Interpolation and Robinson´s lemma for Łukasiewicz predicate logic

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PROPOSITIONAL Ł

Ł stands for [0,1]-valued *Łukasiewicz propositional logic* over $\neg, \to$.
$X, Y, Z, \ldots$ (finite) sets of propositional variables (*languages*)
$Ł_X$ set of formulas built from the variables in X
$[0,1]^X$ set of valuations on X (identifiable with $[0,1]^n$, $n = |X|$)
For $\varphi \in Ł_X$, $v \in [0,1]^X$:
$v(\varphi) :=$ value of φ according to Łukasiewicz interpretation of $\neg, \to$.

$$Mod(\varphi) := \{v \in [0,1]^X : v(\varphi) = 1\}$$

Define similarly $Mod(T)$ for a *theory* $T \subseteq Ł_X$ with possibly infinite X.

Fact

$f_\varphi : [0,1]^X \to [0,1]$, $f_\varphi(v) = v(\varphi)$, *is continuous for any* φ; *hence,* $Mod(T) = \cap_{\varphi \in T} f_\varphi^{-1}(1)$ *is closed in* $[0,1]^X$.

Corollary

(Compactness) if each finite part of a theory T *is satisfiable,* T *is satisfiable.*

Proof. $Mod(T) = \cap_{F \subseteq_{fin} T} Mod(F)$. $\square$

If $X \subseteq Y$ the projection $\pi : [0,1]^Y \to [0,1]^X$, $\pi(v) = v \restriction X$ is continuous.

If $\varphi \in Ł_X \subseteq Ł_Y$, $Ł_Z$ we may speak about $Mod_Y(\varphi)$, $Mod_Z(\varphi)$. The following notions are independent of Y:

$$\begin{array}{rcc} \models \varphi & \text{iff} & Mod_Y(\varphi) = [0,1]^Y \\ \varphi \models \psi & \text{iff} & Mod_Y(\varphi) \subseteq Mod_Y(\psi) \end{array}$$

For a theory $T \subseteq Ł_X \subseteq Ł_Y$:

$$T \models \psi \quad \text{iff} \quad Mod_Y(T) \subseteq Mod_Y(\psi)$$

The completeness theorem says that we may replace above $\models$ with $\vdash$ for a suitable deductive system $\vdash$.(T finite in the last case)

Craig interpolation

X, Y, Z mutually disjoint languages, $\varphi \in \text{Ł}_{X\cup Y}$, $\psi \in \text{Ł}_{X\cup Z}$.
If $\models \varphi \to \psi$ *then there is* $\theta \in \text{Ł}_X$ *such that* $\models \varphi \to \theta$ *and* $\models \theta \to \psi$.

$$\begin{array}{ccccccc}
 & & & \models \varphi \to \psi & & & \\
 & & & \text{Ł}_{X\cup Y\cup Z} & & & \\
 & & \nearrow & & \nwarrow & & \\
\models \varphi \to \theta & \text{Ł}_{X\cup Y} & & & & \text{Ł}_{X\cup Z} & \models \theta \to \psi. \\
 & \varphi & \nwarrow & & \nearrow & \psi & \\
 & & & \text{Ł}_X & & & \\
 & & & \theta & & &
\end{array}$$

It fails in Ł:

$$\models (p \wedge \neg p) \to (q \vee \neg q),$$

the only interpolant is the constant $\frac{1}{2}$, not expressible by a formula of Ł.

Deductive interpolation

$\varphi \models \psi$ instead of $\models \varphi \to \psi$:

$$\begin{tikzcd}
 & & \begin{array}{c}\varphi \models \psi \\ \text{Ł}_{X\cup Y\cup Z}\end{array} & & \\
\varphi \models \theta & \begin{array}{c}\text{Ł}_{X\cup Y} \\ \varphi\end{array} \arrow[ur] & & \begin{array}{c}\text{Ł}_{X\cup Z} \\ \psi\end{array} \arrow[ul] & \theta \models \psi \\
 & & \begin{array}{c}\text{Ł}_{X} \\ \theta\end{array} \arrow[ul] \arrow[ur] & &
\end{tikzcd}$$

holds in Ł.
The lack of a classical *deduction theorem* prevents recovering Craig.

Shown by algebraic means by several people, geometrically by D. Mundici.

The geometric proof

Lemma

The sets $Mod(\varphi)$, $\varphi \in Ł_X$, are exactly the rational polyhedra in $[0,1]^X$.

