Forest Products of MTL-chains

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Preliminaries

A semihoop is an algebra $\mathbf{A} = (A, \cdot, \to, \wedge, \vee, 1)$ of type $(2, 2, 2, 2, 0)$ such that $(A, \wedge, \vee)$ is lattice with 1 as greatest element, $(A, \cdot, 1)$ is a commutative monoid and for every $x, y, z \in A$:

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$$x \cdot y = \begin{cases} x \cdot_i y, & \textit{if} \quad x, y \in A_i \\ y, & \textit{if} \quad x \in A_i, \textit{ and } y \in A_j - \{1\}, \textit{ with } i > j, \\ x, & \textit{if} \quad x \in A_i - \{1\}, \textit{ and } y \in A_j, \textit{ with } i < j. \end{cases}$$

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Forest Products

Definition

Let $\mathbf{F} = (F, \leq)$ be a forest and let $\{\mathbf{M}_i\}_{i \in \mathbf{F}}$ a collection of MTL-chains such that, up to isomorphism, all they share the same neutral element 1. If $\left(\bigcup_{i \in \mathbf{F}} \mathbf{M}_i\right)^F$ denotes the set of functions $h : F \to \bigcup_{i \in \mathbf{F}} \mathbf{M}_i$ such that $h(i) \in \mathbf{M}_i$ for all $i \in \mathbf{F}$, the forest product $\bigotimes_{i \in \mathbf{F}} \mathbf{M}_i$ is the algebra $\mathbf{M}$ defined as follows:

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3. *For all* $i \in \boldsymbol{F}$ *if* $h(i) \neq 1$ *then for all* $i < j$, $h(j) = 0_j$,
4. $\bigcup_{i\in\boldsymbol{F}} h^{-1}(0_j)$ *is an upset of* $\boldsymbol{F}$, $h^{-1}(1)$ *is a downset of* $\boldsymbol{F}$ *and*

   $$C_h = \{i \in \boldsymbol{F} \mid h(i) \notin \{0_i, 1\}\},$$

   is a (possibly empty) antichain of $\boldsymbol{F}$.

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Let $\mathbf{F}$ be a forest and $\{\mathbf{M}_i\}_{i\in\mathbf{F}}$ a collection of MTL-chains. We write $\mathbb{D}(\mathbf{F})$ for the collection of downsets of $\mathbf{F}$. Let S be a proper downset of $\mathbf{F}$ and consider

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Lemma

Let $\boldsymbol{F}$ be a forest and $\{\boldsymbol{M}_i\}_{i\in\boldsymbol{F}}$ a collection of MTL-chains and $S \in \mathbb{D}(\boldsymbol{F})$. Then X_S and $\bigotimes_{i\in\boldsymbol{S}^c}\boldsymbol{M}_i$ are isomorphic semihoops.

Forest Products are Sheaves

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$$\mathcal{P}(S) \cong \mathcal{P}(F)/X_S.$$

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Corollary

Let $\boldsymbol{F}$ be a forest and $\{\boldsymbol{M}_i\}_{i \in \boldsymbol{F}}$ a collection of MTL-chains. Then $\mathcal{P}$ is a sheaf of MTL-chains.

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