Estimaciones anisótropas para Elementos Finitos piramidales con aplicaciones a problemas con singularidades de arista y vértice

Alexis Jawtuschenko [1],
Ariel Lombardi [2]

1) IMAS CONICET - UBA
2) UNGS - UBA - CONICET

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Problema

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$$\Gamma := \partial\Omega \quad \text{caras planas.}$$

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Ω no es convexo.

$$\boldsymbol{u} \notin H^1(\Omega).$$

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$$\|v\|^2_{V^{1,2}_{\beta,\delta}(T_l)} = \sum_{|\alpha|\leqslant 1} \int_{T_l} |R(\mathbf{x})^{\beta - 1 + |\alpha|} \theta(\mathbf{x})^{\delta-1+|\alpha|} D^\alpha v(\mathbf{x})|^2 \, d\mathbf{x}$$

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$$\beta > \frac{1}{2} - \lambda_{\mathbf{v}}.[2]$$

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Formulación débil

$$\begin{array}{ll} \text{Hallar} & (\boldsymbol{u}, p) \in H(\mathrm{div}, \Omega) \times L^2(\Omega) \\ \text{t.q. para todo} & (\boldsymbol{v}, q) \in H(\mathrm{div}, \Omega) \times L^2(\Omega) \end{array}$$

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Hallar $(\boldsymbol{u}, p) \in H(\mathrm{div}, \Omega) \times L^2(\Omega)$
t.q. para todo $(\boldsymbol{v}, q) \in H(\mathrm{div}, \Omega) \times L^2(\Omega)$

$$\int_\Omega \boldsymbol{u} \cdot \boldsymbol{v} \, d\boldsymbol{x} - \int_\Omega p \, \mathrm{div} \, \boldsymbol{v} \, d\boldsymbol{x} = 0$$

$$\int_\Omega q \, \mathrm{div} \, \boldsymbol{u} \, d\boldsymbol{x} = \int_\Omega f \, q \, d\boldsymbol{x}$$

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$$h_1, h_2 \sim \begin{cases} h^{\frac{1}{\mu}} & \text{if } d(K,e) = 0 \\ h\, d(K,e)^{1-\mu} & \text{if } 0 < d(K,e) \lesssim 1 \\ h & \text{if } d(K,e) \sim 1 \end{cases}$$

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$\mu \sim 1 - \delta$,
$\nu \sim 1 - \beta$
Y además: $\mu < 1 \Rightarrow \mu \leqslant \nu$.

Formulación débil discreta

$$\mathbb{V}_h = \mathcal{RT}_k(\tau_h)$$
$$\mathbb{Q}_h = \mathcal{P}_k(\tau_h)$$

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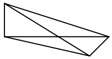
Queremos mallar combinando tetraedros y prismas para

- evitar 

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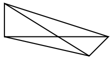
- evitar 

- (1 prisma = 3 tetra $\rightsquigarrow$) usar menos elementos.

Table 1

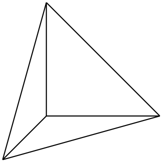


Table 1

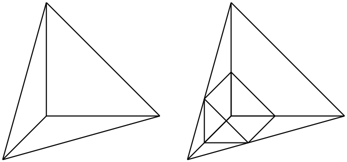
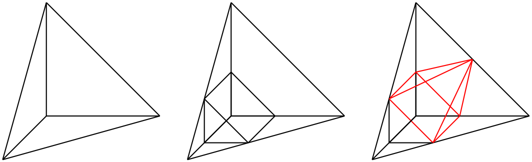


Table 1



Aparecen pirámides en el medio.

Interpolación $H(\mathrm{div})$–conforme en pirámides.