Lemma

Projections of rational polyhedra are rational polyhedra.

$$Mod_{X\cup Y\cup Z}(\varphi) \subseteq Mod_{X\cup Y\cup Z}(\psi)$$

$$\begin{tikzcd}
 & {[0,1]^{X\cup Y\cup Z}} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\circlearrowright"] & \\
{[0,1]^{X\cup Y}} \atop Mod_{X\cup Y}(\varphi) \arrow[dr] & & {[0,1]^{X\cup Z}} \atop Mod_{X\cup Z}(\psi) \arrow[dl] \\
 & {[0,1]^X} \atop \pi Mod_{X\cup Y}(\varphi) = Mod_X(\theta) &
\end{tikzcd}$$

Proof. $Mod_{X\cup Y}(\varphi) \subseteq \pi^{-1}\pi Mod_{X\cup Y}(\varphi) = \pi^{-1}Mod(\theta) = Mod_{X\cup Y}(\theta)$.
$Mod_{X\cup Z}(\theta) \subseteq Mod_{X\cup Z}(\psi)$, using the disjointedness of X, Y, Z.

Uniform interpolation

This proof yields left *uniform* (deductive) interpolation: θ does not depend on ψ, it is a *left uniform interpolant*. Similarly there is a *right* uniform interpolation not depending on φ because:

Fact

Cylindrifications of rational polyhedra are rational polyhedra.

The cylindrification of $R \subseteq [0,1]^{X \cup Z}$ in X is the largest subset C of $[0,1]^X$ such that $C \times [0,1]^Z \subseteq R$.
Thus the cylindrification of $Mod_{X \cup Z}(\psi)$ is the model class of a sentence $\theta \in \text{Ł}_X$, which interpolates.

Theorem

For each $\varphi \in \text{Ł}_{X \cup Y}$ there is $\varphi_* \in \text{Ł}_X$ such that $\varphi \models \varphi_*$ and $\varphi_* \models \psi$ for any $\psi \in \text{Ł}_{X \cup Z}$ such that $\varphi \models \psi$. Dually, for each $\psi \in \text{Ł}_{X \cup Z}$ there is $\psi^* \in \text{Ł}_X$ such that $\psi^* \models \psi$ and $\varphi \models \psi^*$ whenever $\varphi \models \psi$.

Some propositional extensions

Rational Pavelka Ł: add a constant connective $\frac{1}{n}$ for each $n \in \omega$.
Divisible Ł: add a unary connective $\frac{1}{n}x$ for each $n \in \omega$ (Gerla).
Riesz Ł: add a unary connective αx for each $\alpha \in [0, 1]$ (Di Nola, Leustean).

....

Continuous Ł: add all continuous connectives.

The model classes of these logics:

	$Mod(\varphi)$
Ł	rational polyhedra
Rational Pavelka Ł	rational polyhedra
Divisible Ł	rational polyhedra
Riesz Ł	polyhedra
....	....
Continuous Ł	closed sets

Closed under projections and cylindrification.All satisfy uniform deductive interpolation.

Craig again

Divisible Ł satisfies Craig Interpolation and it is the smallest extension doing so (Baaz and Veith 1999). This hinges on:

Lemma

The family $\{f_\varphi : \varphi \in$ Divisible Ł$\}$ is closed under propositional quantification (sup and inf with respect to a variable).

	$Mod(\varphi)$	$\{f_\varphi\}_\varphi$	Craig
Ł	rational polyhedra	$McN_{\mathbb{Z}}$	$-$
Rational Pavelka Ł	rational polyhedra		$-$
Divisible Ł	rational polyhedra	$McN_{\mathbb{Q}}$	$+$
Riesz Ł	polyhedra	$McN_{\mathbb{R}}$	$+$
....	....		$\vdots$
Continuous Ł	closed sets	$\mathcal{C}([\imath,\infty]^X)$	$+$

A similar fact holds for *Riesz Ł* and *Continuous Ł*

Robinson´s joint consistency

$T \subseteq T'$ theories in $Ł_X$ and $Ł_Y$, respectively, with $X \subseteq Y$.

Definition

T' is *conservative* over T if for any $\varphi \in Ł_X$: $T' \models \varphi$ implies $T \models \varphi$.

Theorem

If $T_i \subseteq Ł_{X_i}$, $i = 1, 2$, are satisfiable extensions of $T \subseteq Ł_{X_1 \cap X_2}$ with T_1 (or T_2) conservative over T then $T_1 \cup T_2$ is satisfiable.

If $T \subseteq Ł_X$ is X-complete (maximally satisfiable among theories in $Ł_X$) then any satisfiable extension of T is conservative. Hence, the usual statement of the Robinson´s property.

Lemma

Deductive interpolation implies Robinson's property.

