On the hereditary (p,q)-Helly property of hypergraphs, cliques, and bicliques [1,2]

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Abstract

We prove several characterizations of hereditary (p,q)-Helly hypergraphs, including one by minimal forbidden partial subhypergraphs, and show that the recognition of hereditary (p,q)-Helly hypergraphs can be solved in polynomial time for fixed p but is co-NP-complete if p is part of the input. We also give several characterizations of hereditary (p,q)-clique-Helly graphs, including one by forbidden induced subgraphs, and prove that the recognition of hereditary (p,q)-clique-Helly graphs can be solved in polynomial time for fixed p and q but is NP-hard if p or q is part of the input. We prove similar results for hereditary (p,q)-biclique-Helly graphs.

Keywords: forbidden induced subgraphs, forbidden partial subhypergraphs, maximal bicliques, maximal cliques, (p,q)-Helly property, recognition algorithms

1 Introduction

In this work, we study the (p,q)-Helly property, which originated in the works [7,8]. The *core* of a family $\mathcal{F}$ of sets, denoted core$(\mathcal{F})$, is the intersection of all the sets in $\mathcal{F}$. A family of sets is (p,q)*-intersecting*, if every

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nonempty subfamily consisting of p or fewer sets has core of size at least q. A family of sets has the (p, q)-*Helly property* if every (p, q)-intersecting nonempty subfamily has core of size at least q. The $(p, 1)$-Helly property is known as the p-*Helly property*, which has its origin in the celebrated *Helly's theorem* that states that any finite family of convex sets in $\mathbb{R}^{p-1}$ has the p-Helly property. The 2-Helly property is the usual *Helly property* [1].

A hypergraph $\mathcal{H}$ has the (p, q)-*Helly property* if its edge family has the (p, q)-Helly property and $\mathcal{H}$ is *hereditary* (p, q)-*Helly* if each of its subhypergraphs has the (p, q)-Helly property. Hereditary $(p, 1)$-Helly hypergraphs are called *hereditary* p-*Helly*. In [2,4], characterizations of the class of hereditary p-Helly hypergraphs were given and its recognition was shown to be polynomial-time solvable if p is fixed but NP-hard if p is part of the input.

A graph is (p, q)-*clique-Helly* if the family of its maximal cliques has the (p, q)-Helly property. (In this work, the word *maximal* always means inclusion-wise maximal.) A graph is *hereditary* (p, q)-*clique-Helly* if each of its induced subgraphs is (p, q)-clique-Helly. Hereditary $(p, 1)$-clique-Helly graph are called *hereditary* p-*clique-Helly*. In [4], different characterizations of the class of hereditary p-clique-Helly graphs were given and its recognition was shown to be polynomial-time solvable for fixed p but NP-hard if p is part of the input.

A *biclique* of a graph is a set of vertices inducing a complete bipartite graph, where we regard edgeless graphs as complete bipartite graphs. We say a graph is (p, q)-*biclique-Helly* if the family of its maximal bicliques has the (p, q)-Helly property and *hereditary* (p, q)-*biclique-Helly* if all its induced subgraphs are (p, q)-biclique-Helly. The 'hereditary biclique-Helly graphs' defined in [6] are different from the hereditary $(2, 1)$-biclique-Helly graphs defined here because in [6] edgeless graphs are not regarded as complete bipartite graphs.

In this work, graphs are finite, undirected, and without loops or multiple edges. For undefined hypergraph or graph notions, see [1] or [9], respectively.

This abstact is organized as follows. In Section 2, we give several characterizations of the classes of hereditary (p, q)-Helly hypergraphs, hereditary (p, q)-clique-Helly graphs, and hereditary (p, q)-biclique-Helly graphs. In Section 3, we give polynomial-time bounds and hardness results for the recognition of these classes depending on which of p and q are fixed. Our results generalize structural and algorithmic results for the case $q = 1$ given in [2,4].