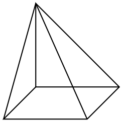
$\boldsymbol{\pi u} \in < \zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5 >$ tal que

$$\int_f \boldsymbol{\pi u} \cdot \boldsymbol{\nu} \, d\sigma = \int_f \boldsymbol{u} \cdot \boldsymbol{\nu} \, d\sigma$$

donde

Table 2

Shape functions en $\hat{P}$



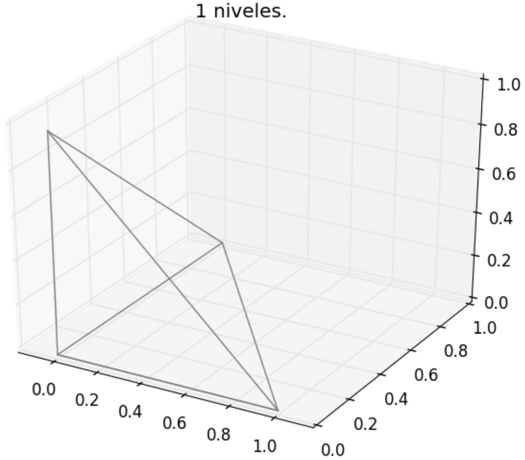
$$\zeta_1{}^3 = \begin{pmatrix} -\frac{xz}{1-z} \\ y - 2 + \frac{y}{1-z} \\ z \end{pmatrix} \quad \zeta_2 = \begin{pmatrix} x - 2 + \dfrac{x}{1-z} \\ -\dfrac{yz}{1-z} \\ z \end{pmatrix}$$

$$\zeta_3 = \begin{pmatrix} x + \dfrac{x}{1-z} \\ -\dfrac{yz}{1-z} \\ z \end{pmatrix} \quad \zeta_4 = \begin{pmatrix} -\dfrac{xz}{1-z} \\ y + \dfrac{y}{1-z} \\ z \end{pmatrix} \quad \zeta_5 = \begin{pmatrix} x \\ y \\ z - 1 \end{pmatrix}$$

Ya no son polinomios! [GH99].

[3] $f_1 = \hat{P} \cap \{y = 0\}, \boldsymbol{\nu}_1 = (0, -1, 0)'$

Programa para mallar macro–elementos.



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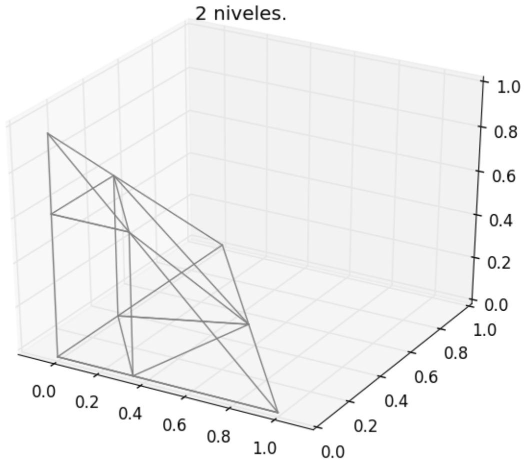

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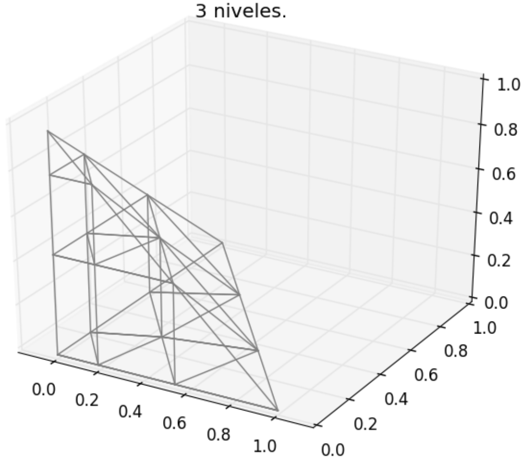

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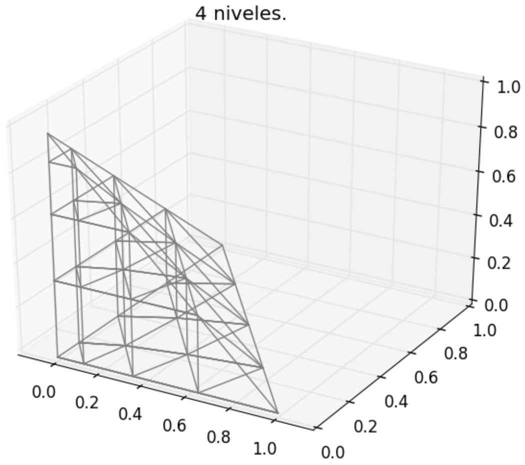

