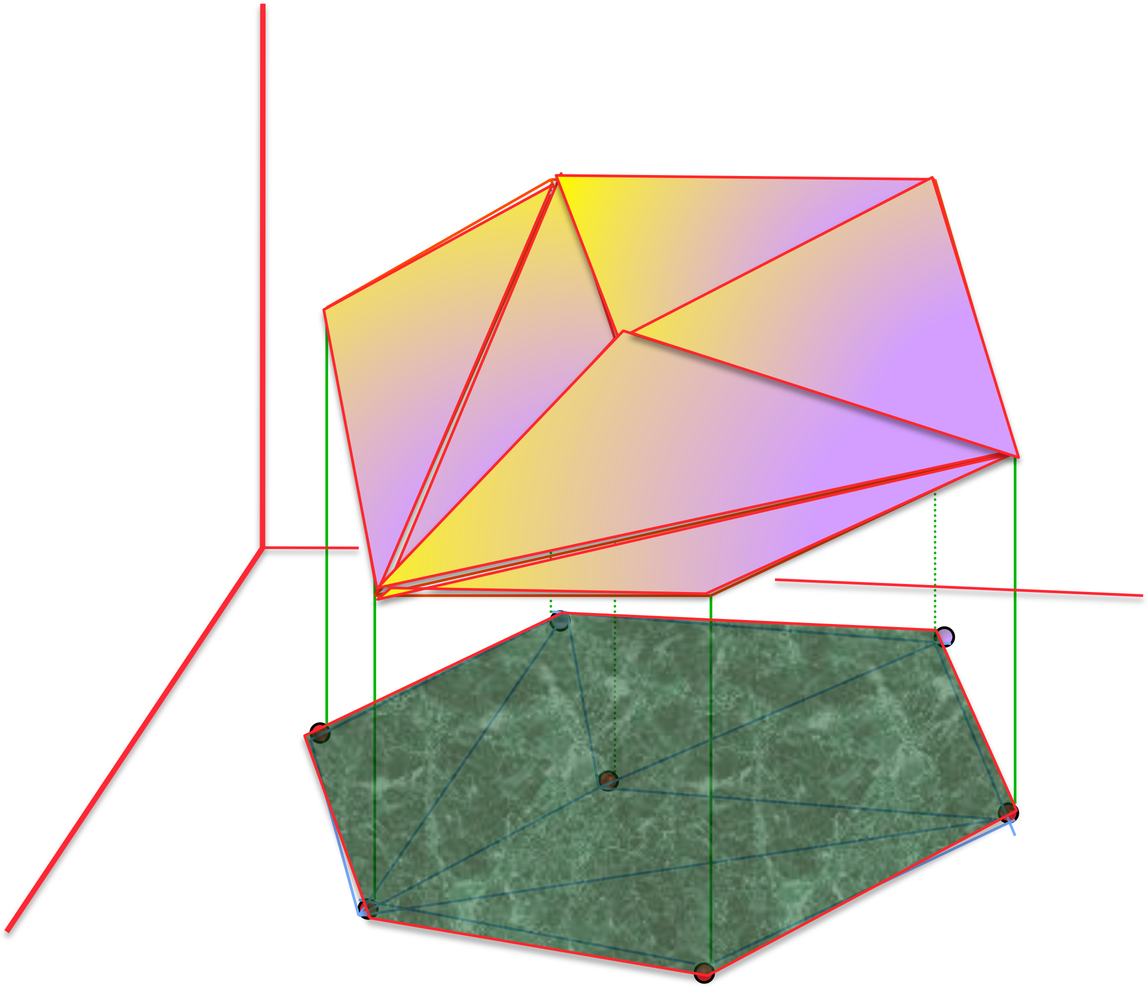
RATIONAL PWL thus our L(X) must be, at least, lattices of continous piecewise linear functions with rational coefficients

there are various possibilities...

A rational polyhedron P in $\mathbf{R}^2 \subseteq \mathbf{R}^3$



various piecewise linear functions on P



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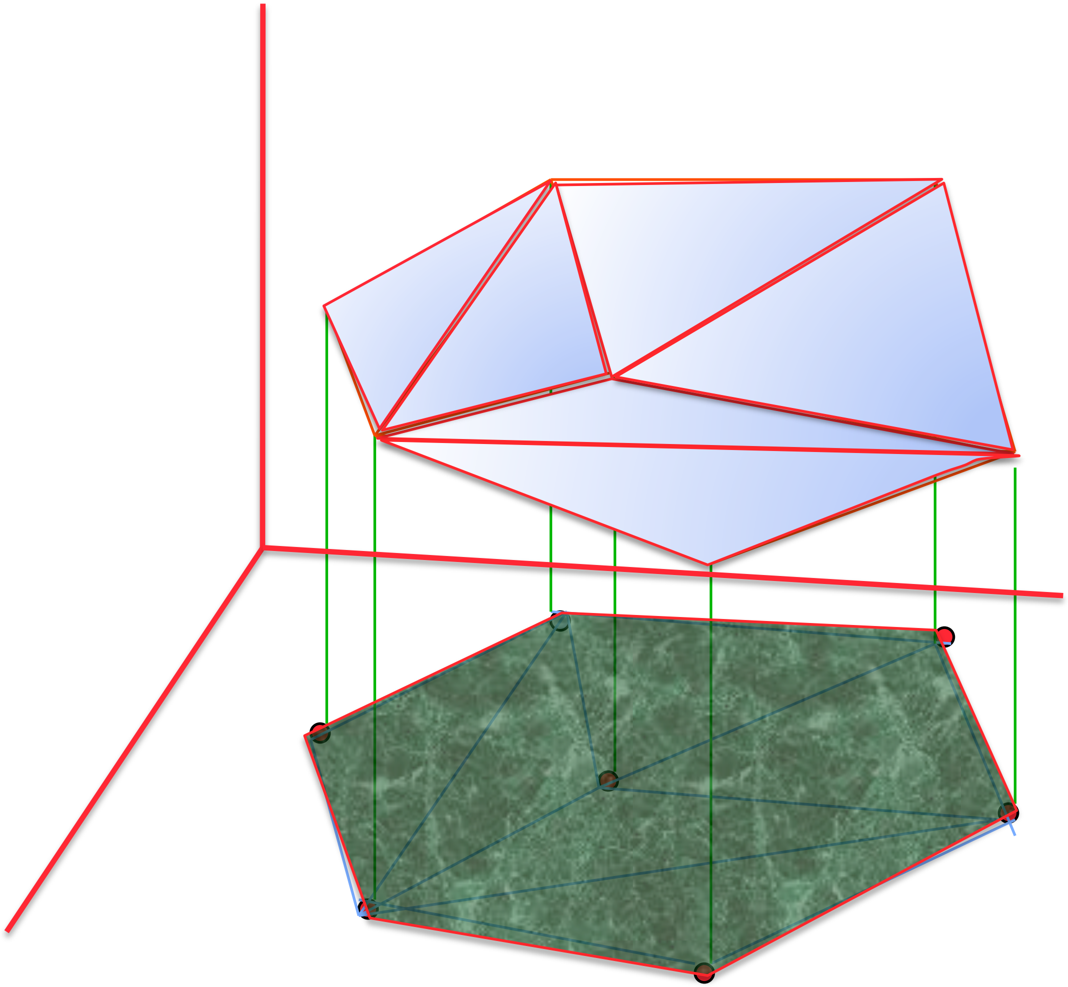

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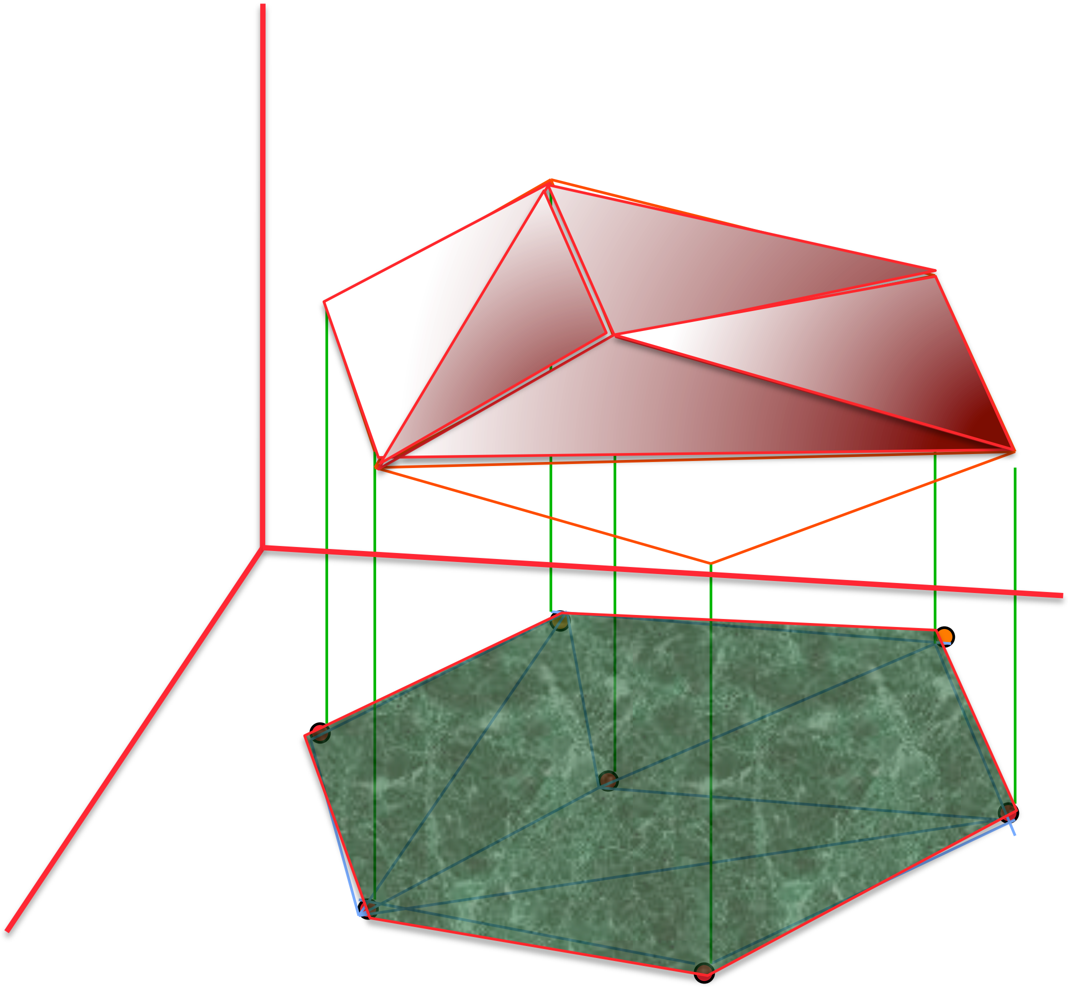

