Esquemas singulares de ecuaciones diferenciales algebraicas

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Haces. Cayley-Bacharach

Let $X_1, X_2, \ldots, X_n$ be hypersurfaces in $\mathbb{P}^n$ of degrees $d_1, d_2, \ldots, d_n$, and suppose that the intersection subscheme $\Gamma = \bigcap X_i$ is zero-dimensional.
Let Γ_0 and Γ_1 be subschemes of Γ residual to one another in Γ, and set

$$e = d_1 + d_2 + \ldots + d_n - n - 1.$$

If $s \leq e$ is a nonnegative integer, then the dimension of the family of hypersurfaces of degree s containing Γ_0 (modulo those containing all of Γ) is equal to the failure of Γ_1 to impose independent conditions on hypersurfaces of complementary degree $e - s$.

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Recall that Γ_1 being residual to Γ_0 in Γ means that $\mathcal{J}_{\Gamma_1} = \mathrm{Ann}(\mathcal{J}_{\Gamma_0}/\mathcal{J}_{\Gamma})$ and that, being Γ locally a complete intersection, that one also has that $\mathcal{J}_{\Gamma_0} = \mathrm{Ann}(\mathcal{J}_{\Gamma_1}/\mathcal{J}_{\Gamma})$.

- (x, y) **self-residual** por the CI $(x^2, y) = (y - x^2, y)$
- $(x, y), (x^2, xy, y^2)$ **mutual residuals** and $(x, y^2), (x^2, y)$ both **self-residual** for the CI (x^2, y^2)

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$$e = d_1 + d_2 + \ldots + d_n - n - 1.$$

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Let X_1, X_2 be curves in $\mathbb{P}^2$ of degrees d_1, d_2, and suppose that the intersection subscheme $\Gamma = X_1 \cap X_2$ is zero-dimensional.
Let Γ_0 and Γ_1 be subschemes of Γ residual to one another in Γ, and set

$$e = d_1 + d_2 - 3.$$

If $s \leq e$ is a nonnegative integer, then the dimension of the family of curves of degree s containing Γ_0 (modulo those containing all of Γ) is equal to the failure of Γ_1 to impose independent conditions on curves of complementary degree $e - s$.

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What subschemes of the base scheme of a pencil of curves of degree r can be removed in such a way that the remainder subscheme determines uniquely the pencil?

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Answer:

One can remove exactly those imposing independent conditions to the curves of degree $r-3$. (Consequence of CB)

In particular:

Any subscheme of at least r^2-r+2 base points determines the pencil.

If the base scheme is reduced, there exist appropiate sets of $\frac{r^2+3r-2}{2}$ base points determining the pencil.

(For $r=10$: Any 92 points, or appropriate 64 points in the reduced case)

Foliaciones. Polaridad

- For $n = 2$, a foliation $\mathcal{F}$ is a null rational map

$$\Phi = \Phi_{\mathcal{F}} : \mathbb{P}^2 \longrightarrow \check{\mathbb{P}}^2$$

(called its *polarity map*), together with the linear system

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- For $n \geq 2$, a foliation $\mathcal{F}$ is a null rational map

$$\Phi = \Phi_{\mathcal{F}} : \mathbb{P}^n = \mathbb{P}(V) \longrightarrow \mathcal{G}(1, n) \subset \mathbb{P}\left(\overset{2}{\wedge} V\right)$$

together with the linear system $\Delta = \Delta_{\mathcal{F}}$ of hypersurfaces giving rise to Φ (called its *polar linear system*).

Esquema singular. Multiplicidades

$$T = U\frac{\partial}{\partial X} + V\frac{\partial}{\partial Y} + W\frac{\partial}{\partial Z}, \quad \Omega = A\,dX + B\,dY + C\,dZ, \quad XA + YB + ZC = 0$$

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$$\mu_q = \dim_k\left(\frac{\mathcal{O}_{\mathbb{P}^2,q}}{(a,b)\cdot\mathcal{O}_{\mathbb{P}^2,q}}\right), \quad \mu_q = \dim_k\left(\frac{\mathcal{O}_{\mathbb{P}^2,q}}{\mathcal{I}_q}\right), \quad \sum_{q\in \mathrm{Sing}(\mathcal{F})} \mu_q = r^2 + r + 1$$

Geometría del esquema singular

Let $\mathcal{F}$ be a reduced foliation of degree $r \geq 0$ on $\mathbb{P}^2$, let $\mathcal{J}_0$ be the sheaf of ideals of its singular subscheme SingS($\mathcal{F}$). *For every integer $s \geq 0$, one has that*

$$h^0(\mathbb{P}^2, \mathcal{J}_0(s)) = \begin{cases} 0 & \text{if } s \leq r\ , \\ (t+1)(t+3) & \text{if } r+1 \leq s = r+1+t \leq 2r\ , \\ \frac{1}{2}(s+1)(s+2) - (r^2+r+1) & \text{if } s > 2r\ . \end{cases}$$

The formula above remains valid if one replaces the bound $2r$ by $2(r-1)$.

