NPc-algebras and Gödel hoops

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Odintsov, S. P.: *On the representation of N4-lattices*. Stud. Log. **76**, 385-405 (2004).

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Busaniche, M., Cignoli, R.: *Residuated lattices as an algebraic semantics for paraconsistent Nelson logic*. J. Log. Comput. **19**, 1019-1029 (2009).

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Residuated lattices form a variety, as the residuation quasiequation can be replaced by equations.

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If the underlying lattice is distributive, we say $\mathbf{L}$ is a *commutative distributive residuated lattice*.

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If e is the maximum element, we say $\mathbf{L}$ is *integral*.

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$\mathbf{L}^- = (L^-, \wedge, \vee, *, \to_e, e)$ is an integral commutative residuated lattice.

Twist structures

By a *full twist-product* of an integral commutative residuated lattice **L** we mean the algebra

$$\mathbf{K}(\mathbf{L}) = (L \times L, \sqcap, \sqcup, \bullet, \Rightarrow, (e, e))$$

with the operations $\sqcup, \sqcap, *, \Rightarrow$ given by

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$$\begin{aligned}
(x, y) \sqcap (x', y') &= (x \wedge x', y \vee y') \\
(x, y) \sqcup (x', y') &= (x \vee x', y \wedge y') \\
(x, y) \bullet (x', y') &= (x * x', (x \to y') \wedge (x' \to y)) \\
(x, y) \Rightarrow (x', y') &= ((x \to x') \wedge (y' \to y), x * y')
\end{aligned}$$

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Every subalgebra $\mathbf{A}$ of $\mathbf{K}(\mathbf{L})$ containing the set $\{(a, e) : a \in L\}$ is called a *twist-product* obtained from $\mathbf{L}$.

Every twist-product satisfies

K-lattices

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- **(e-involution)** $((x, y) \Rightarrow (e, e)) \Rightarrow (e, e) = (x, y)$
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- **(distributivity at (e, e))**

$$(x, y) \sqcup ((x', y') \sqcap (x'', y'')) = ((x, y) \sqcup (x', y')) \sqcap ((x, y) \sqcup (x'', y''))$$
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- $((x,y) \bullet (x',y')) \sqcap (e,e) = ((x,y) \sqcap (e,e)) \bullet ((x',y') \sqcap (e,e))$
- $(((x,y) \sqcap (e,e)) \Rightarrow (x',y')) \sqcap ((\sim (x',y') \sqcap (e,e)) \Rightarrow \sim (x,y)) = (x,y) \Rightarrow (x',y')$

K-lattices

A K-lattice is a commutative residuated lattice satisfying

- **(e-involution)** $(a \to e) \to e = a$
  (then we define $\sim a = a \to e$)
- **(distributivity at e)**

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$
$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

  whenever one of the three a, b, c is replaced with e
- $(a * b) \wedge e = (a \wedge e) * (b \wedge e)$
- $((a \wedge e) \to b) \wedge ((\sim b \wedge e) \to \sim a) = a \to b$

Theorem

Let **A** *be a K-lattice. The map*

$$\phi_{\mathbf{A}} : \mathbf{A} \to \mathbf{K}(\mathbf{A}^{-})$$

given by

$$a \mapsto (a \wedge e, \sim a \wedge e)$$

is an injective homomorphism.

Busaniche, M., Cignoli, R.: *Commutative residuated lattices represented by twist-products*, Algebra Universalis **71**, 5-22 (2014).

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The negative cone of an NPc-lattice is a *Brouwerian algebra*: an integral residuated lattice with $a * b = a \wedge b$ (also called *generalized Heyting algebra* or *implicative lattice*).

Odintsov's approach

Odintsov, S. P.: *Constructive Negations and Paraconsistency*. Trends in Logic-Studia Logica Library 26. Springer. Dordrecht (2008).

Odintsov's approach

$\mathbf{L}$ a Brouwerian algebra, Odintsov defines a weak implication over $\mathbf{L} \times \mathbf{L}^{\partial}$

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$$Tw(L, \nabla, \Delta) = \{(x, y) : x \vee y \in \nabla, x \wedge y \in \Delta\}$$

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  is the universe of a "twist-product" over **L** (with this weak implication).
- **B** a "twist-product" over **L**. Define

$$\nabla = \{\pi_1(b \sqcup \sim b) : b \in B\}, \qquad \Delta = \{\pi_2(b \sqcup \sim b) : b \in B\}.$$

  Then ∇ is a regular filter, Δ an ideal and $B = Tw(L, \nabla, \Delta)$.