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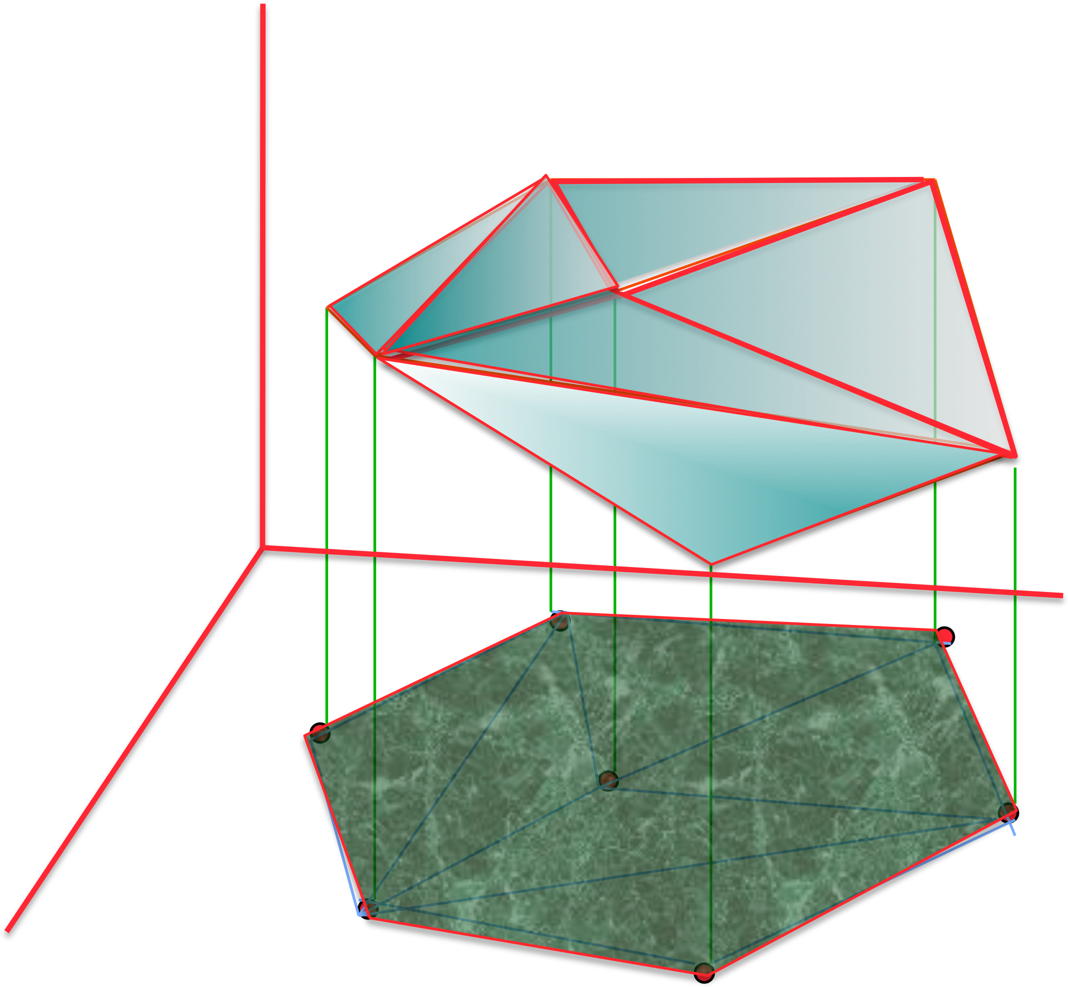

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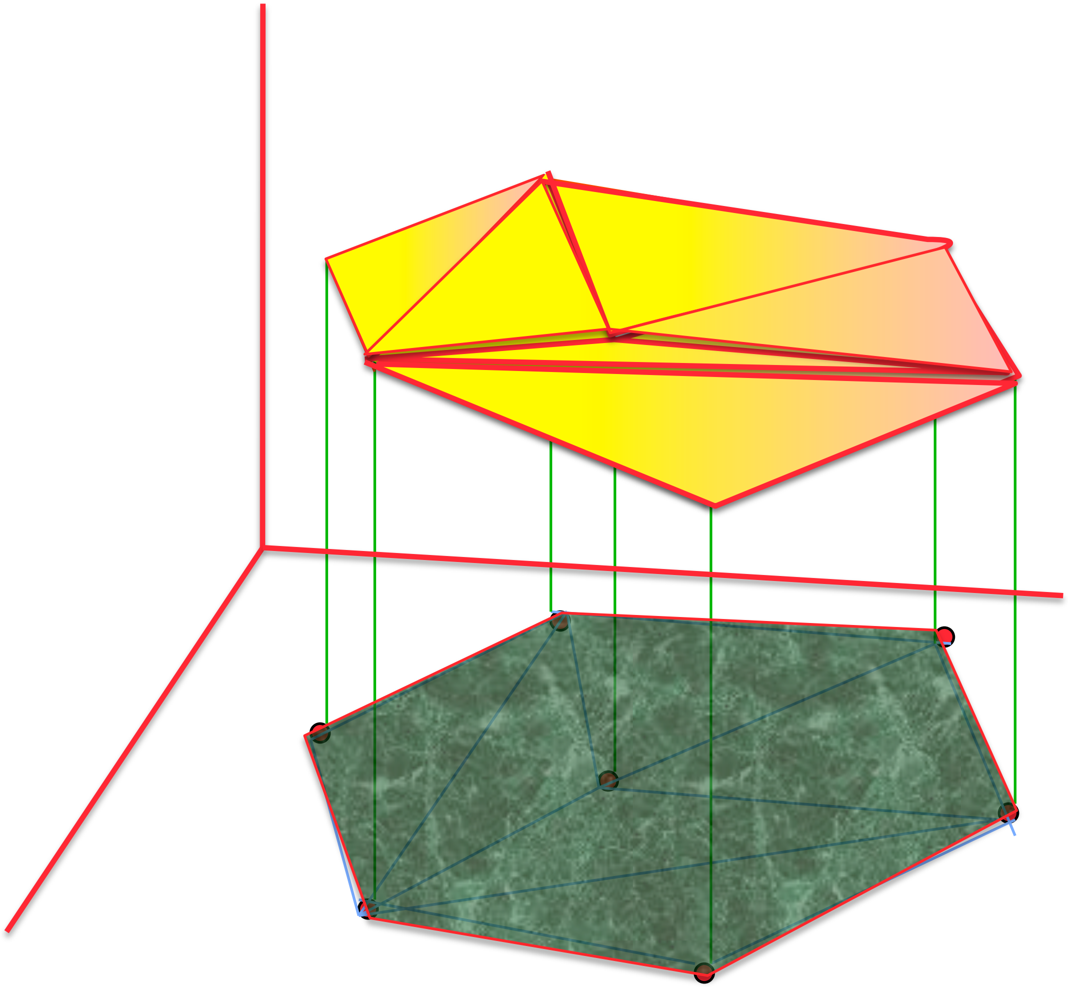
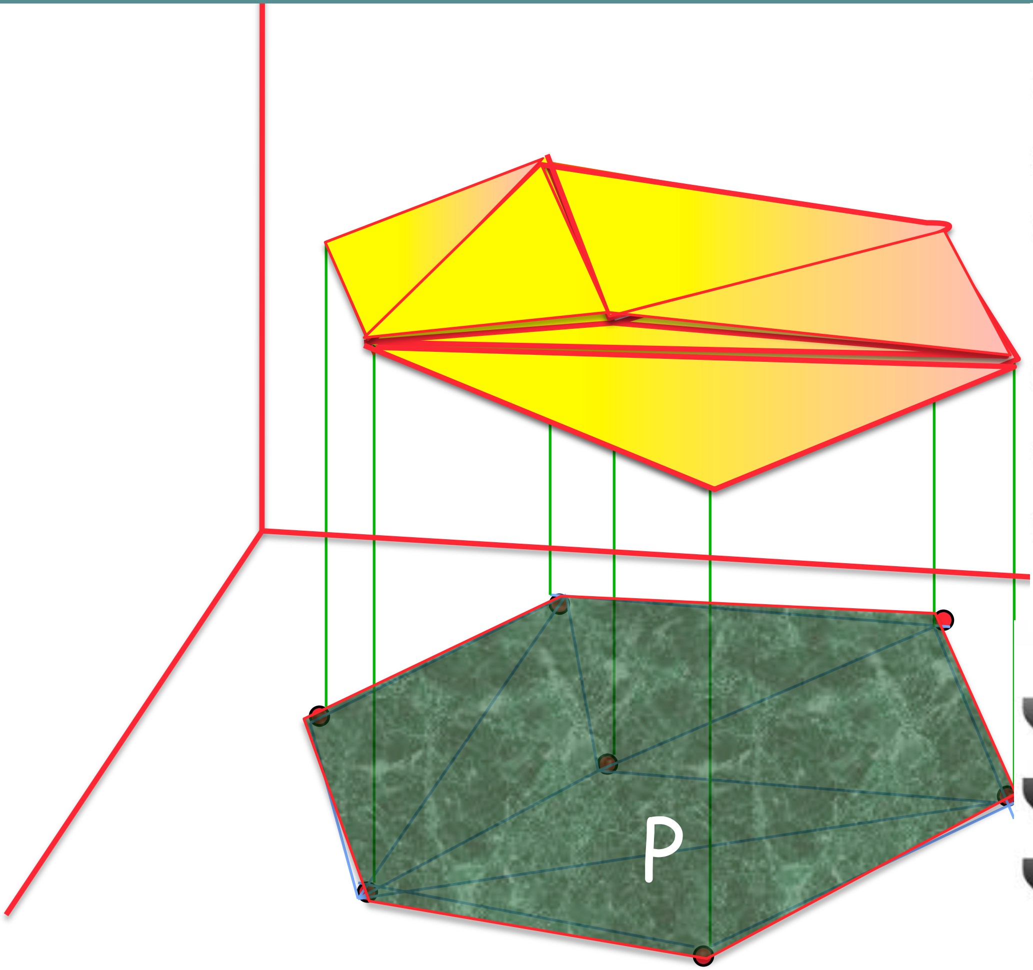
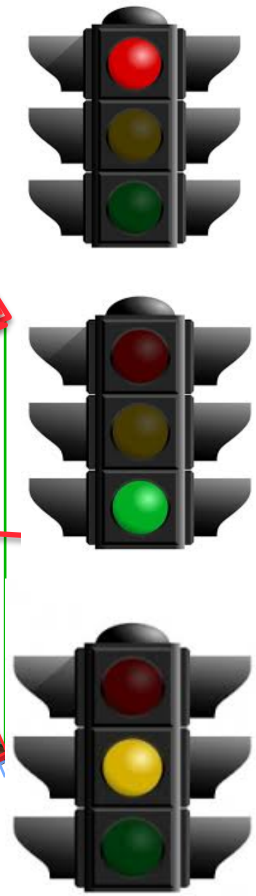


