On the poset product representation of BL-algebras

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Residuated lattices and BL-algebras

A *residuated lattice* is an algebra $\langle L, \wedge, \vee, \cdot, \rightarrow, 0, 1\rangle$ with four binary operations and two constants such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a lattice with minimum 0 and maximum 1 (w.r.t. the lattice ordering $\leq$), $\langle L, \cdot, 1\rangle$ is a commutative monoid with the unit element 1 and

$$z \leq (x \rightarrow y) \iff x \cdot z \leq y \qquad \text{(residuation)}$$

for all $x, y, z \in L$. A residuated lattice $(L, \wedge, \vee, \cdot, \rightarrow, 0, 1)$ is a *BL-algebra* if and only if the following identities hold for all $x, y \in L$

$$x \wedge y = x \cdot (x \rightarrow y) \qquad \text{(divisibility)}$$

$$(x \rightarrow y) \vee (y \rightarrow x) = 1 \qquad \text{(prelinearity)}$$

A *BL-chain* is a totally ordered BL-algebra.

Subvarieties

We will mainly work with two subvarieties of BL-algebras:

- *MV-algebras*

$$\neg\neg x = x \qquad (\text{where } \neg x \text{ is } x \to 0)$$

- *Product algebras*

$$(\neg\neg z \cdot ((x \cdot z) \to (y \cdot z))) \to (x \to y) = 1$$
$$x \wedge \neg x = 0$$

The poset product construction

Given a poset $\mathbf{P} = \langle P, \leq \rangle$ and a collection $\{\mathbf{A}_p : p \in P\}$ of BL-algebras sharing the same neutral element 1 and the same minimum element 0, the *poset product* $\bigotimes_{p \in P} \mathbf{A}_p$ is the residuated lattice $\mathbf{A} = \langle A, \cdot, \to, \vee, \wedge, \bot, \top \rangle$ defined as follows:

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- A is the set of all maps $x \in \prod_{p \in P} A_p$ such that for all $i \in P$, if $x_i \neq 1$, then $x_j = 0$ provided that $j > i$.

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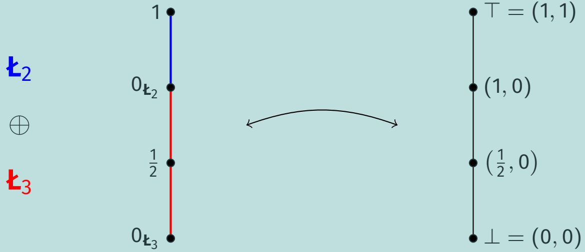
- $\top$ ($\bot$) is the map whose value in each coordinate is 1 (0).
- Monoid and lattice operations are defined pointwise.
- The residual is

$$(x \rightarrow_{\mathbf{A}} y)_i = \begin{cases} x_i \rightarrow_{\mathbf{A}_i} y_i & \text{if } x_j \leq y_j \text{ for all } j < i; \\ 0 & \text{otherwise.} \end{cases}$$

When P is finite and totally ordered, $\bigotimes_{p \in P} \mathbf{A}_p \cong \bigoplus_{p \in P} \mathbf{A}_p$.

If $P = \{a < b\}$, $\mathbf{A}_a = \mathbf{Ł}_3$ and $\mathbf{A}_b = \mathbf{Ł}_2$, then $\mathbf{Ł}_3 \otimes \mathbf{Ł}_2 \cong \mathbf{Ł}_3 \oplus \mathbf{Ł}_2$.



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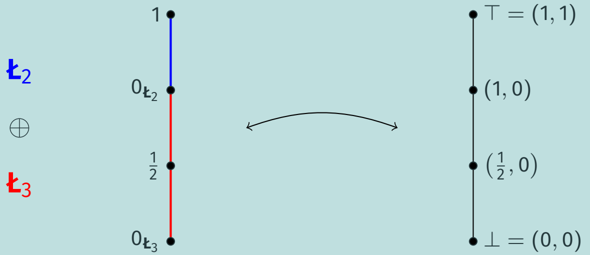

When P is an antichain, $\bigotimes_{p \in P} \mathbf{A}_p = \prod_{p \in P} \mathbf{A}_p$.

If $P = \{a \parallel b\}$ and $\mathbf{A}_a = \mathbf{A}_b = \text{Ł}_2$, then $\text{Ł}_2 \otimes \text{Ł}_2 = \text{Ł}_2 \times \text{Ł}_2$.

Although the class of BL-algebras is not closed under poset product,

if P is a *forest* and $\mathbf{A}_p$ is a BL-chain for all $p \in P$, then $\bigotimes_{p \in P} \mathbf{A}_p$ is a BL-algebra.

BL-algebras and poset product

Given a BL-algebra $\mathbf{A}$, are there a forest P and a family $\{\mathbf{A}_p : p \in P\}$ of MV-chains and product chains such that $\mathbf{A} \cong \bigotimes_{p\in P} \mathbf{A}_p$?