2 Characterizations

It was proved in [4] that hereditary p-Helly hypergraphs coincide with *strong* p-*Helly hypergraphs* [5]. We give the following generalization of the latter, which we will show to coincide with hereditary (p,q)-Helly hypergraphs. A hypergraph $\mathcal{H}$ is *strong* (p,q)*-Helly* if, for each (p,q)-intersecting nonempty partial hypergraph $\mathcal{H}'$ of $\mathcal{H}$, some nonempty subfamily of p or fewer edges of $\mathcal{H}'$ has the same core as $\mathcal{H}'$.

Let q be a positive integer. We denote by $\varphi_q(S)$ the set of all subsets of size q of a set S. For every hypergraph $\mathcal{H}$, we define $\Phi_q(\mathcal{H})$ as the hypergraph whose vertices are the subsets of size q of $V(\mathcal{H})$ that are contained in some edge of $\mathcal{H}$ and whose edge family consists of those sets $\varphi_q(E)$ that are nonempty as E varies over the edge family of $\mathcal{H}$. This hypergraph operator Φ_q mirrors the graph operator Φ_q defined in [3] to characterize (p,q)-clique-Helly graphs.

For each positive integer p and q and each $s \in \{0,1,\dots,q-1\}$, we define $\mathcal{J}_{p+1,q,s}$ as the unique hypergraph $\mathcal{H}$ (up to isomorphism) having $(p+1)(q-s)+s$ vertices and such that $E(\mathcal{H}) = \{V(\mathcal{H}) - T_i : 1 \leq i \leq p+1\}$ where $T_1,\dots,T_{p+1}$ are $p+1$ pairwise disjoint subsets of size $q-s$ of $V(\mathcal{H})$. Since $\mathcal{J}_{p+1,q,s}$ is (p,q)-intersecting but has core of size s, $\mathcal{J}_{p+1,q,s}$ is not (p,q)-Helly.

An *incidence matrix* $M(\mathcal{H})$ of a hypergraph $\mathcal{H}$ is an edge vs. vertex incidence matrix. A matrix P *contains* a matrix Q if Q is a submatrix of P.

A $(p+1,q)$*-basis* of a hypergraph $\mathcal{H}$ is a family $\mathcal{S}$ of $p+1$ pairwise different subsets of size q of $V(\mathcal{H})$. A *support set* of $\mathcal{S}$ is the union of all but exactly one member of $\mathcal{S}$. We denote by $\mathcal{H}^{\cup}_{\mathcal{S}}$ the partial hypergraph of $\mathcal{H}$ formed by those edges that contain some support set of $\mathcal{S}$ each. We say $\mathcal{S}$ is *nontrivial* if each of its support sets is contained in some edge of $\mathcal{H}$. We say $\mathcal{S}$ is *starlike* if every vertex which belongs to at least two members of $\mathcal{S}$ also belongs to $\operatorname{core}(\mathcal{S})$. We denote by $\operatorname{ext}(\mathcal{S})$ the set of vertices belonging to some set of $\mathcal{S}$ but not to $\operatorname{core}(\mathcal{S})$.

Our first result extends to hereditary (p,q)-Helly hypergraphs the characterizations given in [2,4] for hereditary p-Helly hypergraphs.

Theorem 2.1 *If p and q are positive integers, then the following statements are equivalent for each hypergraph $\mathcal{H}$:*

(i) *$\mathcal{H}$ is hereditary (p,q)-Helly;*

(ii) *$\mathcal{H}$ is (p,q')-Helly for every $q' \geq q$;*

(iii) *$\mathcal{H}$ is strong (p,q)-Helly;*

(iv) *every partial $(p+1)$-hypergraph of $\mathcal{H}$ is strong (p,q)-Helly;*

(v) *$\Phi_q(\mathcal{H})$ is hereditary p-Helly;*

(vi) *$M(\mathcal{H})$ contains no incidence matrix of $\mathcal{J}_{p+1,q,s}$ for any $s \in \{0,\dots,q-1\}$;*

(vii) *$\mathcal{J}_{p+1,q,s}$ is not a partial subhypergraph of $\mathcal{H}$ for any $s \in \{0,\dots,q-1\}$;*

(viii) *for each nontrivial starlike* $(p+1,q)$*-basis* $\mathcal{S}$ *of* $\mathcal{H}$*,* $\text{core}(\mathcal{H}_{\mathcal{S}}^{\cup})\cap\text{ext}(\mathcal{S})\neq\emptyset$.