which functions on P should be allowed in L(P)?
which other operations in L(P) should be admitted?





all rational coefficients

lattice operations, pointwise min/max

addition and subtraction

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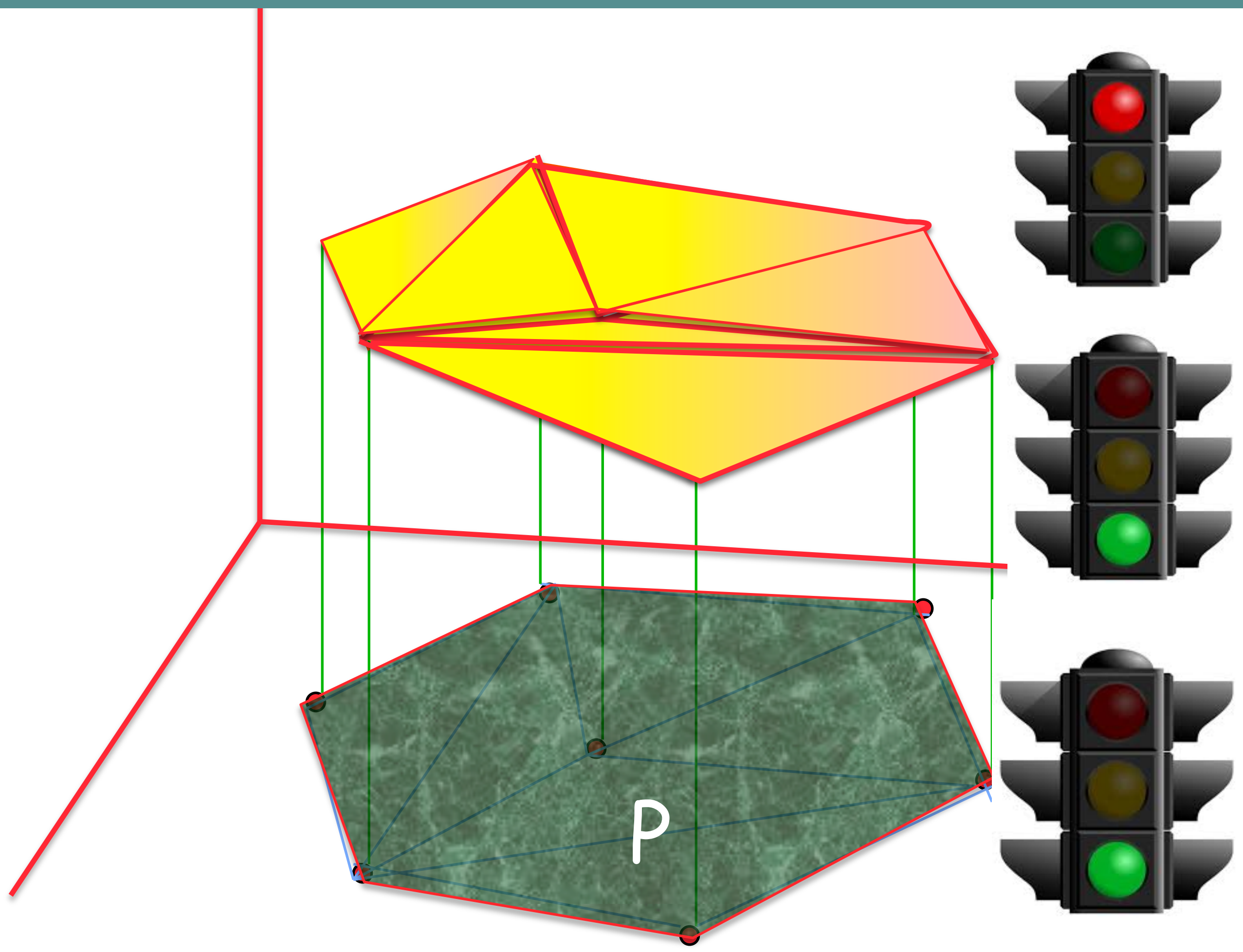

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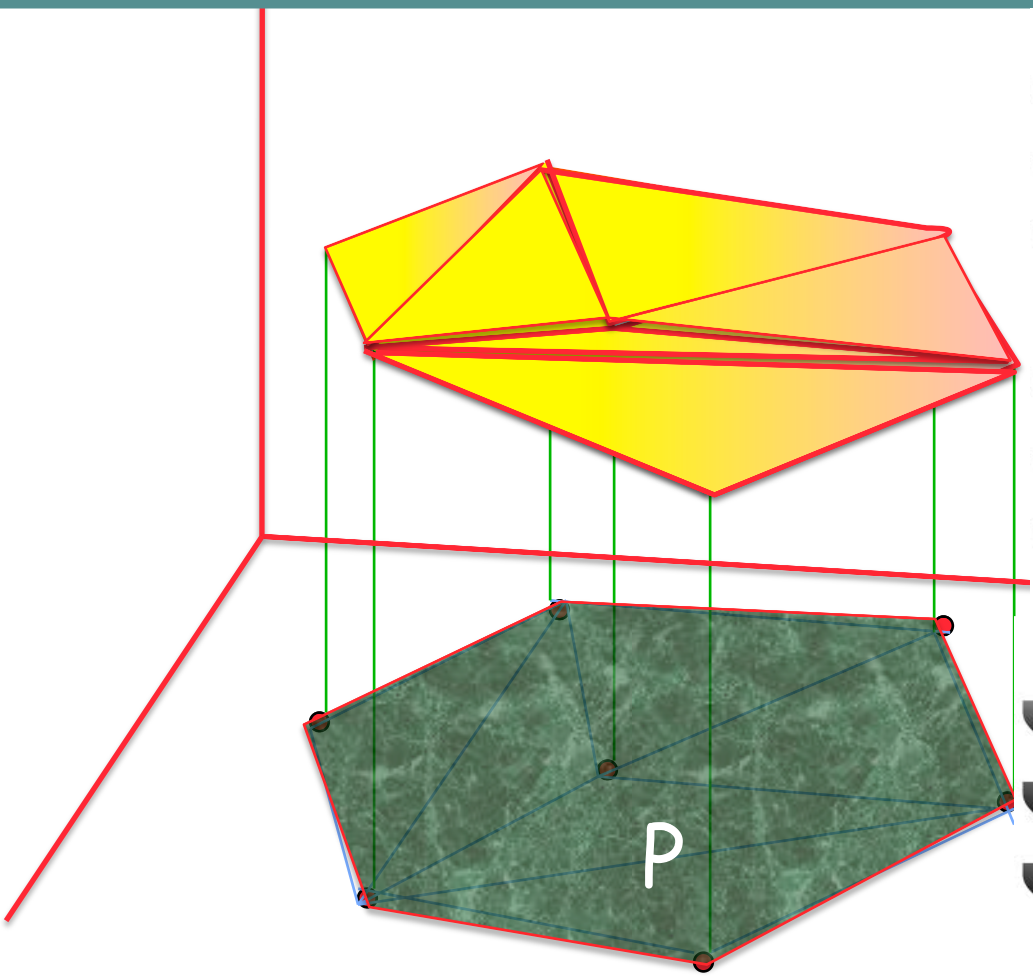