In general, the answer is *no*.

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In general, the answer is *no*.

Consider $\mathbf{A} = \mathbf{Ł_3} \oplus \mathbf{(0,1]_\Pi}$, which is neither an MV-chain nor a product chain. Any attempt to write $\mathbf{A}$ as a poset product would require two *bounded* summands.

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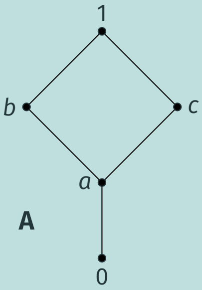
For finite BL-algebras the authors proved that

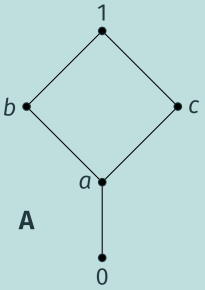
every finite BL-algebra is isomorphic to the poset product of a family of MV-chains.

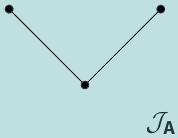
Let $\mathbf{A} = \mathbf{Ł}_2 \oplus (\mathbf{Ł}_2 \times \mathbf{Ł}_2)$, whose diagram is



Let $\mathbf{A} = \text{Ł}_2 \oplus (\text{Ł}_2 \times \text{Ł}_2)$.

Denote $\mathcal{J}_{\mathbf{A}}$ the set of $\vee$-irreducible elements of $\mathbf{A}$.





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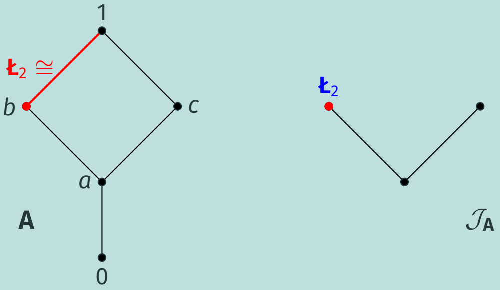

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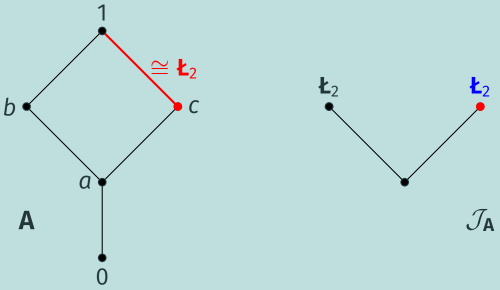

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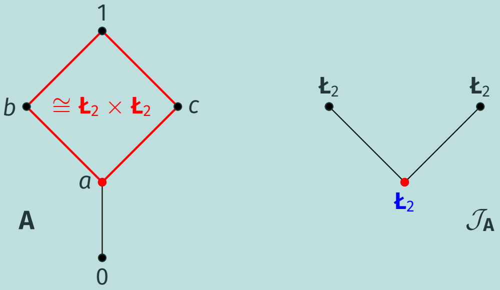

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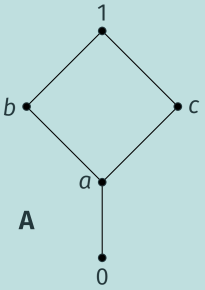

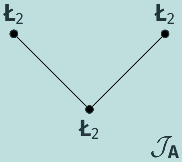

$$\mathbf{A} \cong \bigotimes_{\mathcal{J}_{\mathbf{A}}} \text{Ł}_2$$

Idempotent free BL-algebras

An algebra $\mathbf{A}$ is said to be *poset product indecomposable* if $\mathbf{A}$ is non-trivial and if $\mathbf{A}$ is a poset product of two algebras $\mathbf{A}_1$ and $\mathbf{A}_2$, then either $\mathbf{A}_1$ or $\mathbf{A}_2$ is trivial.

We will say that a BL-chain $\mathbf{A}$ is *idempotent free* if $\mathbf{Id}(\mathbf{A}) \cong \mathbf{Ł}_2$.

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Proposition

Let $\mathbf{A}$ be a non-trivial BL-chain. Then $\mathbf{A}$ is idempotent free if and only if $\mathbf{A}$ is poset product indecomposable.

Representability in the sense of poset product

Given a BL-chain $\mathbf{A}$, if there are a totally ordered set P and a collection of idempotent free BL-chains $\{\mathbf{A}_p : p \in P\}$ such that $\mathbf{A} \cong \bigotimes_{p \in P} \mathbf{A}_p$, we will say that $\mathbf{A}$ is *representable*.

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- Since they are idempotent free BL-chains, MV-chains and every BL-chain of type $\mathbf{Ł}_n \oplus \mathbf{(0,1]_\Pi}$ $(n \geq 2)$ are representable.
- Finite BL-algebras are representable.

