Relation Between BL-Possilistic Logic and Epistemic BL-Algebras

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Possibilistic measures of uncertainty: the Boolean case

Logical setting: $\mathcal{L}$ propositional language
events = propositions (mod. CPL logical equivalence)
$\top$ always true event,
$\bot$ always false event

Possibility and Necessity measures $\Pi, N : \mathcal{L} \to [0,1]$

- $\Pi(\bot) = N(\bot) = 0$
- $\Pi(\top) = N(\top) = 1$
- if $\vdash \varphi \to \psi$, then $\Pi(\varphi) \leq \Pi(\psi)$ and $N(\varphi) \leq N(\psi)$

Possibility: $\Pi(\varphi \vee \psi) = \max(\Pi(\varphi), \Pi(\psi))$

Necessity: $N(\varphi \wedge \psi) = \min(N(\varphi), N(\psi))$

Dual pairs of measures (N, Π): when $\Pi(\varphi) = 1 - N(\neg\varphi)$

Possibility and Necessity Measures: representation

Let $\mathcal{L}$ be the language generated by a set of propositional variables Var and let W be its set of Boolean interpretations (possible worlds).

- $\Pi : \mathcal{L} \to [0,1]$ is a possibility measure iff there is a possibility distribution $\pi : W \to [0,1]$ such that, for every φ

$$\Pi(\varphi) = \sup_{w \models \varphi} \pi(w).$$

  $\Pi(\varphi)$ represents the degree to which the event φ is compatible with the available evidence represented by π.

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- $N : \mathcal{L} \to [0,1]$ is a necessity measure iff there is a possibility distribution $\pi : W \to [0,1]$ such that, for every φ,

$$N(\varphi) = \inf_{w \not\models \varphi} 1 - \pi(w) = 1 - \Pi(\neg\varphi)$$

  $N(\varphi)$ represents the degree to which the event φ is entailed by the available evidence, i.e. the certainty of the occurrence of φ.

Necessity and Possibility measures on BL-algebras

- $\mathcal{L}$ language of propositional logic $(\&, \rightarrow, \neg)$ + modal operators (N, Π).
- W set of $\mathcal{C}$-interpretations, with $\mathcal{C}$ is a BL-algebra.
- $\pi : W \mapsto \mathcal{C}$ normalized possibility distribution.

The tuple $\langle W, \pi \rangle$ is called *Possibilistic Model*. We call its underling logic KD45($\mathcal{C}$).

We choose the following generalizations (compatible with the natural evaluation of $\Box$ and $\Diamond$ in many-valued modal logics):

Necessity measure: $N : \mathcal{L} \rightarrow \mathcal{C}$ defined as

$$N(\varphi) = \inf_{w \in W} \{\pi(w) \Rightarrow w(\varphi)\}$$

Possibility measure: $\Pi : \mathcal{L} \rightarrow \mathcal{C}$ defined as

$$\Pi(\varphi) = \sup_{w \in W} \{\pi(w) \star w(\varphi)\}$$

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Remark: duality is lost, $N(\neg\varphi) = \neg\Pi(\varphi)$ but $N(\varphi) \neq \neg\Pi(\neg\varphi)$

A particular case: S5($\mathcal{C}$)

If $\forall w \in W : \pi(w) = 1$ then the previous semantics is called *Universal*.
Hájek defined the fuzzy modal logic S5($\mathcal{C}$) as the underlying logic of this semantics and he presented an axiomatization for this logic.
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We are able to give a nice translation between KD45($\mathcal{C}$) and S5($\mathcal{C}$).
Given a fixed set of propositional variables Var and $c \notin Var$, we define inductively a map $\varphi \mapsto \varphi^*$ from $\mathcal{L}(Var)$ into $\mathcal{L}(Var \cup \{c\})$ as follows:

$$\begin{aligned}
\varphi^* &:= \varphi \text{ for } \varphi \in Var \cup \{\top, \bot\} \\
(\varphi \circledast \psi)^* &:= \varphi^* \circledast \psi^* \text{ for } \circledast \in \{\wedge, \vee, \&, \rightarrow\} \\
(N\varphi)^* &:= N(c \rightarrow \varphi^*) \\
(\Pi\varphi)^* &:= \Pi(c \& \varphi^*)
\end{aligned}$$

Thus, using their translation, it is easy to prove the following:

Theorem

Let c be a fixed propositional variable not occurring in $\varphi \in \mathcal{L}$ then:

$$\models_{\mathrm{KD45}(\mathcal{C})} \varphi \quad \textit{iff} \quad \Pi c \models_{\mathrm{S5}(\mathcal{C})} \varphi^*$$

Hájek's Axiomatization of S5($\mathcal{C}$)

The logic S5($\mathcal{C}$) was axiomatizated by Hájek taking the axioms for the basic logic BL together with the following modal axioms:

$(\Box 1)$ $\Box\varphi \to \varphi$.