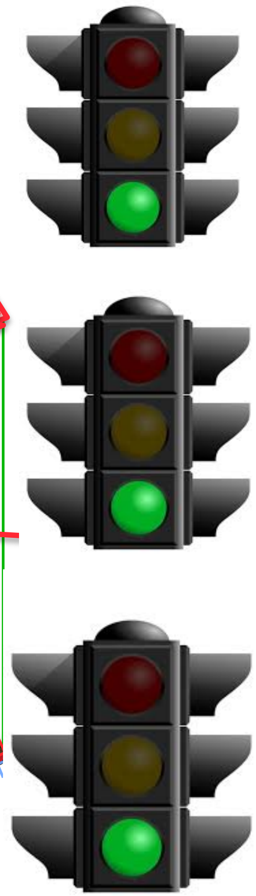

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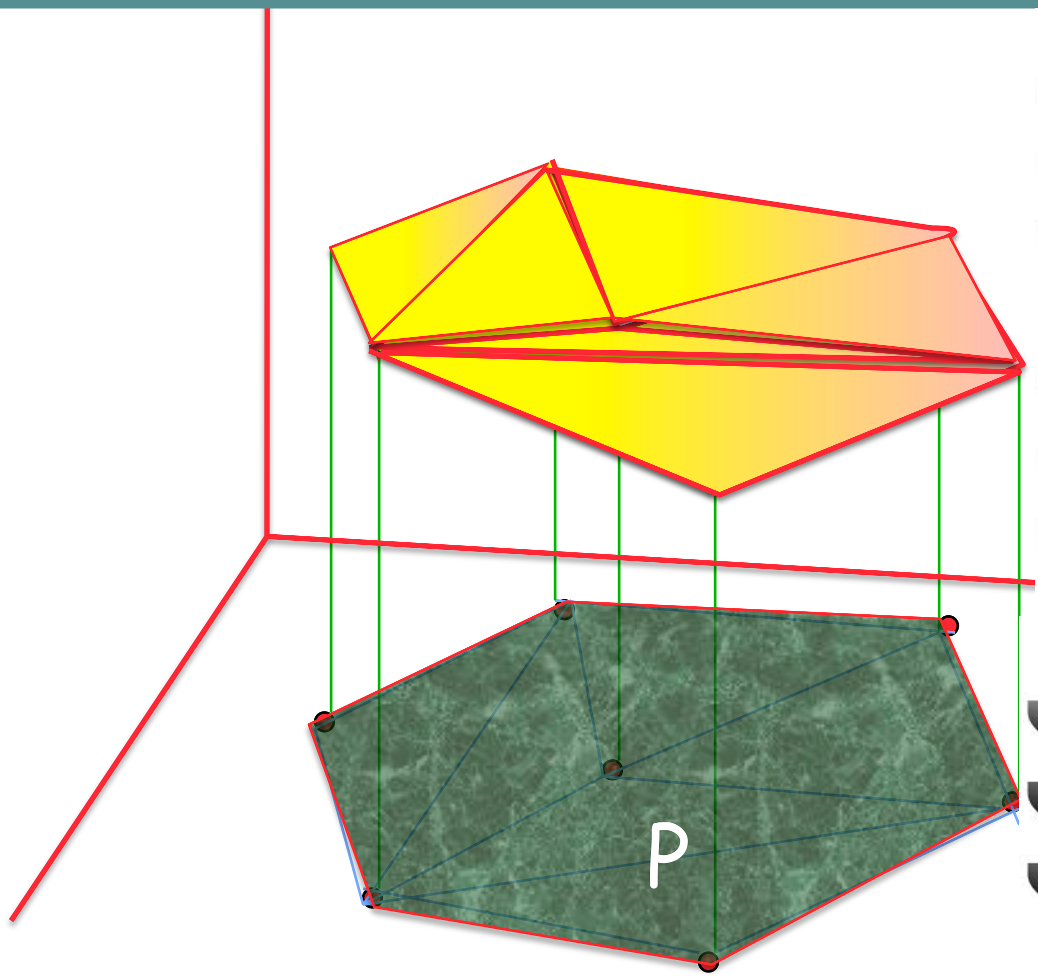

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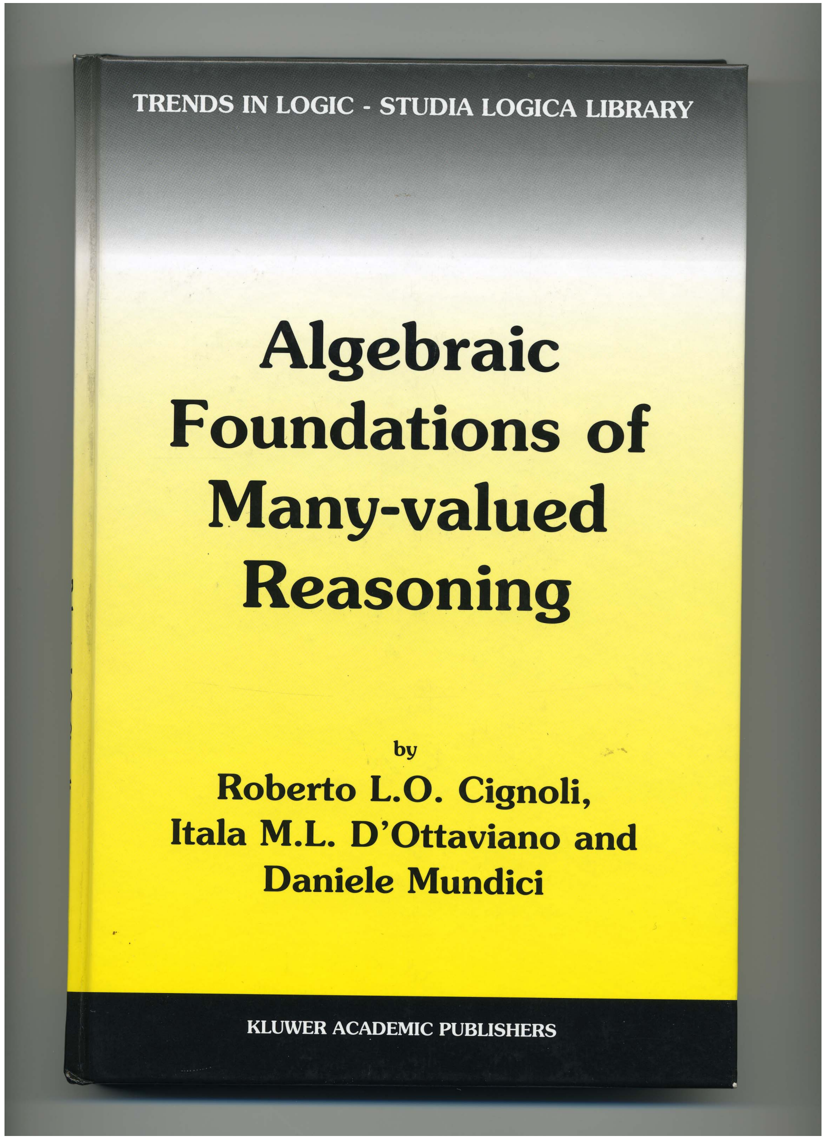
rational polyhedra ≈ finitely presented unital *l*-groups
≈ finitely presented MV-algebras