Our approach

Theorem

Let $\mathbf{L}$ *be a Brouwerian algebra and* ∇ *a regular filter of* $\mathbf{L}$*. Then the subset*

$$Tw(L,\nabla) = \{(x,y) \in L \times L : x \vee y \in \nabla\},$$

of the NPc-lattice $\mathbf{K}(\mathbf{L})$ *is a twist-product obtained from* $\mathbf{L}$*.*

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of the NPc-lattice $\mathbf{K}(\mathbf{L})$ *is a twist-product obtained from* $\mathbf{L}$*.*

Moreover, if $\mathbf{L}'$ *is another Brouwerian algebra and* ∇' *a regular filter in* $\mathbf{L}'$*, for each morphism* $f : \mathbf{L} \to \mathbf{L}'$ *satisfying* $f(\nabla) \subseteq \nabla'$ *we obtain an NPc-lattice morphism*

$$\mathbf{f} : \mathbf{Tw}(\mathbf{L},\nabla) \to \mathbf{Tw}(\mathbf{L}',\nabla')$$

given by $\mathbf{f}((x,y)) = (f(x), f(y))$*.*

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Let $\mathbf{B}$ *be an NPc-lattice. Then the set* $\nabla = \{(b \vee \sim b) \wedge e : b \in B\}$ *is a regular filter in* $\mathbf{B}^-$, *and*

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Moreover, if $\mathbf{B}'$ *is another NPc-lattice, for each NPc-lattice morphism* $\mathbf{f} : \mathbf{B} \to \mathbf{B}'$ *we obtain a Brouwerian morphism* $f : \mathbf{B}^- \to (\mathbf{B}')^-$ *given by* $f = \mathbf{f}|_{\mathbf{B}^-}$*, that satisfies* $f(\nabla) \subseteq \nabla'$*, where* $\nabla' = \{(c \vee \sim c) \wedge e : c \in B'\}$.

Categorical equivalence

Category $\mathbb{BF}$

- objects: pairs $(\mathbf{L}, \nabla)$, $\mathbf{L}$ a Brouwerian algebra and $\nabla \subset L$ a regular filter

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Category $\mathbb{NPC}$ of NPc-lattices and NPc-lattice morphisms.

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Theorem

The functor $Tw : \mathbb{BF} \to \mathbb{NPC}$ that acts on objects as $\mathbf{Tw}(\mathbf{L}, \nabla)$ and on arrows $f : (\mathbf{L}, \nabla) \to (\mathbf{L}', \nabla')$ as $Tw(f) : \mathbf{Tw}(\mathbf{L}, \nabla) \to \mathbf{Tw}(\mathbf{L}', \nabla')$ given by

$$Tw(f)(x, y) = (f(x), f(y)),$$

gives an equivalence of categories.

GNPc-lattices

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Theorem

The restriction of the functor Tw *to the category* $\mathbb{GHF}$ *of pairs consisting of Gödel hoops and regular filters, gives an equivalence of categories between* $\mathbb{GHF}$ *and the full subcategory* $\mathbb{GNPC}$ *of* $\mathbb{NPC}$ *having Gödel NPc-lattices as objects.*

Free algebras

Recall that if a variety of algebras is generated by an algebra $\mathbf{A}$, then the free algebra with n generators is isomorphic to the subalgebra of functions $f : \mathbf{A}^n \to \mathbf{A}$ generated by the projection functions (we use this for $n = 1$).

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The variety $\mathbb{GNPC}$ of Gödel NPc-lattices is generated by the full twist product $\mathbf{K}([0,1]_\mathbf{G})$.

