On Kalman’s functor for bounded hemiimplicative semilattices and hemiimplicative lattices

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XIV Congreso Dr. Antonio Monteiro (2017)
A De Morgan algebra is an algebra \( \langle A, \lor, \land, \sim, 0, 1 \rangle \) of type \((2, 2, 1, 0, 0)\) such that satisfies the following conditions:

- \( \langle A, \lor, \land, 0, 1 \rangle \) is a bounded distributive lattice,
- \( \sim \sim x = x \),
- \( \sim (x \lor y) = \sim x \land \sim y \), \( \sim (x \land y) = \sim x \lor \sim y \).
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A Kleene algebra is a De Morgan algebra which satisfies

$$x \land \sim x \leq y \lor \sim y.$$
Kalman’s functor

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\]

A Kleene algebra is centered if it has a center. That is, an element \( c \) such that \( \sim c = c \) (it is necessarily unique).
Kalman’s functor

In 1958 Kalman proved that if $L$ is a bounded distributive lattice, then

\[ K(L) = \{(a, b) \in L \times L : a \land b = 0\} \]

is a centered Kleene algebra by defining

\[
\begin{align*}
(a, b) \lor (d, e) &:= (a \lor d, b \land e), \\
(a, b) \land (d, e) &:= (a \land d, b \lor e), \\
\sim (a, b) &:= (b, a),
\end{align*}
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$(0, 1)$ as the zero, $(1, 0)$ as the top and $(0, 0)$ as the center.
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Kalman’s functor

For \((a, b) \in K(L)\) we have that

\[(a, b) \land (0, 0) = (a \land 0, b \lor 0) = (0, b),\]

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\]

Therefore, the center give us the coordinates of \((a, b)\).
Kalman’s functor

Later, in 1986 Cignoli proved the following facts:

1. K can be extended to a functor from the category of bounded distributive lattices BDL to the category of centered Kleene algebras.

2. There exists an equivalence between BDL and the full subcategory of centered Kleene algebras which satisfy (IP) (a topological condition).

Later, in 1986 Cignoli proved the following facts:

1. $K$ can be extended to a functor from the category of bounded distributive lattices $\text{BDL}$ to the category of centered Kleene algebras.

2. There exists an equivalence between $\text{BDL}$ and the full subcategory of centered Kleene algebras which satisfy (IP) (a topological condition).


In an unpublished manuscript (2004) M. Sagastume proved that if $T$ is a centered Kleene algebra then

\[ T \text{ satisfies (IP) iff } T \text{ satisfies (CK) (an algebraic condition).} \]
Kalman’s functor

1. If $T$ is a centered Kleene algebra then $C(T) = \{x : x \geq c\} \in \text{BDL}$. 

2. $C$ can be extended to a functor.

Theorem

There is a categorical equivalence $K \dashv C$ between BDL and the full subcategory of centered Kleene algebras whose objects satisfy (CK).

Sagastume, M. Categorical equivalence between centered Kleene algebras with condition (CK) and bounded distributive lattices, 2004.
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A quasi-Nelson algebra is a Kleene algebra such that
\[ x \rightarrow y := x \rightarrow_{\text{HA}} (\sim x \lor y) \]
exists, where \( \rightarrow_{\text{HA}} \) is the Heyting implication.

A Nelson algebra is a quasi-Nelson algebra such that
\[ (x \land y) \rightarrow z = x \rightarrow (y \rightarrow z). \]

A Nelson lattice is an involutive bounded commutative residuated lattice which satisfies an additional equation. The varieties of Nelson algebras and Nelson lattices are term equivalent.
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If \( \to \) is the implication of a Nelson algebra, then the implication as
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Let $H$ be a Heyting algebra where $\to$ is the Heyting implication. In $K(H)$ the implication as Nelson algebra is given by

$$(a, b) \Rightarrow_{\text{NA}} (d, e) = (a \to d, a \land e)$$
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Remark

Notice that the well definition of $\Rightarrow_{NA}$ and $\Rightarrow$ follows from the equation

$$a \land (a \to b) \leq b$$
Hemiimplicative semilattices (lattices)

Definition

An algebra \((H, \land, \to, 1)\) of type \((2, 2, 0)\) is a hemiimplicative semilattice if:

1. \((H, \land, 1)\) is a bounded semilattice.
2. For every \(a, b \in H\), \(a \land (a \to b) \leq b\).
3. For every \(a \in H\), \(a \to a = 1\).

We say that an algebra \((H, \land, \lor, \to, 0, 1)\) of type \((2, 2, 2, 0, 0)\) is a hemiimplicative lattice if \((H, \land, \lor, 0, 1)\) is a bounded distributive lattice and \((H, \land, \to, 1)\) is a hemiimplicative semilattice.
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We say that an algebra \((H, \land, \lor, \rightarrow, 0, 1)\) of type \((2, 2, 2, 0, 0)\) is a hemiimplicative lattice if \((H, \land, \lor, 0, 1)\) is a bounded distributive lattice and \((H, \land, \rightarrow, 1)\) is a hemiimplicative semilattice.