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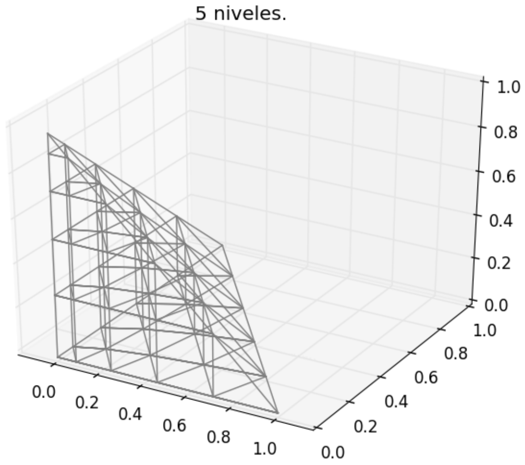

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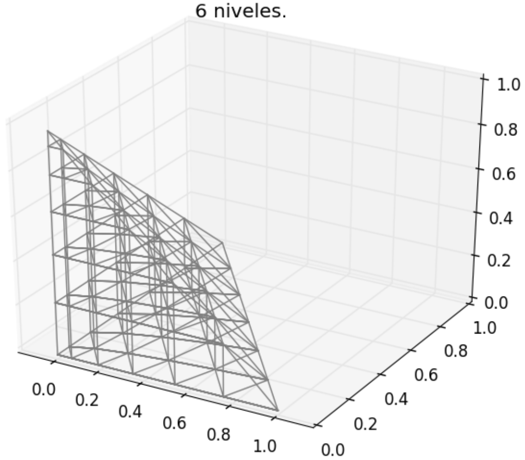

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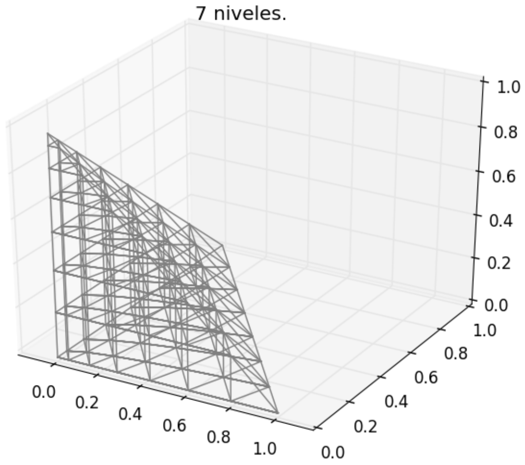

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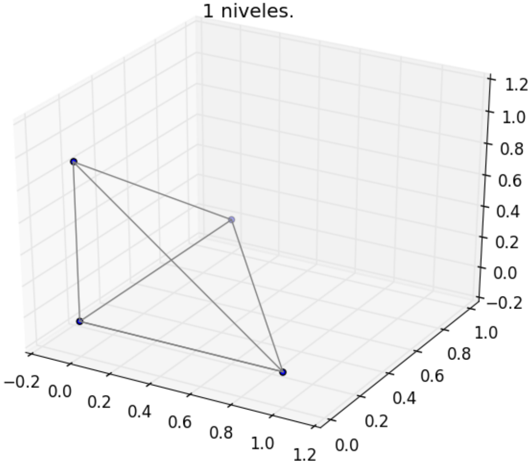

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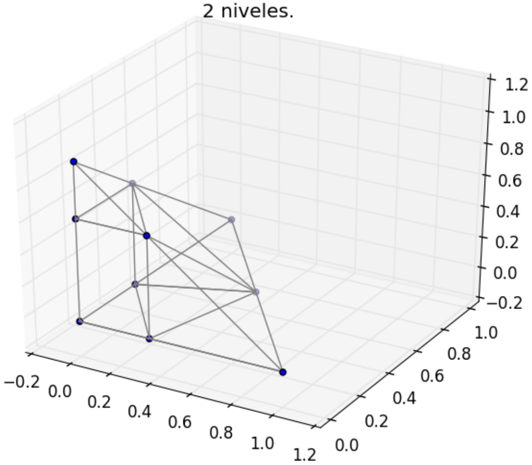