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$$h^0\left(\mathbb{P}^2, \mathcal{J}_0(r+1)\right) = 3$$

Determinación de la foliación

Theorem

If $r \geq 2$*, then there exists a unique triple* A, B, C *(up to a scalar multiple) in* $\mathcal{V} = H^0(\mathbb{P}^2, \mathcal{J}_0(r+1))$ *satisfying Euler's condition* $XA + YB + ZC = 0$*.*

In consequence, if $r \geq 2$*,* $\mathcal{F}$ *is the unique foliation in* $\mathcal{F}ol(r, \mathbb{P}^2)$ *having* $\mathrm{SingS}(\mathcal{F})$ *as singular subscheme, and the same is true if* $r = 0$*.*

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$$\Omega_t = YZ\,dX - t\,XZ\,dY + (t-1)XY\,dZ, \quad \text{for } t \in \mathbb{C} \setminus \{0, 1\}$$

Caracterización del esquema singular

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Let $\mathcal{J} \subset \mathcal{O}_{\mathbb{P}^2}$ *be a local complete intersection ideal sheaf of colength* r^2+r+1, *with* $r \geq 2$. *Then* $\mathcal{J}$ *is the ideal sheaf* $\mathcal{J}_{\Gamma_0}$ *of the singular subscheme* $\Gamma_0 = \mathrm{SingS}(\mathcal{F})$ *of a (therefore unique) reduced foliation* $\mathcal{F}$ *of degree* r *on* $\mathbb{P}^2$, *if and only if the following two conditions hold:*

(i) $h^0\left(\mathbb{P}^2, \mathcal{J}(r+1)\right) \geq 3$,

(ii) $h^0\left(\mathbb{P}^2, \mathcal{J}'_j(r-j)\right) = 0$, *for every* j *such that* $0 \leq j < r$, *and every ideal sheaf* $\mathcal{J}'_j \supset \mathcal{J}$ *of colength* $\ell(\mathcal{J}'_j) = (r-j)(r+j+1)$.

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Espacios de foliaciones

The space $\mathcal{F}\mathrm{ol}(r,\mathbb{P}^n)$ of holomorphic foliations (with singularities) of degree $r \geq 0$ is the projective space of lines through 0 in $H^0\left(\mathbb{P}^n,\Theta_{\mathbb{P}^n}(r-1)\right)=H^0\left(\mathbb{P}^n,\mathcal{H}om(\mathcal{H}_{-r+1},\Theta_{\mathbb{P}^n})\right)$.

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$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(r-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(r)^{\oplus(n+1)} \xrightarrow{\Pi_*} \Theta_{\mathbb{P}^n}(r-1) \longrightarrow 0$$

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$$T = V_0 \frac{\partial}{\partial X_0} + V_1 \frac{\partial}{\partial X_1} + \cdots + V_n \frac{\partial}{\partial X_n}, \qquad A_{i,j} = X_i V_j - X_j V_i$$

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Key example:

$$V_0 = X_0^r,\ V_1 = X_1^r,\ \dots,\ V_n = X_n^r, \quad \begin{cases} \mathbb{K} = \mathbb{F}_q \text{ a finite field;} \\ r = m\,p^t + 1, \quad m \text{ a divisor of } q-1 \end{cases}$$

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Application: Differential Codes

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In particular:

Any subscheme of at least r^2+3 base points determines the foliation.

If the base scheme is reduced, there exist appropiate sets of $\frac{r(r+5)}{2}$ base points determining the foliation.

(For $r=10$: Any 103 points, or appropriate 75 in the reduced case)

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The minimal number $M_{r,n}$ of singular points which can determine any degree r foliation in the n-dimensional projective space is the smallest integer greater or equal than $\frac{e}{n}$, where e is the dimension of the space of foliations of that degree.

Generic foliation in this space have reduced singular scheme and they are uniquely determined by appropriated sets of $M_{r,n}$ singular points.

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$$M_{r,n} = \frac{r(r+5)}{2} - u, \quad u = \left[\frac{r}{2}\right] - 1.$$

(For $r = 10$, appropriated 71 points in the generic case)