unital *l*-groups $\approx$ MV-algebras

unital l-groups $\approx$ MV-algebras

2000



MV-algebras

$$(x \oplus y) \oplus z = x \oplus (y \oplus z)$$

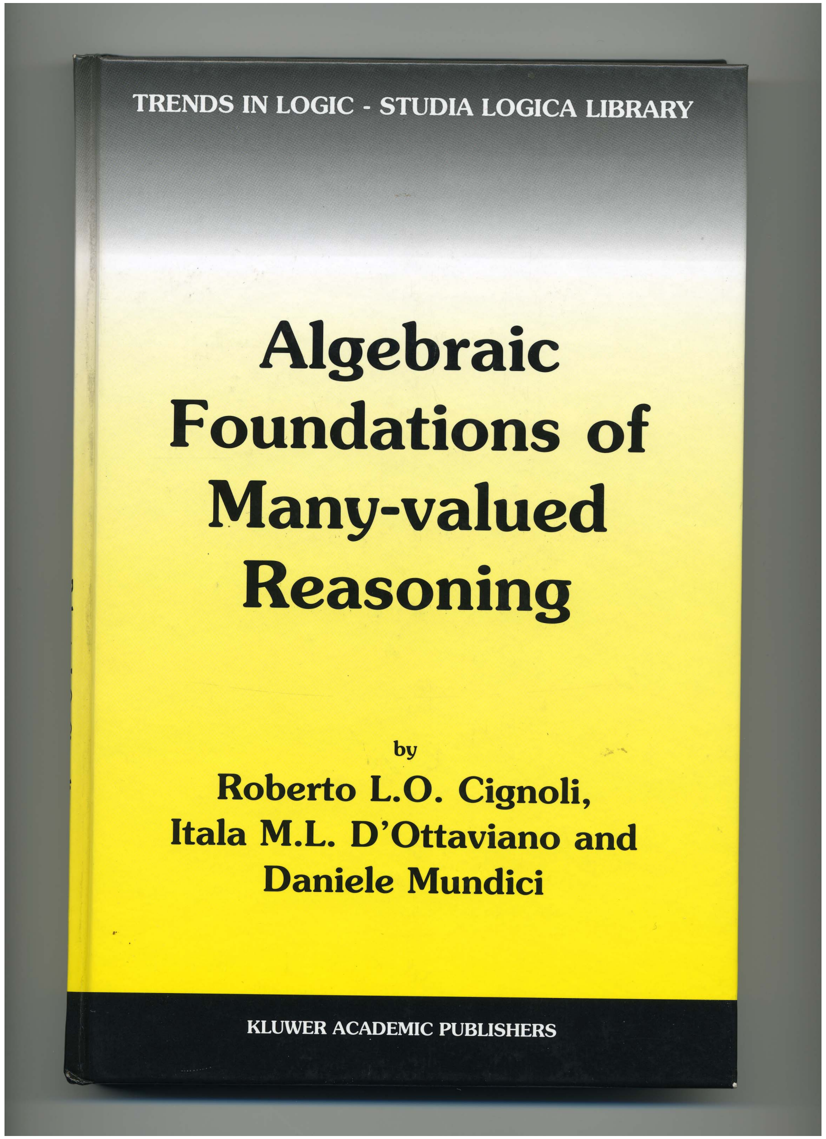
$$x \oplus 0 = x$$

$$\neg\neg x = x$$

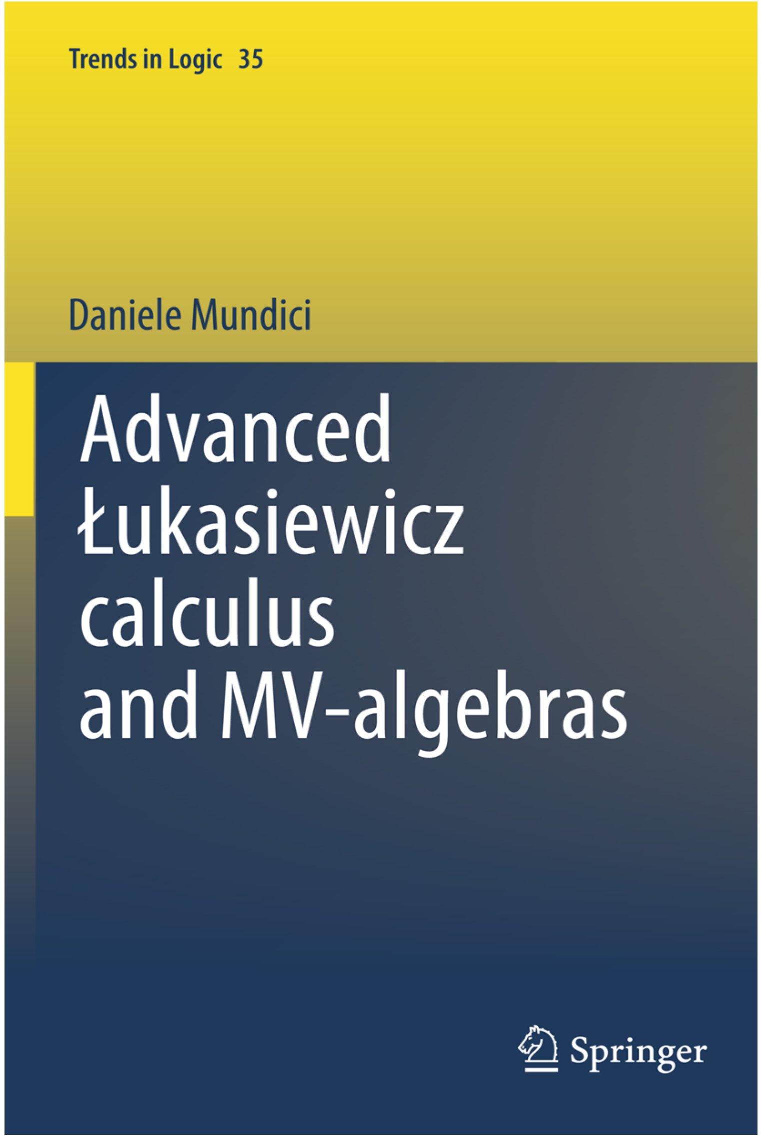
$$x \oplus \neg 0 = \neg 0$$

$$\neg(y \oplus \neg x) \oplus y = \neg(x \oplus \neg y) \oplus x$$

2000


TRENDS IN LOGIC - STUDIA LOGICA LIBRARY
Algebraic Foundations of Many-valued Reasoning
by
Roberto L.O. Cignoli, Itala M.L. D'Ottaviano and Daniele Mundici
KLUWER ACADEMIC PUBLISHERS

2011


Trends in Logic 35
Daniele Mundici
Advanced Łukasiewicz calculus and MV-algebras
Springer

from P to $\mathcal{M}(P)$

For P a rational polyhedron in euclidean space, let $\mathcal{M}(P)$ denote the MV-algebra of all $[0,1]$-valued piecewise linear continuous functions f on P where each piece of f has **integer** coefficients.

from P to $\mathfrak{M}$(P)

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THEOREM *Letting P range over rational polyhedra in euclidean space, the algebras of the form $\mathfrak{M}(P)$ exhaust all finitely presented MV-algebras.*

COROLLARY *The map $P \rightarrow \mathfrak{M}(P)$ is a duality between rational polyhedra with* **integer** *piecewise linear* [0,1]-*valued maps (called* **Z**-maps), *and finitely presented MV-algebras.*

A wealth of new invariants

Differently from polyhedra, ***rational*** polyhedra P with **Z**-maps possess many new invariants under **Z**-homeomorphism:

—the number n_d of points of denominator d lying in P (these n_d yield infinitely many computable invariants for P)

—the smallest number of elements in a basis of $\mathfrak{M}$(P)

—the rational volume of P

—the number of idempotent endomorphisms of F_n onto $\mathfrak{M}$(P)

These invariants make no sense for rational polyhedra with piecewise linear maps with **rational** coefficients

a disanalogy between analogies

Rational polyhedra with continuous piecewise linear maps having **integer** coefficients are dually equivalent to **finitely presented unital l-groups**

paralleling the Baker-Beynon duality:

Rational polyhedra with continuous piecewise linear maps having **rational** coefficients are dually equivalent to **finitely presented *l*-groups**

BUT ***while*** *the isomorphism problem for finitely presented l-groups is* ***undecidable****, the isomorphism problem for finitely presented* unital *l-groups (≈MV-algebras) is still* ***open****.*

main tools for the duality theorem

THEOREM (D.M., 1986) *Unital lattice-ordered abelian groups (unital l-groups) are categorically equivalent to MV-algebras*.

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CHANG COMPLETENESS THEOREM (Chang 1959) *The variety of MV-algebras is generated by the unit real interval* [0,1] *equipped with negation* $\neg x = 1 - x$ *and truncated sum* $\min(1, x+y)$

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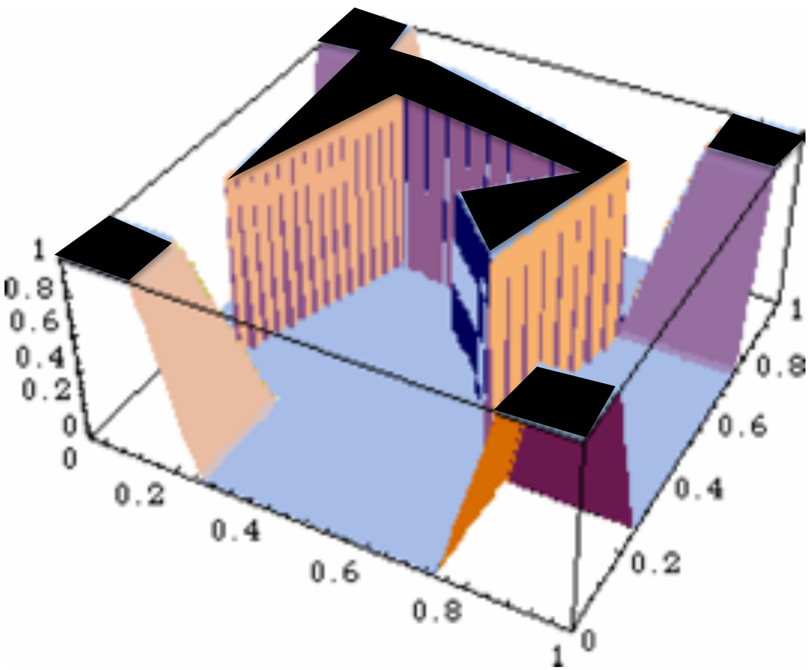
THEOREM (D.M., 1986) *Unital lattice-ordered abelian groups (unital l-groups) are categorically equivalent to MV-algebras.*