$(\Diamond 1)$ $\varphi \to \Diamond\varphi$.

$(\Box 2)$ $\Box(\nu \to \varphi) \to (\nu \to \Box\varphi)$.

$(\Diamond 2)$ $\Box(\varphi \to \nu) \to (\Diamond\varphi \to \nu)$.

$(\Box 3)$ $\Box(\nu \vee \varphi) \to (\nu \vee \Box\varphi)$.

$(\Diamond 3)$ $\Diamond(\varphi \star \varphi) \equiv \Diamond\varphi \star \Diamond\varphi$.

where ν is any formula beginning with $\Box$ or $\Diamond$. The inference rules are:

(MP) $\varphi, \varphi \to \psi \vdash \psi$.

(Nec) $\varphi \vdash \Box\varphi$.

Hájek proved that this axiomatization is strongly complete with respect to Universal Models.

Our goal

To give an algebraic characterization of the fuzzy modal logic KD45(BL).

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Currently, the logic KD45(G) is the unique logic with a known axiomatization.

The axiom K: $\Box(p \to q) \to \Box p \to \Box q$ is not valid

We consider a model in $\mathbf{L}_5$:

$\{u(p) = \frac{1}{2},\ u(q) = \frac{3}{4}\}$ $\pi(u) = 1$ $\qquad$ $\pi(v) = \frac{3}{4}$ $\{v(p) = \frac{1}{4},\ v(q) = 0\ \}$

$$\Box(p \to q) = \min\{\pi(u) \to (u(p) \to u(q)), \pi(v) \to (v(p) \to v(q))\} =$$

$$= \min\{1 - 1 + 1, 1 - \frac{3}{4} + \min(1, 1 - \frac{1}{4} + 0\} = 1$$

$$\Box p = \min\{\pi(u) \to u(p), \pi(v) \to v(p)\} = \min\{1 - 1 + \frac{1}{2}, 1 - \frac{3}{4} + \frac{1}{4}\} = \frac{1}{2}$$

$$\Box q = \min\{\pi(u) \to u(q), \pi(v) \to v(q)\} = \min\{1 - 1 + \frac{3}{4}, 1 - \frac{3}{4} + 0\} = \frac{1}{4}$$

$$\Box p \to \Box q = 1 - \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

Epistemic BL-Algebras

Definition

An algebra $\mathbf{A} = \langle A, \vee, \wedge, \star, \rightarrow, \forall, \exists, 0, 1\rangle$ of type $(2, 2, 2, 2, 1, 1, 0, 0)$ is called a *Epistemic* BL-*algebra* (an EBL-algebra for short) if $\langle A, \vee, \wedge, \star, \rightarrow, 0, 1\rangle$ is a BL-algebra that also satisfies:

$$
\begin{array}{clcl}
(\mathrm{E}\forall) & \forall 1 & = & 1 \\
(\mathrm{E}\exists) & \exists 0 & = & 0 \\
(\mathrm{E1}) & \forall a \rightarrow \exists a & = & 1 \\
(\mathrm{E2}) & \forall(a \rightarrow \forall b) & = & \exists a \rightarrow \forall b \\
(\mathrm{E3}) & \forall(\forall a \rightarrow b) & = & \forall a \rightarrow \forall b \\
(\mathrm{E4}) & \exists a \rightarrow \forall \exists a & = & 1 \\
(\mathrm{E4a}) & \forall(a \wedge b) & = & \forall a \wedge \forall b \\
(\mathrm{E4b}) & \exists(a \vee b) & = & \exists a \vee \exists b \\
(\mathrm{E5}) & \exists(a \star \exists b) & = & \exists a \star \exists b
\end{array}
$$

Hájek's fuzzy modal logic KD45(BL)

Definition

Given a complete BL-algebra $\mathcal{A}$, a *possibilistic* $\Pi\mathcal{A}$ *model* is a triple $\langle W, \pi, e\rangle$ where where W *is a non-empty set of worlds*, $\pi : W \to \mathbf{A}$ (i.e. $\pi \in \mathbf{A}^W$) is normalized possibility distribution over W, that is, such that $\sup_{w\in W} \pi(w) = 1^{\mathcal{A}}$, and $e : W \times Var \mapsto \mathbf{A}$ provides an evaluation of variables in each world. For each $w \in W$, $e(w, -)$ extends to arbitrary formulas in the usual way for the propositional connectives and for modal operators in the following way:

$$e(w, \Box\varphi) := \inf_{w\in W}\{\pi(w) \Rightarrow e(w, \varphi)\}$$
$$e(w, \Diamond\varphi) := \sup_{w\in W}\{\pi(w) \star e(w, \varphi)\}$$

Remark

The map $e : W \times Var \mapsto \mathbf{A}$ *can be turned into a map* $\bar{e} : Var \mapsto \mathbf{A}^W$ *and it may be extended to the whole modal language in the usual way* $\tilde{e} : \mathcal{L} \mapsto \mathbf{A}^W$.