Both properties are equivalent in any extension of Ł satifying compactness and closed under Łukasiewicz connectives. Classical proofs depend heavily

Expressing rational approximations in Ł

For each rational $r \in (0,1)$ there are unary connectives $\Box_r^+$ and $\Box_r^-$ definable in Ł such that

$$\Box_r^+(x) = 1 \textit{ iff } x \geq r \qquad \Box_r^-(x) = 1 \textit{ iff } x \leq r$$

This follows from McNaughton's theorem.
More elegantly, $[r, 1]$ and $[0, r]$ are rational polyhedra and thus model sets.
We will write, suggestively,

$$\varphi_{\geq \frac{n}{m}} \qquad \varphi_{\leq \frac{n}{m}}$$

for $\Box_r^+(\varphi)$ and $\Box_r^-(\varphi)$.

FIRST ORDER ŁUKASIEWICZ $Ł\forall$

The failure of Craig's property lifts to $Ł\forall$ with essentially the same counterexample:

$$\models \exists x(Px \wedge \neg Px) \rightarrow \forall x(Qx \vee \neg Qx).$$

Warning: the interpolant does not need to be a constant sentence, it may contain the identity symbol.

Situation for deductive interpolation and Robinson's property?

- We prove an approximate form of deductive interpolation
- Full Robinson's property.

This extends to any logic between $Ł\forall$ and continuous logic $CŁ\forall$ (the approximate interpolant may be chosen in $Ł\forall$)

First order Łukasiewicz logic

- First order languages: $\tau = \{R, ..; f, ..., c, ..\}$,
- $[0,1]$-valued structures: $\mathfrak{A} = (A, R^{\mathfrak{A}}, ...; f^{\mathfrak{A}} ...; c^{\mathfrak{A}}, ..)$

$$R^{\mathfrak{A}} : A^n \to [0,1], \quad f^{\mathfrak{A}} : A^n \to A, \quad c^{\mathfrak{A}} \in A.$$

Łukasiewicz connectives $\to, \neg$; *quantifiers* $\exists, \forall$ interpreted as suprema and infima:

$$\begin{aligned} [\exists x \varphi(x)]^{\mathfrak{A}}(\mathbf{b}) &:= \sup_{a \in A} \varphi^{\mathfrak{A}}(a, \mathbf{b}) \\ [\forall x \varphi(x)]^{\mathfrak{A}}(\mathbf{b}) &:= \inf_{a \in A} \varphi^{\mathfrak{A}}(a, \mathbf{b}). \end{aligned}$$

Terms, evaluated as in classical logic, give rise to functions $t^A : A^n \to A$.
Formulas $\varphi(x_1, ..., x_n)$ give rise to maps $\varphi^{\mathfrak{A}} : A^n \to [0,1]$.
Sentences give rise to values $\varphi^{\mathfrak{A}} \in [0,1]$.

Identity

A distinguished binary predicate $\approx^{\mathfrak{A}}: A^2 \longrightarrow [0,1]$ satisfying:

$x \approx x$	$d(x,x) = 0$
$x \approx y \rightarrow y \approx x$	$d(x,y) = d(y,x)$
$(x \approx y \rightarrow (y \approx z \rightarrow x \approx z)$	$d(x,z) \leq d(x,y) \oplus d(y,z)$
$x \approx y \rightarrow (R(x,..) \leftrightarrow R(y,..))$	$\lvert R(x,..) - R(y,..) \rvert \leq d(x,y)$
$x \approx y \rightarrow f(x,..) \approx f(y,..)$	$d(f(x,..), f(y,..)) \leq d(x,y)$

- $d(x,y) := \neg x \approx y$ defines a *pseudo-metric* for which R and f are 1-Lipschitz continuous.
- If we assume that the maximum degree of identity of two elements imply their true identity, the pseudometric d becomes a metric. We will assume this is always the case.

- The schema $x \approx y \to (\varphi(x,..) \leftrightarrow \varphi(y,..))$ is not inherited by all formulas

$x \approx y \to (R(x) \leftrightarrow R(y))$ does not imply

$$x \approx y \to (R(x) \oplus R(x) \leftrightarrow R(y) \oplus R(y))$$

but

$$(x \approx y)^2 \to (R(x) \oplus R(x) \leftrightarrow R(y) \oplus R(y)).$$

the congruence axioms for basic predicates and operations imply that for any formula φ or term t of Ł$\forall$ there is a constant k such that

$$\begin{array}{ll} (x \approx y)^k \to (\varphi(x,..) \leftrightarrow \varphi(y,..)) & |R(x,..) - R(y,..)| \leq kd(x,y) \\ (x \approx y)^k \to (t(x,..) \leftrightarrow t(y,..)) & d(f(x,..), f(y,..)) \leq kd(x,y). \end{array}$$

- Their interpretations become uniformly continuous with a Lipschitz constant depending on the formula only.