CHANG COMPLETENESS THEOREM (Chang 1959) *The variety of MV-algebras is generated by the unit real interval* [0,1] *equipped with negation* $\neg x = 1 - x$ *and truncated sum* $\min(1, x+y)$

COROLLARY *Free MV-algebras* $\mathfrak{M}_n$ *are algebras of piecewise linear continuous functions with integer coefficients, defined over the cube* $[0,1]^n$. *In symbols,* $\mathfrak{M}_n = \mathfrak{M}([0,1])^n$

from presentations by MV-terms to rational polyhedra

the function f_t of an MV-term t



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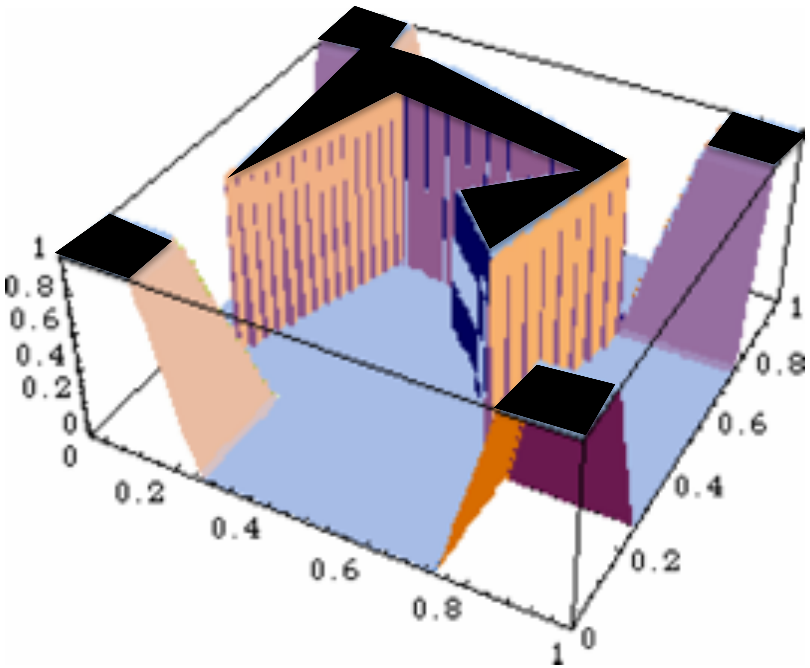

this duality casts a new light on projectives, hopficity, recognition of free generating sets, for MV-algebras and l-groups (with or without a unit), and much more...

2015-2017

L.M. Cabrer, D.M., Severi-Bouligand tangents, Frenet frames and Riesz spaces, *Advances in Applied Mathematics*, 64 (2015) 1-20.

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D.M., Hopfian *l*-groups, MV-algebras and AF C*-algebras, *Forum Mathematicum*, 28(6) (2016) 1111-1130.

L.M. Cabrer, D.M., Classifying orbits of the affine group over the integers, *Ergodic Theory and Dynamical Systems*, 37 (2017) 440-453.

L.M. Cabrer, D.M., Germinal theories in Lukasiewicz logic, *Annals of Pure and Applied Logic*, 168 (2017) 1132-1151.

L.M. Cabrer, D.M., Idempotent endomorphisms of free MV-algebras and unital l-groups, *Journal of Pure and Applied Algebra*, 221 (2017) 908-934.

D.M., Fans, decision problems and generators of free abelian *l*-groups, *Forum Mathematicum*, DOI: 10.1515/forum-2016-0255

APPLICATIONS 1:

Jónsson Tarski for l-groups via polyhedra

the Jónsson-Tarski theorem (repeated)

THEOREM *Let K be a class of algebras such that every equation which is satisfied in all finite algebras of K is also satisfied in all algebras of K.*

Then for any algebra $A \in K$ having a free generating set of n elements, every generating set of A with n elements is a free generating set of A.

COROLLARY *Every n-element generating set of the free n-generator MV-algebra $\mathfrak{M}_n$ freely generates $\mathfrak{M}_n$.*

PROOF. Indeed, for any equation $t(y_1,\ldots,y_n) = 0$ that fails in some MV-algebra, **the proof of Chang's completeness theorem yields a finite MV-algebra where the equation fails**. Now apply the Jónsson-Tarski theorem.

for l-groups we need much more...

THEOREM (D.M., Forum Mathematicum, 2017)
Any generating set $\{g_1,\ldots,g_n\}$ *of the free n-generator abelian l-group freely generates it*.

REMARK A direct proof from the Jónsson-Tarski and residual finiteness theorem is not available, because the only finite *l*-group is the trivial singleton $\{0\}$. The proof is geometrical, using the Baker-Beynon duality.

APPLICATIONS 2

(D.M., Forum Math. 2016):

hopficity in MV-algebras and *l*-groups

algebraic-topological theorems (A.Monteiro)

X is a normal space iff each prime filter of L(X) is contained in a unique maximal filter.

In the lattice **Z** of integers with divisibility, this property is obvious because prime filters correspond to prime powers, and maximal filters to prime numbers.

STRONGER PROPERTY: X is a completely normal (also known as hereditarily normal) space if and only if for any prime filter P of L(X), the set of filters F of L(X) such that $P \subseteq F$ is totally ordered by inclusion.

in the spirit of A. Monteiro

Theorem (D.M., Forum Mathematicum, 2016)

For any finitely generated MV-algebra A the following conditions are equivalent:

(i) A is residually finite.

(ii) A is semisimple and its maximal ideals of finite rank form a dense subset of its maximal spectral space $\mu(A)$.

in the spirit of A. Monteiro

Corollary *Any finitely presented, as well as any finitely generated projective unital l-group (G, u) is hopfian.*

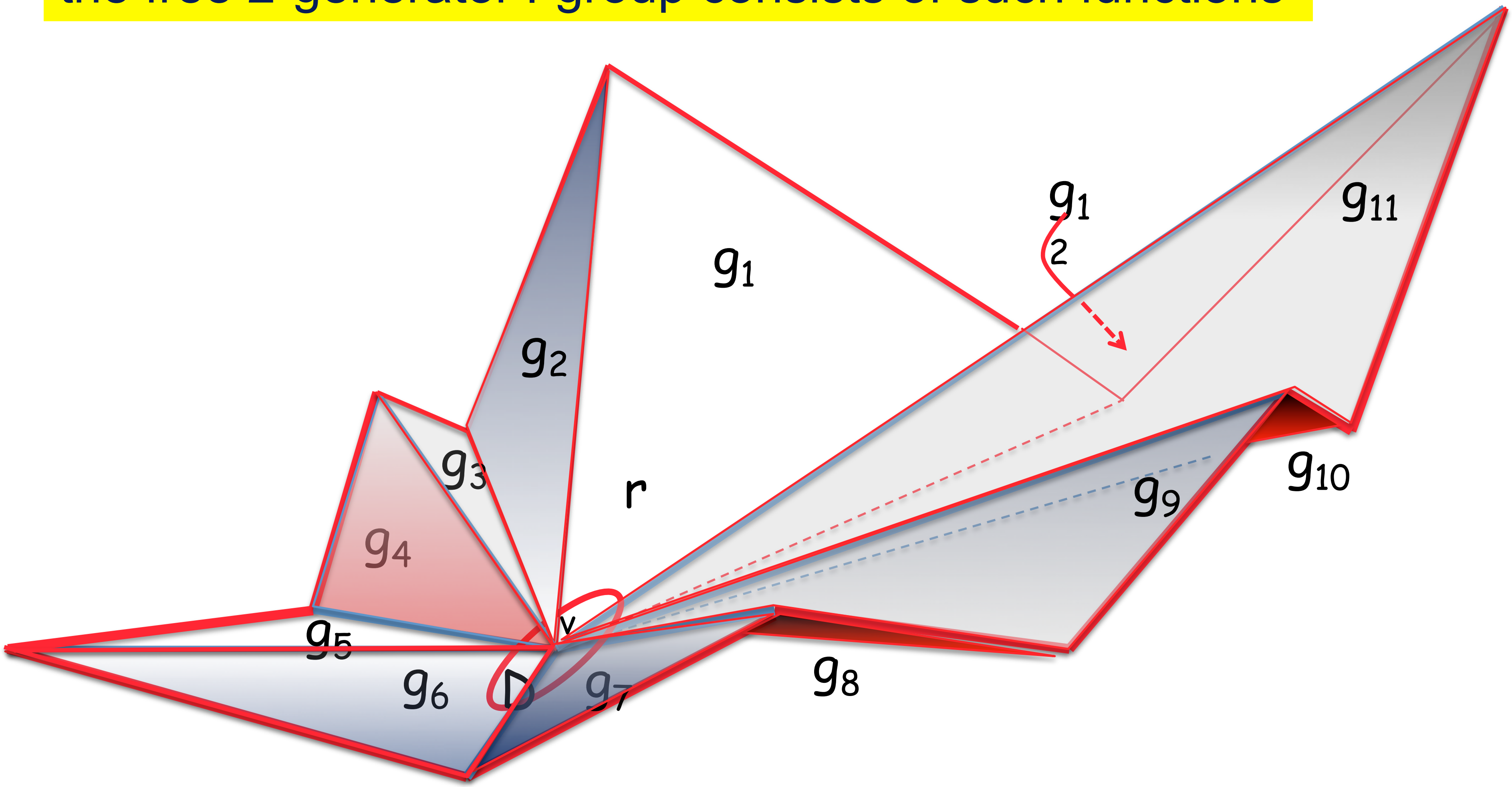
Corollary *The following classes of unital l-groups are hopfian:*
(i) simple unital l-groups.
(ii) finitely generated unital l-groups with finitely many prime ideals.