Complex Epistemic BL-Algebras (1)

Considering a $\Pi\mathcal{A}$-frame $\mathcal{P} = \langle W, \pi \rangle$ and remembering that $\pi \in \mathbf{A}^W$, we can define its associated complex $\mathcal{A}$-algebra $\mathcal{A}^{\mathcal{P}} = \langle \mathbf{A}^W, \forall^{\mathcal{P}}, \exists^{\mathcal{P}} \rangle)$ where $\mathbf{A}^W$ is the product algebra, and for each map $f \in \mathbf{A}^W$:

$$\begin{aligned} \forall^{\mathcal{P}}(f) &= \inf_{w \in W} \{\pi(w) \Rightarrow f(w)\} \\ \exists^{\mathcal{P}}(f) &= \sup_{w \in W} \{\pi(w) \star f(w)\} \end{aligned}$$

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On the other direction, given an complex $\mathcal{A}$-algebra $\mathcal{A} = \langle \mathbf{A}^W, \forall, \exists\rangle$, we can define its associated possibilistic frame $\mathcal{P}' = \langle W, \pi'\rangle$, where:

$$\pi'(w) = \inf_{f\in \mathbf{A}^W}\{\min((\forall f)(w) \to f(w), f(w) \to (\exists f)(w))\}$$

Complex Epistemic BL-Algebras (2)

Theorem

Given a $\Pi\mathbf{A}$-framel $\mathcal{M} = \langle W, \pi \rangle$, the associated complex $\mathcal{A}$-algebra $\mathcal{A}^{\mathcal{P}} = \langle \mathbf{A}^W, \forall^{\mathcal{P}}, \exists^{\mathcal{P}} \rangle)$ is an Epistemic $\mathcal{A}$-algebra.

Complex Epistemic BL-Algebras (2)

Theorem

Given a $\Pi\mathbf{A}$*-framel* $\mathcal{M} = \langle W, \pi \rangle$*, the associated complex* $\mathcal{A}$*-algebra* $\mathcal{A}^{\mathcal{P}} = \langle \mathbf{A}^W, \forall^{\mathcal{P}}, \exists^{\mathcal{P}} \rangle)$ *is an Epistemic* $\mathcal{A}$*-algebra. Furthermore,* $\mathcal{M} = \langle W, \pi \rangle$ *is its associated possibilistic frame.*

We call an algebra with universe $\mathbf{A}^W$ a complex Epistemic $\mathcal{A}$-algebra.

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Theorem

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We call an algebra with universe $\mathbf{A}^W$ a complex Epistemic $\mathcal{A}$-algebra. In particular, we can show the following result:

Theorem

Let $\mathcal{P} = \langle W, \pi, e \rangle$ *be a* $\Pi\mathcal{A}$*-model. Then the set* **E** $= \{\tilde{e}(\varphi) | \varphi \in \mathcal{L}\} \subseteq \mathbf{A}^W$ *is the universe of a complex Epistemic* BL*-algebra.*

Optimal Possibilistic Models

Definition

Given a $\Pi\mathbf{A}$-model $\mathcal{M} = \langle W, \pi, e\rangle$, define a new accessibility relation as follows:

$$\pi^{+}(w) = \inf_{\varphi \in Fm_{\Box\Diamond}} \{\min(e(\Box\varphi, w) \Rightarrow e(\varphi, w), e(\varphi, w) \Rightarrow e(\Diamond\varphi, w))\}$$

Call $\mathcal{M}$ *optimal* whenever $\pi^{+} = \pi$.

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The following lemma shows that any $\Pi\mathbf{A}$-model is equivalent to an optimal one.

Lemma

The model $\mathcal{M}^+ = \langle W, \pi^+, e^+ \rangle$ is optimal. Moreover, if e^+ is the extension of e in $\mathcal{M}^+$, then $e^+(\varphi, w) = e(\varphi, w)$ for any $\varphi \in Fm_{\Box\Diamond}$ and any $w \in W$.

Conclusions and future works

- We have showed a relation between Possibilistic Models and EBL-algebras.
- However, we have not be able to give an axiomatization for the logic $KD45(BL)$ yet.
- Studying the algebraic characterization of $KD45(BL)$.

Thank you for your attention


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