On graphs with a single large Laplacian eigenvalue [1]

L. Emilio Allem [a,2] Antonio Cafure [b,c,d,2] Ezequiel Dratman [b,e,2]
Luciano N. Grippo [e,2] Martín D. Safe [f,2] Vilmar Trevisan [a,2]

[a] *Instituto de Matemática, Universidade Federal do Rio Grande do Sul, Brazil*

[b] *CONICET, Argentina*

[c] *Instituto del Desarrollo Humano, Universidad Nacional de General Sarmiento, Argentina*

[d] *Depto. de Matemática, CBC, Universidad de Buenos Aires, Argentina*

[e] *Instituto de Ciencias, Universidad Nacional de General Sarmiento, Argentina*

[f] *Departamento de Matemática, Universidad Nacional del Sur, Argentina*

Abstract

We address the problem of characterizing those graphs G having only one Laplacian eigenvalue greater than or equal to the average degree of G. Our conjecture is that these graphs are stars plus a (possible empty) set of isolated vertices.

Keywords: anticomponents, Laplacian eigenvalues, stars

[1] A. Cafure and E. Dratman were partially supported by CONICET PIP 112-2013-010-0598. A. Cafure, E. Dratman, L. N. Grippo, and M. D. Safe were partially supported by CONICET UNGS-144-20140100027-CO. V. Trevisan acknowledges the support of CNPq - Grant 303334/2016-9.

[2] Email addresses: emilio.allem@ufrgs.br (L. E. Allem), acafure@ungs.edu.ar (A. Cafure), edratman@ungs.edu.ar (E. Dratman), lgrippo@ungs.edu.ar (L. N. Grippo), msafe@uns.edu.ar (M. D. Safe), trevisan@mat.ufrgs.br (V. Trevisan)

1 Introduction

Let G be a graph on n vertices and m edges and let $d_1 \geq \cdots \geq d_n$ be its degree sequence. Let $A(G)$ be its adjacency matrix and $D(G)$ its diagonal matrix of vertex degrees. The *Laplacian matrix* of G is the positive semidefinite matrix $L(G) = D(G) - A(G)$. The spectrum of $L(G)$ is called the *Laplacian spectrum* of G and is denoted by $Lspec(G) = \{\mu_1, \mu_2, \ldots, \mu_n\}$, where $n \geq \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$.

Understanding the distribution of Laplacian eigenvalues of graphs is a problem that is both relevant and difficult. It is relevant due to the many applications related to Laplacian matrices (see, for example [8,9]). It seems to be difficult because little is known about how the n Laplacian eigenvalues are distributed in the interval $[0, n]$.

Our main motivation is understanding the structure of graphs that have few large Laplacian eigenvalues. In particular, we would like to characterize graphs that have a single large Laplacian eigenvalue. What do we mean by a large Laplacian eigenvalue? A reasonable measure is to compare this eigenvalue with the average of all eigenvalues. Since the average of Laplacian eigenvalues equals the average degree $\overline{d}(G) = \frac{2m}{n}$ of G, we say that *a Laplacian eigenvalue is large if it is greater than or equal the average degree.*

Inspired by this idea, the paper [2] introduces the spectral parameter $\sigma(G)$ which counts the number of Laplacian eigenvalues greater than or equal to $\overline{d}(G)$. Equivalently, $\sigma(G)$ is the largest index i for which $\mu_i \geq \frac{2m}{n}$.

There is evidence that $\sigma(G)$ plays an important role in defining structural properties of a graph G. For example, it is related to the clique number ω of G (the number of vertices of the largest induced complete subgraph of G) and it also gives insight about the Laplacian energy of a graph [10,2]. Moreover several structural properties of a graph are related to σ (see, for example [1,2]).

In this paper we are concerned with furthering the study of $\sigma(G)$. In particular, we deal with a problem posed in [2] which asks for characterizing all graphs having $\sigma(G) = 1$, *i.e.* having only one large Laplacian eigenvalue. We conjecture that these graphs are some stars plus a (possible empty) set of isolated vertices ($K_{1,r}$ denotes the star on $r+1$ vertices and $+$ the disjoint union):

Conjecture 1.1 *Let G be a graph. Then $\sigma(G) = 1$ if and only if G is isomorphic to K_1, $K_2 + sK_1$ for some $s \geq 0$, or $K_{1,r} + sK_1$ for some $r \geq 2$ and $0 \leq s < r - 1$.*

In this work, we show that this conjecture is true if it holds for graphs which are simultaneously connected and co-connected (Conjecture 4.3) and prove that Conjecture 1.1 is true for forests and extended P_4-laden graphs [4] (a common superclass of split graphs and cographs). The main tool for proving our results is an interesting link we have found between σ and the number ℓ of nonempty anticomponents of G (see Section 2). The interesting feature of this result is that it relates a spectral parameter with a classical structural parameter. Studying structural properties of the anticomponents of G may shed light on the distribution of Laplacian eigenvalues and, reciprocally, the distribution of Laplacian eigenvalues should give insight about the structure of the graph.