Corollary *Let (G, u) be a semisimple unital l-group. If its maximal spectral space is a manifold without boundary, then (G, u) is hopfian.*

Corollary *For all $n = 1, 2, \ldots,$ the free n-generator l-group is hopfian*

a piecewise linear homogeneous function on R^2 with integer coefficients

the free 2-generator l-group consists of such functions



APPLICATIONS 3

(D.M., Forum Math. 2016):

decidable and undecidable recognition problems for free generating sets of l-groups

A constant that one can perceive through the whole mathematical work done by Monteiro is his preference for the finitistic methods, that allow concrete constructions and algorithms. This tendency is already noticeable in his doctoral thesis, a long part of which is dedicated to finite matrices, as used to approximate the kernels of integral equations. When considering a new class of algebras, it was a basic question for him to decide if the finitely generated free algebras were finite, and if so, to find explicitly the number of their elements as a function of the number of generators. In general, to achieve this goal it is necessary to have a deep understanding of the structure of the algebras in the given class.

R. Cignoli: Actas del IX Congreso Dr. Antonio Monteiro, 2007, pp.3-8

4 decision problems for K (repeated)

Let K be an equational class of algebras and F_n the free n - generator K -algebra.
For any set $t_1,...,t_n$ of K-terms, all in the same variables $X_1,...,X_n$, let $t'_1,...,t'_n$ be their respective interpretations as elements of F_n .

Consider the following **decision problems**:

1: Does $\{t'_1,...,t'_n\}$ generate F_n ?
2: Does $\{t'_1,...,t'_n\}$ freely generate F_n ?
3: Does $\{t'_1,...,t'_n\}$ generate an isomorphic copy of F_n ?
4: Does $\{t'_1,...,t'_n\}$ freely generate an isomorphic copy of F_n ?

positive result for l-groups

Theorem *The following problem is* ***decidable****:*

INSTANCE : l-group terms $t_1,\ldots,t_n$ *all in the same variables* $x_1,\ldots,x_n$ *, with their interpretations as piecewise linear homogeneous functions* $t'_1,\ldots,t'_n$ *with integer coefficients in the free l-group* A_n

QUESTION : Is $\{t'_1,\ldots,t'_n\}$ *a free generating set of the l-group it generates in* A_n*?*

the crux of the proof is to find a certificate that $\{t'_1,\ldots,t'_n\}$ *does* ***not*** *(freely) generate the l-group it generates in* A_n

positive result for MV-algebras

Theorem *The following problem is* ***decidable****:*

INSTANCE : MV-terms $t_1,\ldots,t_n$ *all in the same variables* $x_1,\ldots,x_n$ *with their interpretations as piecewise linear functions* $t'_1,\ldots,t'_n$ *with integer coefficients. This is the free MV-algebra* $\mathfrak{M}_n$

QUESTION : Is $\{t'_1,\ldots,t'_n\}$ *a free generating set of* $\mathfrak{M}_n$*?*

the crux of the proof is to find a certificate that $\{t'_1,\ldots,t'_n\}$ *does* ***not*** *(freely) generate the free MV-algebra* $\mathfrak{M}_n$

a negative result for l-groups

THEOREM *The following problem is* ***undecidable****:*

INSTANCE : l-group terms $t_1,\ldots,t_n$ in the same variables $x_1,\ldots,x_m$, and an integer $k > 0$.

QUESTION : Is the l-group generated by $t'_1,\ldots,t'_n$ isomorphic to the free abelian l-group A_k ?

there is no *certificate that* $\{t'_1 ,\ldots, t'_n\}$ *does* ***not*** *(freely) generate the l-group* A_k

APPLICATIONS 4

(L.M. CABRER, arXiv 1405.7118, and L. CABRER, D.M., Communications in Contemporary Mathematics, 2012):

projective MV-algebras

The integers **Z** form a lattice when ordered by divisibility. The meet of two numbers is their greatest common divisor, and the join is their smallest common multiple.The filters of **Z** as a lattice are precisely the ideals of **Z** as a ring.
The maximal filters are the sets of multiples of prime numbers and the prime filters are the sets of multiples of prime powers.
The basic arithmetic properties of **Z** can be expressed in terms of filters. For instance,the decomposition of an integer into prime factors is equivalent to the fact that each filter in the lattice **Z** is a finite intersection of prime filters.

Thus lattices can be considered as generalization of the integers, and the study of the properties of the filters of a lattice can be considered as an “arithmetic” for this lattice.

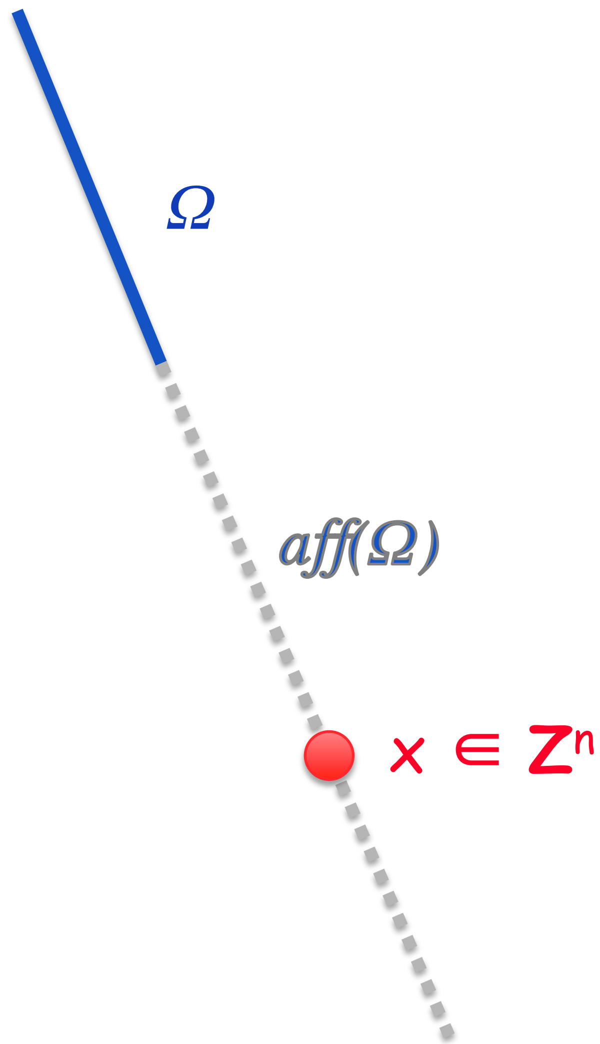
This was Monteiro’s point of view.

For instance Monteiro proved that a lattice is distributive if and only if every proper filter is an intersection of prime filters. Hence distributive lattices are those in which the analogue of the factorization of an integer holds.

R. Cignoli: Actas del IX Congreso Dr. Antonio Monteiro, 2007, pp.3-8

polyhedral topology + arithmetics

DEFINITION A rational polyhedron P is **strongly regular** if for some (equivalently, for every) regular triangulation Ω of P, the affine hull of every maximal simplex of Ω contains an integer point.
Equivalently: the denominators of the vertices of every maximal simplex in Ω are relatively prime



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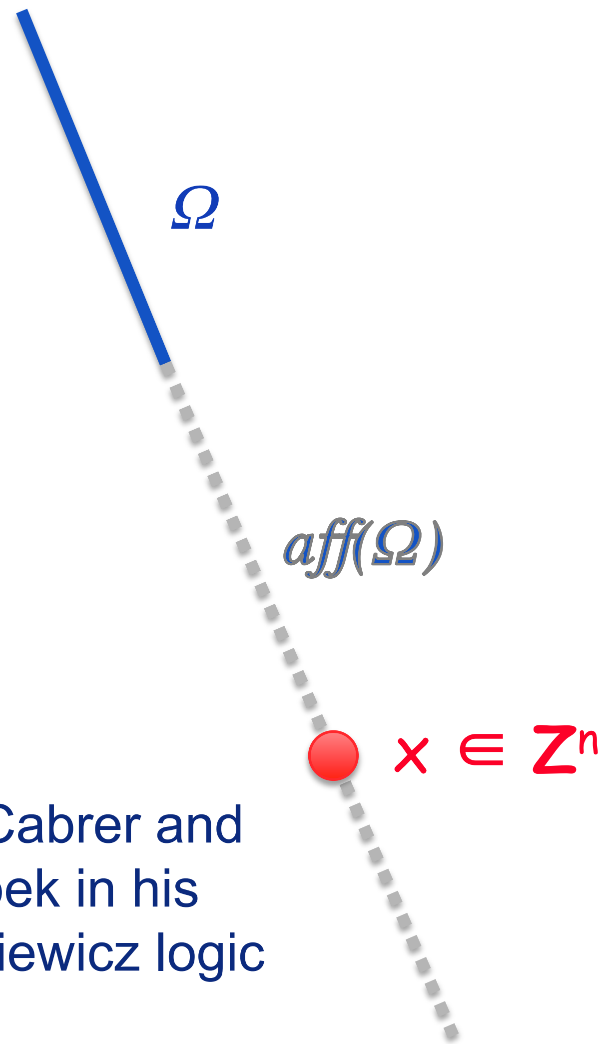

This notion was independently introduced by L.M. Cabrer and D.M. in their analysis of projectives, and by E. Jeřábek in his analysis of admissibility in the proof-theory of Łukasiewicz logic

the geometry of projective MV-algebras

THEOREM (L. CABRER, D.M., Comm. Contemporary Math. 2012)

If A is a finitely generated projective MV-algebra, then writing A=$\mathfrak{M}$(P) for some rational polyhedron P in $[0,1]^n$, ***as given by duality,*** *we have*

(i) P contains some vertex of $[0,1]^n$,

(ii) P is contractible, and

(iii) P is strongly regular.

the geometry of projective MV-algebras

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The **converse of this theorem** has been proved by L.M.Cabrer.