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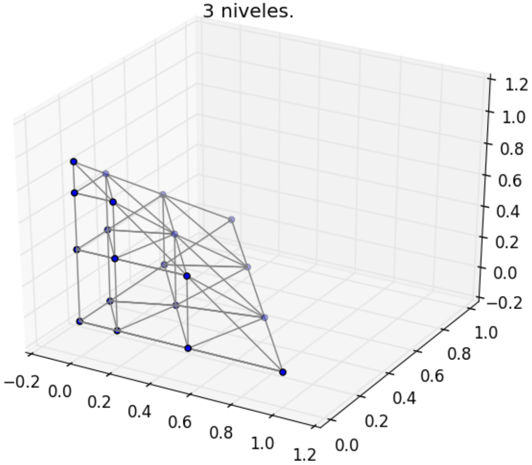

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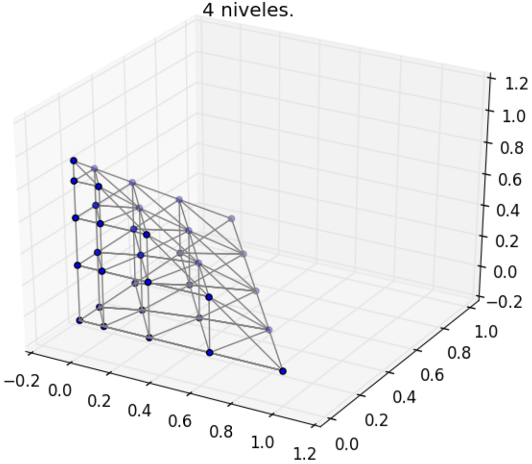

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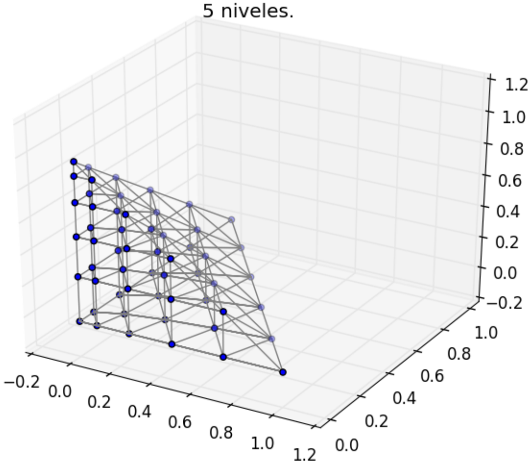

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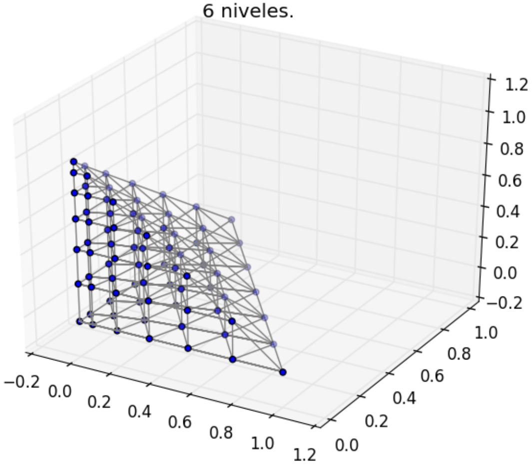

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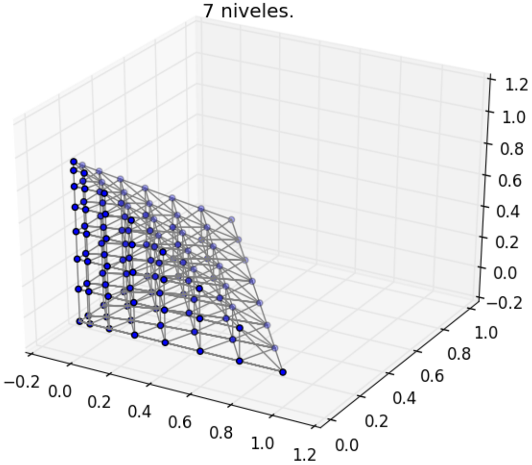

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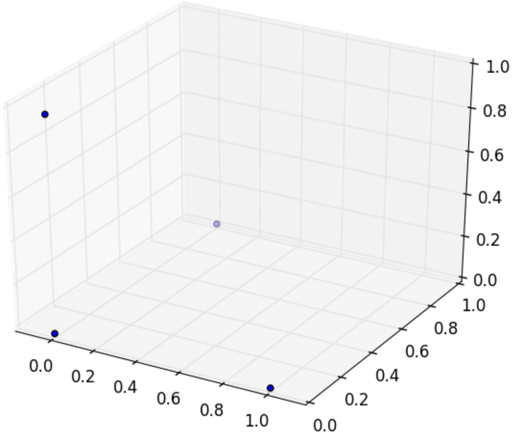