Ultraproducts

• $\{A_i\}_{i \in I}$ a family of $[0,1]$-valued structures of type τ, U and ultrafilter over I.
• $\Pi_i A_i /_U$ ordinary ultraproduct for the algebraic part of the A_i.
• For each predicate symbol $R \in \tau$, including d, the ultraproduct of $\{A_i^n \stackrel{R^{A_i}}{\rightarrow} [0,1]\}_i$

$$(\Pi_i A_i /_U)^n \stackrel{R^*}{\rightarrow} [0,1]^I /_U$$

Compose with $st : [0,1]^I /_U \rightarrow [0,1]$:

$$(\Pi_i A_i /_U)^n \stackrel{R^{**}}{\rightarrow} [0,1]$$

$(\Pi_i A_i /_U)^2 \stackrel{d^{**}}{\rightarrow} [0,1]$ becomes a pseudometric. Divide out by infinitesimals:

Definition. $\overset{*}{\Pi}_i A_i /_U := (\Pi_i A_i /_U)/\sim$, where $f_{/U} \sim g_{/U}$ if and only if $d^{**}(f_{/U}, g_{/U}) = 0$ iff $\{i \in I : d_i(f(i), g(i)) < \varepsilon\} \in U$ for all positive ε.

Ultraproduct theorem

Theorem

For $\varphi \in Ł$ and $f, .. \in \Pi_i A_i$

$\overset{*}{\Pi}_i A_i/_U \models \varphi[f_{/U}, ...]$ *iff* $\{i \in I : A_i \models \varphi_{\geq r}[f(i), ...]\} \in U$ *for any rational r*

Hence,

$$\{i \in I : A_i \models \varphi[f(i), ...]\} \in U \quad \text{implies } \overset{*}{\Pi}_i A_i/_U \models \varphi[f_{/U}, ...]$$

- Model classes are closed under ultraproducts
- Projections of model classes are closed under ultraproducts:

$A_i \in K$ implies $(A_i, R_i) \in Mod(T)$ then

$\overset{*}{\Pi}_i(A_i, R_i)/U = (\overset{*}{\Pi}_i A_i/U, R^*) \in Mod(T)$, thus $\overset{*}{\Pi}_i A_i/U$

Keisler-Shelah

Definition

$A \equiv_{Ł\forall} B$ if and only if $\varphi^A = \varphi^B$ for any sentence φ

If $A_i = A$ for all $i \in I$, then $A^{*I}/_U := \overset{*}{\Pi}_i A_i/_U$ is called a (metric) ultrapower of A.

Theorem

$Ł\forall$-equivalent models have isomorphic ultrapowers

The Ł∀-topology

Let Γ_τ the topology on $St_\tau(Ł\forall)$ obtained by taking the classes $Mod(\theta)$, $\theta \in L_\tau$, as a sub-basis of closed classes. This topology

- Is invariant under isomorphism
- Has for closed classes the classes $Mod(T)$ for $T \subseteq Ł\forall_\tau$
- If $\tau \subseteq \mu$, the reduct map $St_\mu(Ł\forall) \to St_\tau(Ł\forall)$ is continuous.
- It is a *regular* topology (separation of closed clases and points)
- $(St_\tau(Ł\forall), \Gamma_\tau)$ is compact.

Regularity follows from the fact that rational approximations are expressible. Assume $A \notin Mod(T)$ then $A \not\models \varphi$ for some $\varphi \in T$, hence $A \models \varphi_{\leq r}$ for some $r < 1$. If $r < s < 1$ then $Mod(\varphi_{\leq s})^c$ and $Mod(\varphi_{\geq s})^c$ are disjoint open classes containing, respecatively, $Mod(T)$ and $\{A\}$.

Compactness

Given a topological space, X, $\{x_i\}_{i \in I} \subseteq X$, $x \in X$, and an ultrafilter U over I

Definition

$\{x_i\}_{i \in I} \rightarrow_F x$ iff $\{i \in I : x_i \in V\} \in U$, for any open neinghborhood V of x.

Theorem

X is compact if and only if all ultrafilter limits exist for all families in X.

Fact

In $X_\tau = (St_\tau(Ł\forall), \Gamma_\tau)$

$$\{A_i\}_{i \in I} \rightarrow_F \Pi_i A_i / U$$

Proof. $\Pi_i A_i / F \in Mod(\varphi)^c = V$ implies $\{i : A_i \models \varphi_{\geq r}\} \notin U$ for $r < 1$ then $\{i : A_i \models \varphi_{\leq r}\} \in U$ then $\{i : A_i \not\models \varphi\} \in U$, that is $\{i : A_i \in V\} \in U$.