Gödel algebras representation

Given a Gödel algebra $\mathbf{A}$ we set $\mathcal{J}_{\mathbf{A}}$ to be the poset of completely $\vee$-irreducible elements of $\mathbf{A}$.

Theorem

If $\mathcal{J}_{\mathbf{A}}$ is a well partial order, then $\mathbf{A} \cong \bigotimes_{\mathcal{J}_{\mathbf{A}}} \text{Ł}_2$.

Gödel algebras representation

Let $\mathcal{J}_{\mathbf{A}}$ be the poset of completely $\vee$-irreducible elements of a Gödel algebra $\mathbf{A}$.

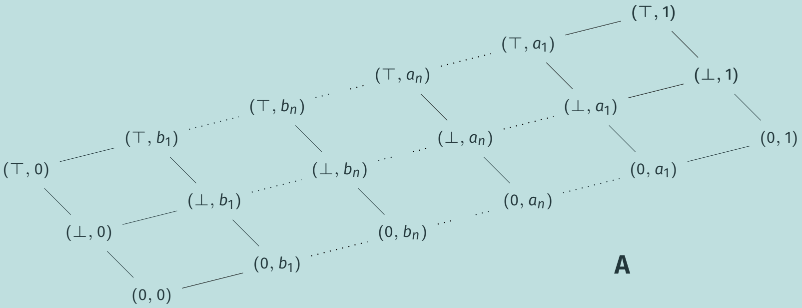
- If every prime filter of $\mathbf{A}$ is a principal filter, then each connected component of the poset $\mathcal{J}_{\mathbf{A}}$ has a minimum element. In addition,
- if $\mathcal{J}_{\mathbf{A}}$ has no infinite antichains, then

each connected component of the poset $\mathcal{J}_{\mathbf{A}}$ is partially well-ordered.

The BL-chain $\mathbf{A} = (\bigoplus_{\mathbb{N}} \mathbf{Ł_2}) \oplus \mathbf{(0,1]_\Pi}$ is not representable.

- Since $\mathcal{J}_\mathbf{A} \cong \mathbb{N}$ as a poset, $\mathcal{J}_\mathbf{A}$ is a well-ordered set.
- Essentially, $\mathbf{A}$ is not representable because $\mathbf{(0,1]_\Pi}$ is unbounded.

Let $\mathbf{A} = (\mathbf{Ł}_2 \oplus \mathbf{Ł}_2) \times \mathbf{C}$, where $\mathbf{C}$ is the Chang's MV-chain.

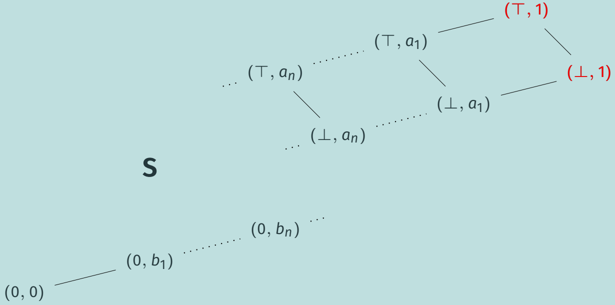


From Gödel algebras to BL-algebras - Issues

S is subalgebra of **A** that is not representable.

$\mathbf{Id}(\mathbf{S}) = \{(0,0) < (\bot, 1) < (\top, 1)\}$.

$\mathcal{J}_{\mathbf{S}} = \{(\bot, 1), (\top, 1)\}$.



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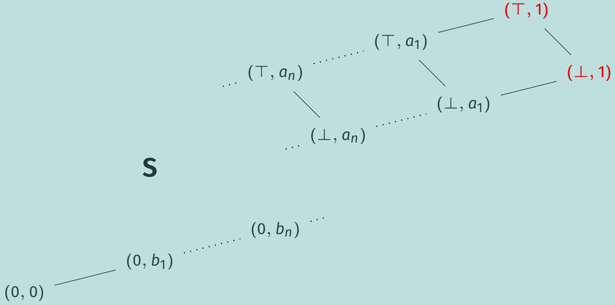

$\mathcal{J}_{\mathbf{S}}$ is not a suitable set for indexing BL-chains and recover **S**.

Sufficient conditions for representability

Last examples suggest the introduction of additional conditions.

Theorem

Let $\mathbf{A}$ be a BL-algebra such that each connected component of the poset of idempotent completely $\vee$-irreducible elements $\mathcal{J}_{\mathbf{A}}$ is a partially well-ordered set. If

(a) every $i \in \mathcal{J}_{\mathbf{A}}$ is a $\vee$-irreducible element in $\mathbf{A}$ and

(b) $\bigcap_{i \in \mathcal{J}_{\mathbf{A}}} [i) = \{1\}$, then

$\mathbf{A} \cong \bigotimes_{i \in \mathcal{J}_{\mathbf{A}}} \mathbf{A}_i$, where each $\mathbf{A}_i$ is an idempotent free BL-chain.