This extended abstract is organized as follows. In Section 2, we give some definitions. In Section 3, we present some new results which establish the connection between $\sigma(G)$ and the number of nonempty anticomponents of G. In Section 4, we present some evidence on the validity of Conjecture 1.1.

2 Definitions

In this abstract, all graphs are finite, undirected, and without multiple edges or loops. All definitions and concepts not introduced here can be found in [11]. We say that a graph is *empty* if it has no edges. A *trivial* graph is a graph with precisely one vertex; every trivial graph is isomorphic to the graph which we will denote by K_1. A graph is *nontrivial* if it has more than one vertex.

Assume that $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two graphs on disjoint set of vertices. The *disjoint union* of G_1 and G_2 is the graph $G_1 + G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. The *join* $G_1 \vee G_2$ is the graph obtained from $G_1 + G_2$ by adding new edges from each vertex of G_1 to every vertex of G_2. The disjoint union $G + \cdots + G$ of k copies of a graph G will be denoted by kG. A vertex v of a graph G is a *twin* of another vertex w of G if they both have the same neighbors in $V(G) \setminus \{v, w\}$.

The *anticomponents* of a graph G are each of the subgraphs of G induced by the vertex set of a connected component of $\overline{G}$, where $\overline{G}$ denotes the complement of G. Notice that if $G_1, G_2, \ldots, G_k$ are the anticomponents of G, then $G = G_1 \vee \cdots \vee G_k$. A graph is *co-connected* if $\overline{G}$ is connected.

The chordless path (resp. cycle) on k vertices is denoted by P_k (resp. C_k). A *forest* is a graph with no cycles and a *tree* is a connected forest. A *cograph* is a graph with no induced P_4. A *spider* [5] is a graph whose vertex set can be partitioned into three sets S, C, and R, where $S = \{s_1, \ldots, s_k\}$ is a stable set and $C = \{c_1, \ldots, c_k\}$ is a clique for some $k \geq 2$; either each s_i is adjacent to c_j if and only if $i = j$ (a *thin spider*), or s_i is adjacent to c_j if and only if $i \neq j$ (a *thick spider*); and R is allowed to be empty and its vertices are adjacent to all the vertices in C and nonadjacent to all the vertices in S. The sets C and S are called *body* and *legs* of the spider, respectively. A graph is *split* [3] if its vertex set can be partitioned into a clique and a stable set.

3 Relating σ and the number of anticomponents

In this section we establish a link between the two parameters.

Lemma 3.1 *If G has k anticomponents, then $k \leq \sigma(G) + 1$.*

Theorem 3.2 *Let G be a graph having k anticomponents where $k = \sigma(G)+1$. If ℓ denotes the number of nonempty anticomponents, then $\ell \leq \sigma$. Moreover, if $k = \ell + 1$, then the remaining anticomponent of G is empty but nontrivial.*

Let G be a graph with $\sigma(G) = 1$ and such that $\overline{G}$ is disconnected. By virtue of Lemma 3.1, G has at most two anticomponents. As $\overline{G}$ is disconnected, G has exactly two anticomponents G_1 and G_2. Hence, $G = G_1 \vee G_2$. Moreover, it can be proved that G_1 and G_2 are empty and thus G is complete bipartite.

Corollary 3.3 *If G is a graph with $\sigma(G) = 1$ and $\overline{G}$ is disconnected, then G is a complete bipartite graph.*

4 Graphs with $\sigma = 1$

We first verify Conjecture 1.1 for graphs having disconnected complement; namely, we prove that the only graphs having $\sigma(G) = 1$ and disconnected complement are the stars (including the trivial star K_1). Then, we prove that Conjecture 1.1 can be reduced to proving that the only connected and co-connected graph with $\sigma = 1$ is K_1. We then verify Conjecture 1.1 for extended P_4-laden graphs, a common superclass of the classes of cographs and split graphs.

4.1 Reduction to co-connected graphs

We first obtain a result which proves the validity of Conjecture 1.1 for graphs having disconnected complement.

Theorem 4.1 *Let G be a graph with n vertices such that $\overline{G}$ is disconnected. Then $\sigma(G) = 1$ if and only if G is isomorphic to $K_{1,n-1}$.*

Because of Theorem 4.1, Conjecture 1.1 holds for graphs G whose complement is disconnected. Hence, the validity of Conjecture 1.1 is equivalent to the validity of the following weaker conjecture.