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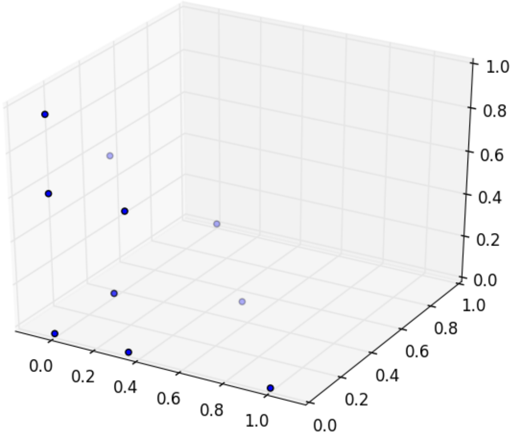

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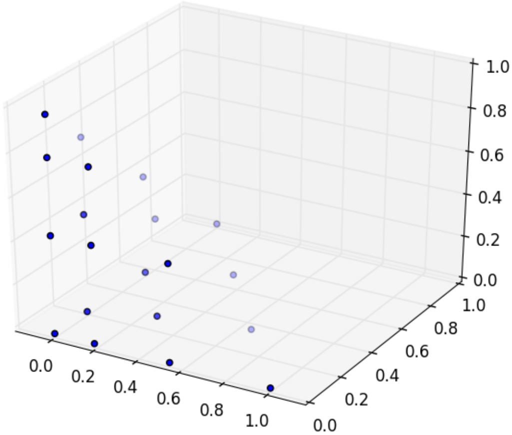

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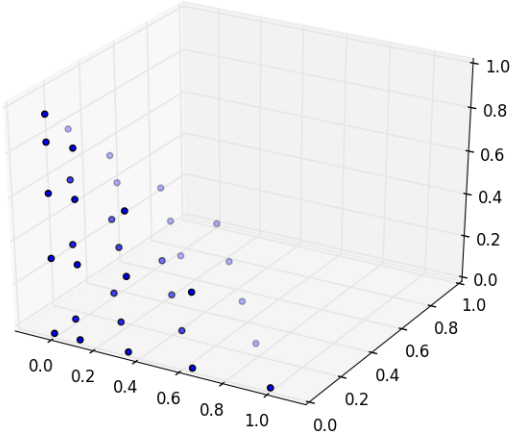

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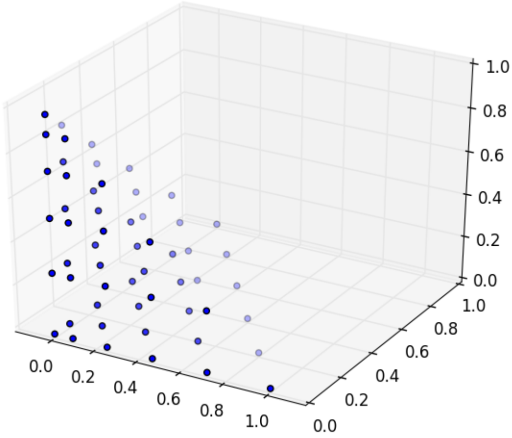

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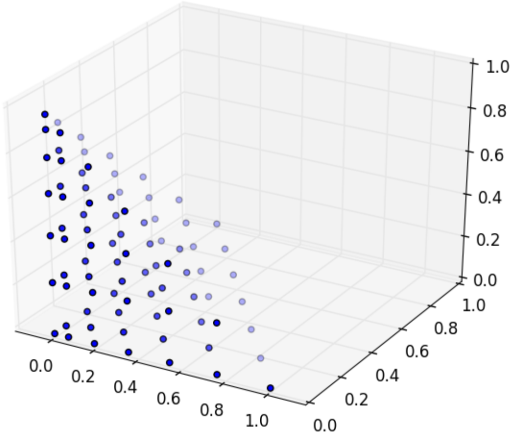

Los nodos de la malla

Fijamos:

$$\begin{aligned} T &= [\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3] \\ \mathbf{p}_0 &= 0 \\ [\mathbf{p}_0, \mathbf{p}_3] &= \text{la arista singular} \\ \mathbf{p}_3 &= \text{vértice singular} \\ [\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2] &\subseteq \{z = 0\}. \end{aligned}$$

λ_0, λ_1 y λ_2: coordenadas baricéntricas con respecto a $\mathbf{p}_0, \mathbf{p}_1$ y $\mathbf{p}_2$.