Sufficient conditions for representability - The factors

If $i \in \mathcal{J}_{\mathbf{A}}$,

- $[i) = \{x \in A : x \geq i\}$ is a prime filter of **A**.

Thus the quotient algebra $\mathbf{A}/[\boldsymbol{i})$ is isomorphic to the BL-chain

$$[\mathbf{0}, \boldsymbol{i}]$$

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If $i \in \mathcal{J}_{\mathbf{A}}$,

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- i has a lower cover $j \in \mathrm{Id}(\mathbf{A})$.

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We set $\mathbf{A}_{\boldsymbol{i}} = [\boldsymbol{j}, \boldsymbol{i}]$, which is an idempotent free BL-chain.

Sufficient conditions for representability - The hypothesis

Given a BL-algebra $\mathbf{A}$, the theorem requires its Gödel subalgebra $\mathbf{Id}(\mathbf{A})$ to be representable.

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The condition every $i \in \mathcal{J}_{\mathbf{A}}$ is a $\vee$-irreducible element in **A** ensures that the prime spectrum is preserved.

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The condition every $i \in \mathcal{J}_{\mathbf{A}}$ is a $\vee$-irreducible element in $\mathbf{A}$ ensures that the prime spectrum is preserved.

Hypothesis $\bigcap_{i \in \mathcal{J}_{\mathbf{A}}} [i) = \{1\}$ guarantees injectivity.

Sufficient conditions for representability - The hypothesis

If $\mathbf{A} = (\bigoplus_{\mathbb{N}} \mathbf{Ł_2}) \oplus \mathbf{[0,1]_\Pi}$,

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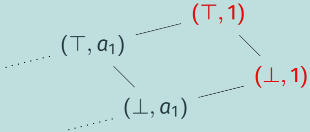
If $\mathbf{S}$ is the subalgebra of $(\text{Ł}_2 \oplus \text{Ł}_2) \times \mathbf{C}$ that we have defined, then condition (b) clearly holds. On the other hand,



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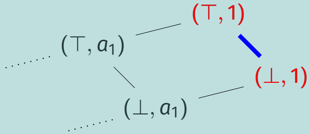

$(\top, 1) \in \mathcal{J}_{\mathbf{S}}$

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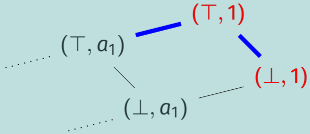

$(\top, 1) \in \mathcal{J}_{\mathbf{S}}$, but it is not $\vee$-irreducible in $\mathbf{S}$.

References and further readings

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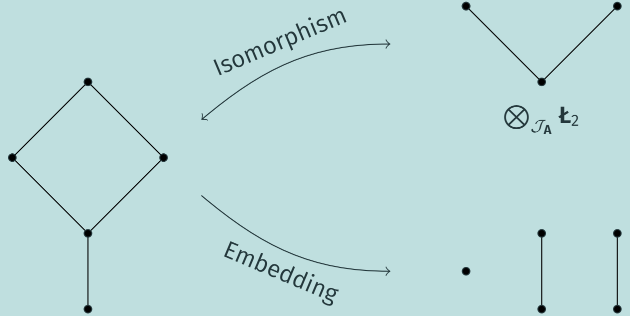
Thank you

Appendix - Embedding and representation theorems



$\bigotimes_{\mathcal{J}_{\mathbf{A}}} \mathbf{Ł}_2$

$\mathbf{A} = \mathbf{Ł}_2 \oplus (\mathbf{Ł}_2 \times \mathbf{Ł}_2)$

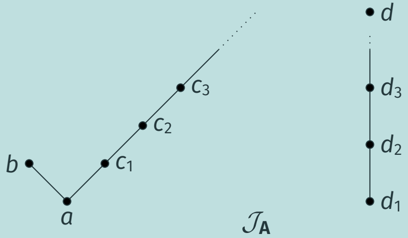
$\mathbf{Ł}_2 \times (\mathbf{Ł}_2 \oplus \mathbf{Ł}_2) \times (\mathbf{Ł}_2 \oplus \mathbf{Ł}_2)$

Appendix - Example

The Gödel algebra (with infinite spectrum)

$$\mathbf{A} = \left(Ł_2 \oplus \left(Ł_2 \times \bigoplus_{\mathbb{N}} Ł_2 \right) \right) \times \bigoplus_{\mathbb{N} \cup \{d\}} Ł_2$$

is representable. The forest $\mathcal{J}_{\mathbf{A}}$ looks like



We showed that $\mathbf{A} \cong \bigotimes_{\mathcal{J}_{\mathbf{A}}} Ł_2$.