Item 2 is equivalent to the following condition:

For every \(a, b, c \in H\), if \(a \leq b \rightarrow c\) then \(a \land b \leq c\).
Example 1: Implicative semilattices

Definition

An implicative semilattice is an algebra \((H, \land, \rightarrow)\) of type \((2, 2)\) such that \((H, \land)\) is semilattice, and for every \(a, b, c \in H\) we have that \(a \land b \leq c\) if and only if \(a \leq b \rightarrow c\).

Every implicative semilattice has a greatest element.

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Every implicative semilattice has a greatest element.

Example 2: Hilbert algebras with infimum

Definition

A Hilbert algebra with infimum is an algebra \((H, \rightarrow, \land, 1)\) of type \((2, 2, 0)\) such that satisfies the following conditions:

1. \(a \rightarrow (b \rightarrow a) = 1\).
2. \(a \rightarrow (b \rightarrow d) = (a \rightarrow b) \rightarrow (a \rightarrow d)\).
3. If \(a \rightarrow b = b \rightarrow a = 1\), then \(a = b\).
4. \((H, \land, 1)\) is a bounded semilattice.
5. For every \(a, b \in H\), \(a \leq b\) if and only if \(a \rightarrow b = 1\), where \(\leq\) is the semilattice order.
A *Hilbert algebra with infimum* is an algebra \((H, \to, \wedge, 1)\) of type \((2, 2, 0)\) such that satisfies the following conditions:

1. \(a \to (b \to a) = 1\).
2. \(a \to (b \to d) = (a \to b) \to (a \to d)\).
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4. \((H, \wedge, 1)\) is a bounded semilattice.
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Example 3: Semi-Heyting algebras

Definition

An algebra \((H, \land, \lor, \rightarrow, 0, 1)\) is a semi-Heyting algebra if the following conditions hold for every \(a, b, c \in H\):

1. \((H, \land, \lor, 0, 1)\) is a bounded distributive lattice,
2. \(a \land (a \rightarrow b) = a \land b\),
3. \(a \land (b \rightarrow c) = a \land [(a \land b) \rightarrow (a \land c)]\),
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Extending Kalman’s functor

Consider the algebraic category KhBDL whose objects are algebras \((T, \wedge, \vee, \rightarrow, \sim, c, 0, 1)\) of type \((2, 2, 2, 1, 0, 0, 0)\) such that

1. \((T, \wedge, \vee, \sim, c, 0, 1)\) is a centered Kleene algebra,
2. \(\rightarrow\) is a binary operation on \(T\) which satisfies certain equations involving the other operations.

Theorem

There is an equivalence \(K \dashv C\) between the full subcategory of KhBDL whose objects satisfy \((\text{CK})\) and the algebraic category of hemiimplicative lattices.

A similar result is obtained for the category of semi-Heyting algebras (the equivalent category to the category of semi-Heyting algebras is a variety).
Extending Kalman’s functor

Consider the category $\text{KhIS}$ whose objects are structures $(T, \leq, \rightarrow, \sim, c, 0, 1)$ such that

1. $(T, \leq)$ is a poset with first element 0 and last element 1,
2. $(T, \rightarrow, \sim, c, 0, 1)$ is an algebra which satisfies certain conditions involving the order and the operations.

Theorem

There is an equivalence $K \dashv C$ between the full subcategory of $\text{KhIS}$ whose objects satify the condition $(\mathcal{C}K)$ (over the corresponding existing infima and suprema) and the algebraic category of hemiimplicative lattices with first element.

A similar result is obtained for the category of Hilbert algebras with infimum and first element, and for the category of implicative semilattices with first element.
Extending Kalman’s functor

In the paper

- *Semi-Nelson Algebras*, from Juan Manuel Cornejo and Ignacio Viglizzo (to appear in Order),

it was introduced and studied the variety of semi-Nelson algebras.
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it was introduced and studied the variety of semi-Nelson algebras.

- The varieties of Nelson algebras and Nelson lattices are term equivalent. Is there any relation between the variety of semi-Nelson algebras and the variety whose algebras are the objects of our equivalent category of the category of semi-Heyting algebras?
If $T \in \text{KhBDL}$ then

$$\text{Con}(T) \cong \text{Con}(C(T)).$$

In particular, if $C(T)$ is a semi-Heyting algebra then we characterize the principal congruences of $T$.

Let $T \in \text{KhIS}$. We say that an equivalence relation $\theta$ of $T$ is a well-behaved congruence of $T$ if it satisfies the following conditions:

1. $\theta \in \text{Con}((T, \rightarrow, \sim))$.
2. For $x, y \in T$, $(x, y) \in \theta$ if and only if $(x \lor c, y \lor c) \in \theta$ and $(\sim x \lor c, \sim y \lor c) \in \theta$.
3. For $x, y, z$ and $w$ in $C(T)$, if $(x, y) \in \theta$ and $(z, w) \in \theta$, then $(x \land z, y \land w) \in \theta$.

Let $\text{Con}_{\text{wb}}(T)$ be the set of well-behaved congruence of $T$. Then,

$$\text{Con}_{\text{wb}}(T) \cong \text{Con}(C(T)).$$

In particular, if $C(T)$ is an implicative semilattice then we characterize the principal well-behaved congruences of $T$. 