Free(1)

Theorem

The free algebra with one generator in the variety $\mathbb{GNPC}$ *satisfies*

$$\begin{aligned}\mathrm{Free}_{\mathbb{GNPC}}(1) &\cong \mathbf{Tw}(\mathbf{G}_3, \mathbf{G}_2) \times \mathbf{K}(\mathbf{G}_2) \times \mathbf{Tw}(\mathbf{G}_3, \mathbf{G}_2) \\ &\cong \mathbf{Tw}(\mathbf{G}_3 \times \mathbf{G}_2 \times \mathbf{G}_3, \mathbf{G}_2 \times \mathbf{G}_2 \times \mathbf{G}_2) \\ &\cong \mathbf{Tw}(\mathrm{Free}_{\mathbb{GH}}(2), \nabla),\end{aligned}$$

where $\nabla = \mathbf{G}_2 \times \mathbf{G}_2 \times \mathbf{G}_2$ *and* $\mathbf{G}_k$ *denotes the Gödel hoop chain of* k *elements.*

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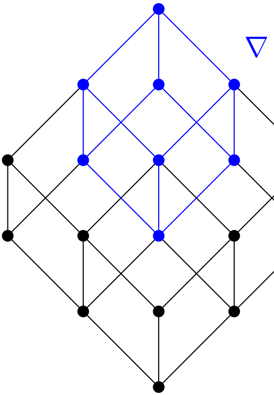
Free(1)

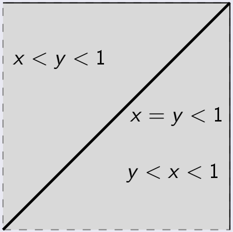


$$\mathrm{Free}_{\mathbb{GNPC}}(1) = Tw\left(\mathrm{Free}_{\mathbb{GH}}(2), \nabla\right)$$

Free(1)

Idea of the proof .

Following the ideas in *A note on functions associated with Gödel formulas* by B. Gerla, the behaviour of the 2-variable terms φ is independent in the following regions of $[0,1]^2$:



In our case, in the regions $x < y < 1$ and $x < y = 1$ we cannot have different behaviours. The same is true for the regions $y < x < 1$ and $y < x = 1$, and the regions $x = y < 1$ and $x = y = 1$. □

A duality result

Given a finite tree T, a subtree t of T is an **atomic upward closed** subtree of T if t contains the root of T and whenever an atom a of T belongs to t and $b \in T$ with $b \geq a$, then $b \in t$.

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Category $\mathcal{T}_{t,fin}$: objects are pairs (T, t) with T a finite tree and t an atomic upward closed subtree; arrows $\phi : (T, t) \to (T', t')$ open maps $\phi : T \to T'$ with $\phi(t) \subseteq t'$.

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Category $\mathcal{T}_{t,fin}$: objects are pairs (T, t) with T a finite tree and t an atomic upward closed subtree; arrows $\phi : (T, t) \to (T', t')$ open maps $\phi : T \to T'$ with $\phi(t) \subseteq t'$.

Theorem

$\mathcal{T}_{t,fin}$ is the dual of the category $\mathbb{GNPC}_{fin}$ of finite Gödel NPc-lattices.

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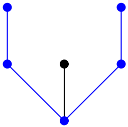

The dual of $\mathrm{Free}_{\mathbb{GNPC}}(1)$

As

$$\mathrm{Free}_{\mathbb{GNPC}}(n) = \coprod_{i=1}^{n} \mathrm{Free}_{\mathbb{GNPC}}(1),$$

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where T_n is the dual of $\mathrm{Free}_{\mathbb{GNPC}}(n)$, H_i is the dual of $\mathrm{Free}_{\mathbb{GH}}(i)$, and

$$a_{i,n} = \binom{2n}{i} - c_{i,n} \qquad b_{i,n} = c_{i,n}$$

where for $i \leq n-1$, $c_{i,n} = 0$ and for $i \geq n$, $c_{i,n} = 2^{2n-i}\binom{n}{2n-i}$.

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Theorem

$$\mathrm{Free}_{\mathbb{GNPC}}(n) \cong \prod_{i=0}^{2n-1} \mathbf{K}((\mathrm{Free}_{\mathbb{GH}}(i))_{\perp})^{a_{i,n}} \times \prod_{i=n}^{2n-1} \mathbf{Tw}\left((\mathrm{Free}_{\mathbb{GH}}(i))_{\perp}, \mathrm{Free}_{\mathbb{GH}}(i)\right)^{b_{i,n}}$$
$$\cong \mathbf{Tw}\left(\mathrm{Free}_{\mathbb{GH}}(2n), \nabla\right),$$

where $\nabla = \prod_{i=0}^{2n-1} ((\mathrm{Free}_{\mathbb{GH}}(i))_{\perp})^{a_{i,n}} \times \prod_{i=n}^{2n-1} (\mathrm{Free}_{\mathbb{GH}}(i))^{b_{i,n}}$.

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Thank you!!!