$\mathbf{p}_{i,j}$ es tal que

$$\lambda_0(\mathbf{p}_{i,j}) = 1 - \lambda_1(\mathbf{p}_{i,j}) - \lambda_2(\mathbf{p}_{i,j})$$

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\lambda_1(\mathbf{p}_{i,j}) &= \frac{i}{n}\left(\frac{i+j}{n}\right)^{\frac{1}{\mu}-1} \\
\lambda_2(\mathbf{p}_{i,j}) &= \frac{j}{n}\left(\frac{i+j}{n}\right)^{\frac{1}{\mu}-1}.
\end{aligned}$$

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$$0 \leqslant i \leqslant n$$

$$0 \leqslant j \leqslant n - i$$

Los puntos son la unión de

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& \vdots & \\
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& \vdots & \\
\left\{\mathbf{p}_{i,j} \,:\, 0 \leqslant i \leqslant 1, \quad 0 \leqslant j \leqslant 1-i\right\} & + & \left[1-\left(\frac{1}{n}\right)^{1/\mu}\right]\mathbf{e}_3 \\
\{\mathbf{e}_3\} & & ,
\end{array}
$$

i.e.:

$$\mathcal{P} = \bigcup_{k=0}^{n} \left\{ \mathbf{p}_{i,j} \,:\, 0 \leqslant i \leqslant n-k, \quad 0 \leqslant j \leqslant n-k-i \right\}$$

$$+ \left[1-\left(\tfrac{n-k}{n}\right)^{1/\mu}\right] \mathbf{e}_3.$$

Theorem

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(\Omega)^3} \leqslant C\, h \, \|f\|_{L^2(\Omega)}$$

$$\|p - p_h\|_{L^2(\Omega)} \leqslant C\, h \, \|f\|_{L^2(\Omega)}.$$

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$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(\Omega)^3} \leqslant C \inf_{v \in \mathbb{V}_h} \|\boldsymbol{u} - \mathbf{v}\|_{\mathbb{V}}$$

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$$\leqslant C\, \|\boldsymbol{u} - \pi \boldsymbol{u}\|_{\mathbb{V}}$$

Theorem

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(\Omega)^3} \leqslant C\, h\, \|f\|_{L^2(\Omega)}$$

$$\|p - p_h\|_{L^2(\Omega)} \leqslant C\, h\, \|f\|_{L^2(\Omega)}.$$

Esbozo de la demostración.

$$\begin{aligned}
\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(\Omega)^3} &\leqslant C \inf_{v \in \mathbb{V}_h} \|\boldsymbol{u} - \mathbf{v}\|_{\mathbb{V}} \\
&\leqslant C\, \|\boldsymbol{u} - \pi \boldsymbol{u}\|_{\mathbb{V}} \\
\|p - p_h\|_{\mathbb{Q}} &\leqslant C\, [\inf_{v \in \mathbb{V}_h} \|\boldsymbol{u} - \mathbf{v}\|_{\mathbb{V}} + \inf_{q \in \mathbb{Q}_h} \|p - q\|_{\mathbb{Q}}]
\end{aligned}$$

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&\leqslant C\,[\|\boldsymbol{u}-\pi\boldsymbol{u}\|_{\mathbb{V}} + \|p-\pi_h^{\perp}p\|_{\mathbb{Q}}].
\end{aligned}$$

$$\|\boldsymbol{u} - \pi\boldsymbol{u}\|_{\mathbb{V}}^2 = \|\boldsymbol{u} - \pi\boldsymbol{u}\|_{L^2(\Omega)^3}^2 + \|\mathrm{div}\,(\boldsymbol{u} - \pi\boldsymbol{u})\|_{L^2(\Omega)^3}^2.$$