See his paper in arXiv 1405.7118

A deep theorem and a tour de force in arithmetic algebraic topology revisiting our three initial problems on retractions of free MV-algebras:

(L.M.Cabrer, D.M., Annals of Pure and Applied Logic, 2016)

PROBLEM 1. Necessary and sufficient conditions for the existence of only finitely many many idempotent endomorphisms of the free algebra $\mathfrak{M}_n$ onto an MV-algebra A

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THEOREM *Suppose the MV-algebra A is the image of an idempotent endomorphism of the free MV-algebra $\mathfrak{M}_n$. Then the number of idempotent endomorphisms of $\mathfrak{M}_n$ onto A is finite iff the maximal space of A coincides with the closure of its own interior in* $[0,1]^n$. ***(Kuratowski regularity property)***

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algebra meets topology: in the spirit of A.Monteiro

PROBLEM 2: An MV-algebra D which is the image of infinitely many idempotent endomorphisms of free MV-algebras

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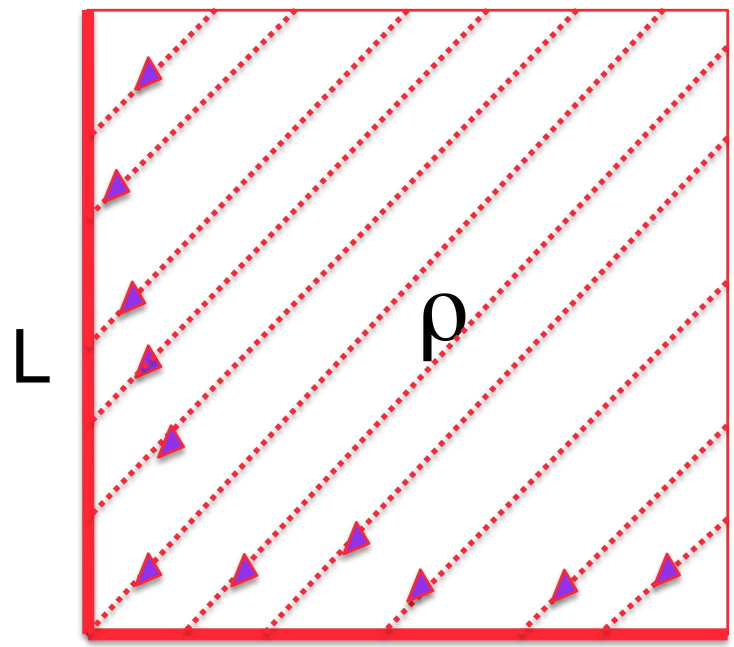

The MV-algebra $\mathfrak{M}$(L), with L the red region above.
Note that L is not a Kuratowski regular domain.

Problem 3. For every $i=1,2,...$, exhibit an MV-algebra A_i such that there are $> i$ (finitely many) idempotent endomorphisms of the free MV-algebra of $\mathcal{M}_n$ onto A_i

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There are ≥ 16 idempotent endomorphisms of the 2-generator free MV-algebra $\mathcal{M}_2$ onto the MV-algebra $\mathcal{M}(P)$, with P the polyhedron in the following picture:

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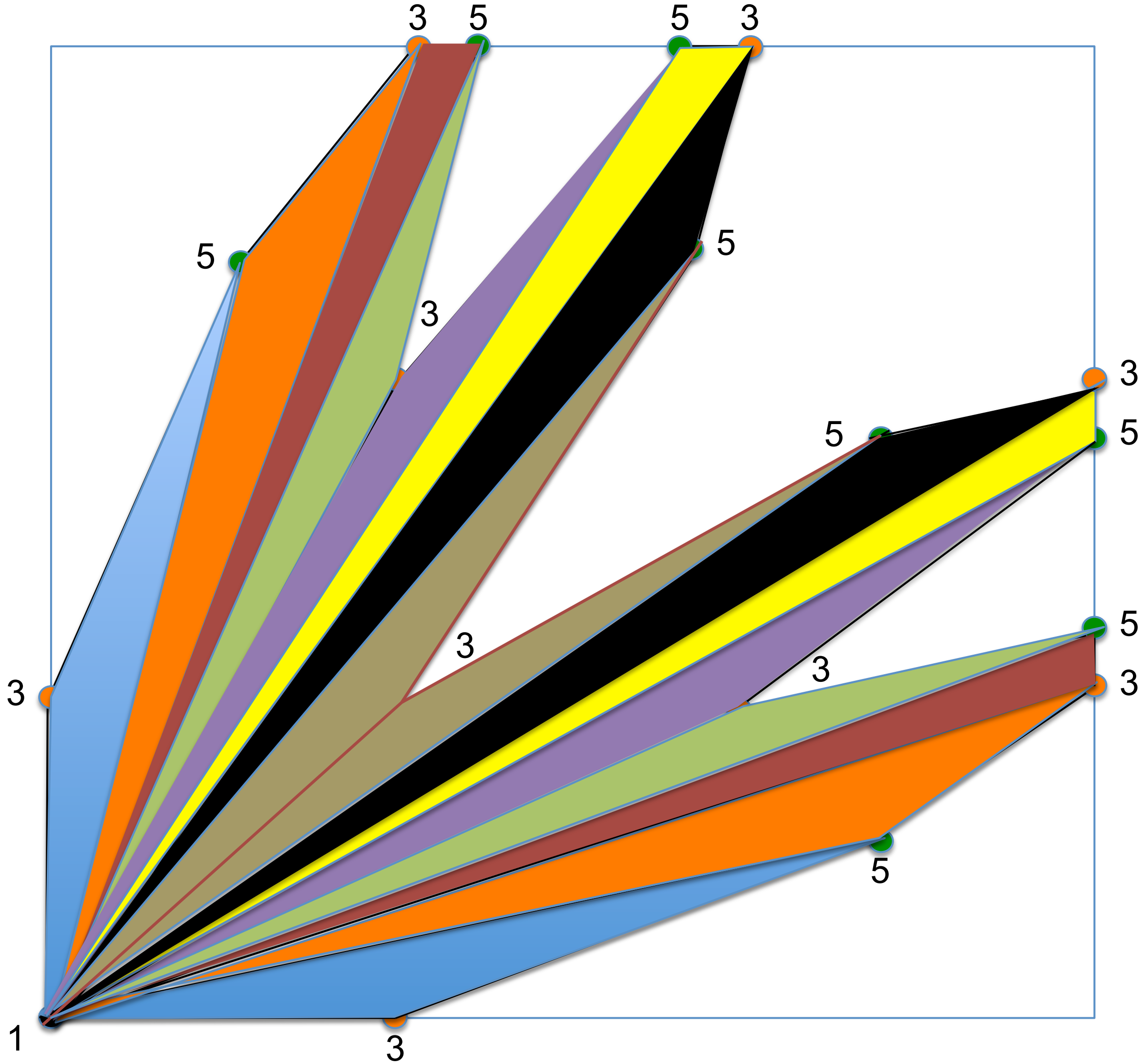

There are ≥ 16 idempotent endomorphisms of the 2-generator free MV-algebra $\mathcal{M}_2$ onto the MV-algebra $\mathcal{M}(P)$, with P the polyhedron in the following picture:

beyond algebraic logic

Riesz spaces: Cabrer, Di Nola, Leustean
Differential geometry: Busaniche, Cabrer
Semirings, tropical and idempotent mathematics: Di Nola
Interval Algebras: Cabrer
Probability: Flaminio, Keimel, Montagna, Rieçan
Games: Herzberg, Kroupa, Teheux
Multisets: Cignoli, Dubuc, Marra, Nganou
Semantics of Łukasiewicz logic: Caicedo
Model-theory of Łukasiewicz logic: Caicedo
Proof-theory of Łukasiewicz logic: Cabrer, Jeràbek, Metcalfe
Modal logic, Belief: Kroupa, Godo, Teheux
Quantum structures: Dvureçenskij, Pulmannovà
Polyhedral topology: Busaniche, Cabrer, Marra, Spada
Topological groups: H.Weber
Categories, Morita equivalence, coordinatization:
Caramello, Cignoli, Dubuc, Lawson, Marra, Poveda, Scott

thank you