(ix) *for each starlike* $(p+1,q)$*-basis* $\mathcal{S}$ *of* $\mathcal{H}$*, either* $\mathcal{H}_{\mathcal{S}}^{\cup}$ *is empty or* $\text{core}(\mathcal{H}_{\mathcal{S}}^{\cup})\cap\text{ext}(\mathcal{S})\neq\emptyset$.

Characterization (vii) above is by minimal forbidden partial subhypergraphs.

We now turn to the problem of characterizing hereditary (p,q)-clique-Helly graphs. Let G be a graph. The graph operator Φ_q was introduced in [3] as follows: $\Phi_q(G)$ is the graph whose vertices are the cliques of size q of G and two cliques of size q of G are adjacent in $\Phi_q(G)$ if and only if they are contained in a common clique of G. The *clique hypergraph* $\mathcal{C}(G)$ of G is the hypergraph whose vertices are those of G and whose edge family is the set of maximal cliques of G. A *clique-matrix* $C(G)$ of G is an incidence matrix of $\mathcal{C}(G)$. We say that G is *strong* (p,q)*-clique-Helly* if $\mathcal{C}(G)$ is strong (p,q)-Helly.

We generalize $(p+1)$-oculars, which were used in [4] to characterize hereditary p-clique-Helly graphs. If p and q are positive integers and $s\in\{0,\dots,q-1\}$, a $(p+1,q,s)$*-ocular* is a graph whose vertex set is the union of two disjoint sets U and W where U has size $(p+1)(q-s)+s$ and $T_1,\dots,T_{p+1}$ are $p+1$ pairwise disjoint subsets of size $q-s$ of U such that one of the following holds:

(α_1) $p=1$, $W=\emptyset$, and $U-T_i$ is a clique but $(U-T_i)\cup\{v_i\}$ is not a clique for each $v_i\in T_i$ and each $i\in\{1,2\}$;

(α_2) $p\geq 2$, $W=\{w_1,\dots,w_{p+1}\}$, U is a clique, and w_i is adjacent all vertices in $U-T_i$ and nonadjacent to all vertices in T_i for each $i\in\{1,\dots,p+1\}$.

Observe that if $p\geq 2$ then the vertices of W may induce an arbitrary graph.

The theorem below generalizes to hereditary (p,q)-clique-Helly graphs the characterizations for hereditary p-clique-Helly graphs given in [2,4].

Theorem 2.2 *If* p *and* q *are positive integers, then the following statements are equivalent for each graph* G*:*

(i) G *is hereditary* (p,q)*-clique-Helly;*

(ii) G *is* (p,q')*-clique-Helly, for every* $q'\geq q$*;*

(iii) G *is strong* (p,q)*-clique-Helly;*

(iv) *Every family of* $p+1$ *maximal cliques of* G *is strong* (p,q)*-Helly;*

(v) $\Phi_q(G)$ *is hereditary* p*-clique-Helly;*

(vi) $C(G)$ *contains no incidence matrix of* $\mathcal{J}_{p+1,q,s}$ *for any* $s\in\{0,\dots,q-1\}$*;*

(vii) G *contains no induced* $(p+1,q,s)$*-ocular for any* $s\in\{0,\dots,q-1\}$.

Characterization (vii) above is by forbidden induced subgraphs.

The remaining of this section is devoted to the characterization of hereditary (p,q)-biclique-Helly hypergraphs. If G is a graph, the *biclique hypergraph* $\mathcal{B}(G)$ is the hypergraph whose vertices are those of G and whose edges are the maximal bicliques of G. A *biclique-matrix* $B(G)$ of G is an incidence matrix of $\mathcal{B}(G)$. We say G is *strong* (p,q)*-biclique-Helly* if $\mathcal{B}(G)$ is strong (p,q)-Helly.