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Primero:

$$\|\mathrm{div}\,(\boldsymbol{u} - \pi\boldsymbol{u})\|_{0,\Omega}^2 = \|\mathrm{div}\ \boldsymbol{u} - \mathrm{div}\ \pi\boldsymbol{u}\|_{0,\Omega}^2$$

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$$\text{(elementwise proj.)} = \|\mathrm{div}\ \boldsymbol{u} - \boldsymbol{p}_{\tau_h}\mathrm{div}\ \boldsymbol{u}\|_{0,\Omega}^2$$

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&\leqslant h^2\,\|\mathrm{div}\ \boldsymbol{u}\|_{0,\Omega}^2
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&\leqslant h^2\,\|\mathrm{div}\ \boldsymbol{u}\|_{0,\Omega}^2 \\
&= h^2\,\|f\|_{0,\Omega}^2.
\end{aligned}$$

Segundo:

$$\|\boldsymbol{u} - \pi \boldsymbol{u}\|_{L^2(\Omega)^3} \leqslant \|\boldsymbol{u}_r - \pi \boldsymbol{u}_r\|_{L^2(\Omega)^3} + \|\boldsymbol{u}_s - \pi \boldsymbol{u}_s\|_{L^2(\Omega)^3}$$

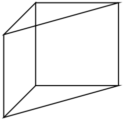
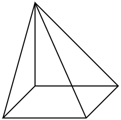
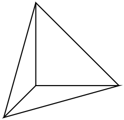
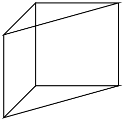
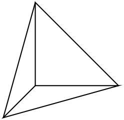
Segundo:

$$\|\boldsymbol{u} - \pi\boldsymbol{u}\|_{L^2(\Omega)^3} \leqslant \|\boldsymbol{u}_r - \pi\boldsymbol{u}_r\|_{L^2(\Omega)^3} + \|\boldsymbol{u}_s - \pi\boldsymbol{u}_s\|_{L^2(\Omega)^3}$$

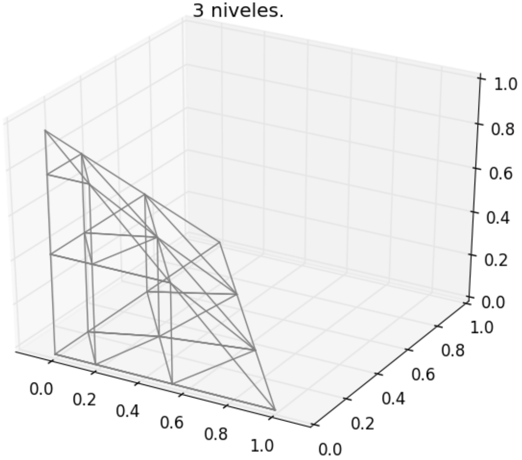
Comentamos una cuenta para la parte singular.

Table 3

Parte singular.

$d(K, \mathbf{e}) > 0$			
$d(K, \mathbf{e}) = 0$	 $d(K, \mathbf{v}) > 0$		 $d(K, \mathbf{v}) = 0$

Recordar



Teorema (Interp. loc.)

K es una pirámide como recién: $\qquad k_3 \geqslant k_1,\, k_2.$

$\pi = \pi_K$ es el interpolador local sobre caras.

$$\|\boldsymbol{u}_s - \pi \boldsymbol{u}_s\|_{0,K} \lesssim \sum_i k_i \|\partial_{\eta_i} \boldsymbol{u}_s\|_{0,K} + h_K \|div\, \boldsymbol{u}_s\|_{0,K} \qquad (1)$$

Teorema (Interp. loc.)

K es una pirámide como recién: $\qquad k_3 \geqslant k_1,\, k_2.$

$\pi = \pi_K$ *es el interpolador local sobre caras.*

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Ahora acotamos cada término a la derecha de (1) por $h\|f\|_{L^2(\Omega)}$.