A topological disgression

Define in any space X,

$$x \equiv y \Leftrightarrow \overline{\{x\}} = \overline{\{y\}}.$$

x and y belong to the same closed (open) subsets of X. We may form the quotient space $X/\equiv$

Lemma

If X is regular, the quotient space $X_{/\equiv}$ is Hausdorff (excercise).

Lemma

If K_1 and K_2 are disjoint compact subsets of a regular topological space X which can not be separated by a finite intersection of basic closed sets, then there exist $x_i \in K_i$, $i = 1, 2$, such that $x_1 \equiv x_2$.

Clearly, $\equiv$ is $Ł\forall_\tau$-equivalence in the space $(St_\tau(Ł\forall), \Gamma_\tau)$.

Lemma

Any pair of disjoint PC_Δ-classes K_1, K_2 of the same signature τ are separable by a sentence $\theta \in Ł\forall_\tau$, that is,

$$K_1 \subseteq Mod(\theta), \quad K_2 \cap Mod(\theta) = \emptyset.$$

Proof. The K_i are compact (being continuous images of compact classes). If separation is not possible, we obtain by the topological lemma above

$$A \equiv_{Ł\forall} B \text{ with } A \in K_1,\ B \in K_2.$$

Utilizing the Keisler-Shelah theorem for $Ł\forall$ we obtain:

$$A \approx B \text{ with } A \in K_1,\ B \in K_2.$$

yielding a contradiction.

Approximate deductive interpolation

Theorem

If $\varphi \models \psi$ then for any $r < 1$ there is θ_r such that $\varphi \models \theta_r \models \psi_{\leq r}$.

Proof. If $\varphi \models \psi$ then φ and $\psi_{\leq r}$ are jointly unsatisfiable. Therefore $K_1 = Mod(\varphi) \restriction X$ and $K_2 = Mod(\psi_{\leq r}) \restriction X$ are disjoint and thus there is $\theta_r \in Ł\forall(\tau \cap \mu)$ such that $\varphi \models \theta_r$ and $Mod(\theta_r) \cap K_2 = \emptyset$, thus $\theta_r \models \psi_{\geq r}$ $\square$

Taking $\Theta = \{\theta_r\}_r$

Corollary

If $\varphi \models \psi$ there is a countable theory $\Theta \subseteq Ł\forall(\tau(\varphi) \cap \tau(\psi))$ such that $\varphi \models \Theta \models \psi$.

Both versions are equivalent.

Robinson

Approximate interpolation implies Robinson:

Theorem

If $T_i \subseteq Ł\forall_{\tau_i}$, $i = 1, 2$, are conservative extensions of a theory $T \subseteq Ł\forall_{\tau_1 \cap \tau_2}$ then $T_1 \cup T_2$ is satisfiable.

Proof. $T_1 \cup T_2$ is unsatisfiable so is $\{\delta_1, \delta_2\}$ where δ_i is the conjunction of a finite subset of T_i. Then $\{\delta_1, \delta_{2 \geq r}\}_{r<1}$ is unsatisfiable and by compactness again $\delta_1 \models (\delta_2)_{\leq r}$ for some $r < 1$. By approximate interpolation there is $\Theta \subseteq Ł\forall_{\tau_1 \cap \tau_2}$ such that $\delta_1 \models \Theta \models_S \delta_{2 \leq r}$. Then, $T_1 \models_S \Theta$ and $T \models \Theta$ by conservativity; hence, $T_2 \models \Theta \models \delta_{2 \leq r}$ which yields unsatisfiability of T_2. $\Box$

In fact, both propeties are equivalent in any compact extension of $Ł\forall$ closed under Lukasiewicz connectives.

For the other direction, notice that $\varphi \models \psi$ implies $K_1 = Mod(\varphi) \restriction X$ and $K_2 = Mod(\psi_{\leq r}) \restriction X$ are disjoin for any $r < 1$. If these classe where not separable, then we would have $A \equiv_{Ł\forall} B$ with $A \in K_1$, $B \in K_2$. Then,

Questions

• Does sharp deductive interpolation holds in Ł∀? or in continuous logic? Do proof-theoretical methods could shed any light?
• Does DivŁ∀ enjoys Craig interpolation
• Does CŁ∀ (continuous logic) enjoys Craig interpolation
(Ben Yaacov has shown that it holds in CL_{ω_1}, and infinitary version of CL)

THANKS!