Conjecture 4.2 *Let G be a graph with connected complement. Then, $\sigma(G) = 1$ if and only if G is isomorphic to K_1, K_2+sK_1 for some $s > 0$, or $K_{1,r}+sK_1$ for some $r \geq 2$ and $0 < s < r-1$.*

4.2 Reduction to connected and co-connected graphs

We next show that the validity of Conjectures 1.1 and 4.2 can be reduced to the validity of the following even weaker conjecture.

Conjecture 4.3 *Let G be a connected graph with connected complement. Then, $\sigma(G) = 1$ if and only if G is isomorphic to K_1.*

Moreover, below we prove that the reduction from Conjecture 1.1 to Conjecture 4.3 holds even when restricted to any graph class closed by taking components. A graph class $\mathcal{G}$ is *closed by taking components* if every connected component of every graph in $\mathcal{G}$ belongs to $\mathcal{G}$. In particular, the class of all graphs is closed by taking components.

Theorem 4.4 *Let $\mathcal{G}$ be a graph class closed by taking components. If Conjecture 4.3 holds for $\mathcal{G}$, then Conjecture 1.1 also holds for $\mathcal{G}$.*

It is well known that the only connected and co-connected cograph is K_1. Hence, Conjecture 4.3 holds trivially for cographs and, by Theorem 4.4, Conjecture 1.1 holds for cographs.

4.3 Characterizing forests and extended P_4-laden graphs with $\sigma = 1$

In this section, we verify Conjecture 1.1 for forests and extended P_4-laden graphs (a common superclass of cographs and split graphs).

Theorem 4.5 *Conjecture 1.1 holds for forests.*

A graph is *pseudo-split* [7] if it is $\{2P_2, C_4\}$-free. The class of pseudo-split graphs is a superclass of split graphs [3]. A graph is *extended P_4-laden* [4] if and only if every induced subgraph on at most six vertices that contains more than two induced P_4's is a pseudo-split graph. By definition, the class of extended P_4-laden graphs is a superclass of the class of pseudo-split graphs and hence also of split graphs. Moreover, the class of extended P_4-laden graphs is a superclass of different superclasses of cographs defined by restricting the number of induced P_4's, including P_4-lite graphs [6] and P_4-tidy graphs [5].

We first obtain the following results.

Lemma 4.6 *If G is a spider or a graph that arises from a spider by adding a twin of a vertex of the body or the legs, then $\sigma(G) \geq 2$.*

Theorem 4.7 *Conjecture 1.1 holds for split graphs.*

In [4], it was proved that a connected and co-connected extended P_4-laden graph G satisfies one of the following conditions: G is isomorphic to K_1, P_5, $\overline{P_5}$, or C_5, or G is a spider or arises from a spider by adding a twin of a vertex of the body or the legs; or G is a split graph. By combining this result, Lemma 4.6, and Theorem 4.7, we obtain the following result.

Theorem 4.8 *Conjecture 1.1 holds for extended P_4-laden graphs.*

References

[1] Das, K. C., S. A. Mojallal and I. Gutman, *On Laplacian energy in terms of graph invariants*, Appl. Math. Comput. **268** (2015), pp. 83–92.

[2] Das, K. C., S. A. Mojallal and V. Trevisan, *Distribution of Laplacian eigenvalues of graphs*, Linear Algebra Appl. **508** (2016), pp. 48–61.

[3] Földes, S. and P. L. Hammer, *Split graphs*, in: *Proceedings of the Eighth Southeastern Conference on Combinatorics, Graph Theory and Computing (Louisiana State Univ., Baton Rouge, La., 1977)*, Utilitas Math., Winnipeg, Man., 1977 pp. 311–315.

[4] Giakoumakis, V., *P_4-laden graphs: a new class of brittle graphs*, Inform. Process. Lett. **60** (1996), pp. 29–36.

[5] Giakoumakis, V., F. Roussel and H. Thuillier, *On P_4-tidy graphs*, Discrete Math. Theor. Comput. Sci. **1** (1997), pp. 17–41.

[6] Jamison, B. and S. Olariu, *A new class of brittle graphs*, Stud. Appl. Math. **81** (1989), pp. 89–92.

[7] Maffray, F. and M. Preissmann, *Linear recognition of pseudo-split graphs*, Discrete Appl. Math. **52** (1994), pp. 307–312.

[8] Mohar, B., *The Laplacian spectrum of graphs*, in: *Graph theory, combinatorics, and applications. Vol. 2 (Kalamazoo, MI, 1988)*, Wiley-Intersci. Publ., Wiley, New York, 1991 pp. 871–898.

[9] Mohar, B., *Laplace eigenvalues of graphs—a survey*, Discrete Math. **109** (1992), pp. 171–183.

[10] Pirzada, S. and H. A. Ganie, *On the Laplacian eigenvalues of a graph and Laplacian energy*, Linear Algebra Appl. **486** (2015), pp. 454–468.

[11] West, D. B., "Introduction to Graph Theory," Prentice Hall, 2000, 2nd edition.