Recordar (direcciones según singularidades)

$$h_1, h_2 \sim \begin{cases} h^{\frac{1}{\mu}} & \text{if } d(K,e) = 0 \\ h\, d(K,e)^{1-\mu} & \text{if } 0 < d(K,e) \lesssim 1 \\ h & \text{if } d(K,e) \sim 1 \end{cases}$$

$$h_3 \sim \begin{cases} h^{\frac{1}{\nu}} & \text{if } d(K,v) = 0 \\ h\, d(K,v)^{1-\nu} & \text{if } 0 < d(K,v) \lesssim 1 \\ h & \text{if } d(K,v) \sim 1 \end{cases}$$

$$h_1 \|\partial_{\eta_1} \boldsymbol{u}_s\| = h_1 \sum_{i=1}^{3} \|\partial_{\eta_1} \boldsymbol{u}_{s,i}\|$$

$$
\begin{aligned}
h_1 \|\partial_{\eta_1} \boldsymbol{u}_s\| &= h_1 \sum_{i=1}^{3} \|\partial_{\eta_1} \boldsymbol{u}_{s,i}\| \\
&\leqslant h \sum_{i=1}^{3} \|r^{1-\mu} \partial_{\eta_1} \boldsymbol{u}_{s,i}\|
\end{aligned}
$$

$$
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&\leqslant h\sum_{i=1}^{3}\|r^{1-\mu}\partial_{\eta_1}\boldsymbol{u}_{s,i}\| \\
&\leqslant h\left(\sum_{i=1}^{2}\|R^{1-\mu}\theta^{1-\mu}\partial_{\eta_1}\boldsymbol{u}_{s,i}\| + \|R^{1-\mu}\partial_{\eta_1}\boldsymbol{u}_{s,3}\|\right)
\end{aligned}
$$

$$
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&\leqslant h \left(\sum_{i=1}^{2} \|R^{1-\mu} \theta^{1-\mu} \partial_{\eta_1} \boldsymbol{u}_{s,i}\| + \|R^{1-\mu} \partial_{\eta_1} \boldsymbol{u}_{s,3}\| \right) \\
(\mu \leqslant \nu) \qquad &\leqslant h \left(\sum_{i=1}^{2} \|R^{1-\nu} \theta^{1-\mu} \partial_{\eta_1} \boldsymbol{u}_{s,i}\| + \|R^{1-\nu} \partial_{\eta_1} \boldsymbol{u}_{s,3}\| \right)
\end{aligned}
$$

$$h_1 \|\partial_{\eta_1} \boldsymbol{u}_s\| \leqslant h \left(\sum_{i=1}^{2} \|R^\beta \theta^\delta \partial_{\eta_1} \boldsymbol{u}_{s,i}\| + \|R^\beta \partial_{\eta_1} \boldsymbol{u}_{s,3}\| \right)$$

$$h_1\|\partial_{\eta_1}\boldsymbol{u}_s\| \leqslant h\left(\sum_{i=1}^{2}\|R^{\beta}\theta^{\delta}\partial_{\eta_1}\boldsymbol{u}_{s,i}\| + \|R^{\beta}\partial_{\eta_1}\boldsymbol{u}_{s,3}\|\right)$$

$$\leqslant h\left(\sum_{i=1}^{2}\|\boldsymbol{u}_{s,i}\|_{\beta,\delta} + \|\boldsymbol{u}_{s,3}\|_{\beta,0}\right)$$

$$
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h_1 \|\partial_{\eta_1} \boldsymbol{u}_s\| &\leqslant h \left(\sum_{i=1}^{2} \|R^{\beta} \theta^{\delta} \partial_{\eta_1} \boldsymbol{u}_{s,i}\| + \|R^{\beta} \partial_{\eta_1} \boldsymbol{u}_{s,3}\| \right) \\
&\leqslant h \left(\sum_{i=1}^{2} \|\boldsymbol{u}_{s,i}\|_{\beta,\delta} + \|\boldsymbol{u}_{s,3}\|_{\beta,0} \right) \\
&\lesssim h \|f\|_{L^2(\Omega)}.
\end{aligned}
$$

$\blacksquare$

Conclusión

$$
\begin{aligned}
\|\boldsymbol{u}_s - \pi \boldsymbol{u}_s\|_{0,K} &\lesssim \sum_i k_i \|\partial_{\eta_i} \boldsymbol{u}_s\|_{0,K} + h_K \|\operatorname{div} \boldsymbol{u}_s\|_{0,K} \\
\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(\Omega)^3} &\leqslant C\, h \,\|f\|_{L^2(\Omega)} \\
\|p - p_h\|_{L^2(\Omega)} &\leqslant C\, h \,\|f\|_{L^2(\Omega)}.
\end{aligned}
$$