We define the analogue of oculars, which we call *bioculars*. For each positive integers p and q and each $s \in \{0,\dots,q-1\}$ such that $(p,q) \neq (1,1)$, a $(p+1,q,s)$*-biocular* is a graph whose vertex set is the union of two disjoint sets U and W where U has size $(p+1)(q-s)+s$ and $T_1,\dots,T_{p+1}$ are pairwise disjoint subsets of size $q-s$ of U such that one of the following holds:

(β_1) $p \in \{1,2\}$, $W=\emptyset$, $U-T_i$ is a biclique but $(U-T_i)\cup\{v_i\}$ is not a biclique for each $v_i \in T_i$ for each $i \in \{1,\dots,p+1\}$, and either $p=1$ or $s=0$;

(β_2) $p \geq 2$, $(p,q)\neq(2,1)$, $W=\{w_1,\dots,w_{p+1}\}$, U is a biclique, and $(U-T_i)\cup\{w_i\}$ is a biclique but $(U-T_i)\cup\{w_i,v_i\}$ is not a biclique for each $v_i \in T_i$ for each $i \in \{1,\dots,p+1\}$.

If $p \geq 2$ then the vertices of W may induce in G an arbitrary graph. For $(p,q)=(1,1)$, we define the $(2,1,0)$*-bioculars* as the graphs $\overline{P_3}$ and K_3, where P_3 and K_3 are the chordless path and the complete graph on 3 vertices each.

We give several characterizations of hereditary (p,q)-biclique-Helly graphs.

Theorem 2.3 *If p and q are positive integers, then the following statements are equivalent for each graph G:*

(i) *G is hereditary (p,q)-biclique-Helly;*

(ii) *G is (p,q')-biclique-Helly for each $q' \geq q$;*

(iii) *G is strong (p,q)-biclique-Helly;*

(iv) *Each family of $p+1$ maximal bicliques of G is strong (p,q)-Helly;*

(v) *$\Phi_q(\mathcal{B}(G))$ is hereditary p-Helly;*

(vi) *$B(G)$ contains no incidence matrix of $\mathcal{J}_{p+1,q,s}$ for any $s \in \{0,\dots,q-1\}$.*

(vii) *G contains no induced $(p+1,q,s)$-biocular for any $s \in \{0,\dots,q-1\}$.*

Characterization (vii) above is by forbidden induced subgraphs.

3 Algorithmic results

We denote by n and m the number of vertices and edges. For hypergraphs, r denotes the maximum size of an edge and M the sum of the sizes of all edges. For graphs, ω (resp. ψ) denotes the maximum size of a clique (resp. biclique).

First, we extend results for hereditary p-Helly hypergraphs in [2] and [4].

Theorem 3.1 *The recognition of hereditary (p, q)-Helly hypergraphs: (i) can be solved in $O(rn^{(p+1)q} + Mn^{pq})$ time if p and q are fixed; (ii) can be solved in $O(rm^{p+1})$ time if p is fixed (even if q is part of the input); (iii) is co-NP-complete if p is part of the input (even for fixed q).*

Next, we generalize results in [4] for hereditary p-clique-Helly graphs.

Theorem 3.2 *The recognition of hereditary (p, q)-clique-Helly graphs: (i) can be solved in $O(m^{pq/2+1} + \omega m^{(p+1)q/2} + n)$ time if p and q are fixed; (ii) is NP-hard if p or q is part of the input (even if the other is fixed).*

We also have similar results for hereditary (p, q)-biclique-Helly graphs.

Theorem 3.3 *The recognition of hereditary (p, q)-biclique-Helly graphs: (i) can be solved in $O(\psi n^{(p+1)q})$ time if p and q are fixed; (ii) is co-NP-complete if p or q is part of the input (even if the other is fixed).*

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