Fully nonlinear elliptic equations of degenerate/singular type with strong absorption condition

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Outline

Introduction

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$$F(x, Du, D^2u) = \lambda_0(x) f(u) \quad \text{in} \quad \Omega,$$

where

- ✓ $F : \Omega \times \mathbb{R}^n_* \times \mathrm{Sym}(n) \to \mathbb{R}$ is a fully nonlinear operator of degenerate/singular type;
- ✓ $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-decreasing (non-Lipschitz function at origin) with $f(0) = 0$;
- ✓ $\lambda_0 : \Omega \to \mathbb{R}_+$ is a bounded function away from zero and infinity (*Thiele modulus*).

Introduction

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A standard model, known in the Literature as *problems with strong absorption* is given by

$$F\left(x, Du, D^2u\right) = \lambda_0(x).u_+^{\mu}(x) \quad \text{in} \quad \Omega, \tag{1.1}$$

where $0 \leq \mu < \gamma + 1$ is the *absorption factor*.

Second order fully nonlinear operators

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1. (**γ-ellipticity condition**) For $\gamma > -1$ there exist $\Lambda \geq \lambda > 0$ s.t.

$$|\overrightarrow{p}|^{\gamma}\mathcal{P}^{-}_{\lambda,\Lambda}(P) \leq F(x, \overrightarrow{p}, M+P) - F(x, \overrightarrow{p}, M) \leq |\overrightarrow{p}|^{\gamma}\mathcal{P}^{+}_{\lambda,\Lambda}(P)$$

2. (**$(\gamma, 1)$-Homogeneity condition**) For all $s \in \mathbb{R}^*$, $t \geq 0$

$$F\left(x, s\overrightarrow{p}, rM\right) = |s|^{\gamma} rF\left(x, \overrightarrow{p}, M\right).$$

3. (**Continuity condition**) There exists $\omega_1 : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|F\left(x, \overrightarrow{p}, M\right) - F\left(y, \overrightarrow{p}, M\right)| \leq \omega_1\left(|x-y|\right)|\overrightarrow{p}|^{\gamma}\|M\|.$$

Examples of fully nonlinear operators

- ✓ $F[u] = |Du|^{\gamma}(\mathcal{P}^{\pm}_{\lambda,\Lambda}(D^2u) \pm \mathfrak{b}(x).Du)$ with $\gamma > -1$ and $\mathfrak{b}$ a smooth function.
- ✓ $F[u] = |Du|^{\gamma}\left(p_1 \mathrm{tr}\,(D^2u) + p_2\left\langle D^2u\frac{Du}{|Du|}, \frac{Du}{|Du|}\right\rangle\right)$ with $p_1 > 0$ and $p_1 + p_2 > 0$.
- ✓ $F[u] = |Du|^{p-2}\left[\mathrm{tr}\,(\mathcal{B}_1(x)D^2u) + C_0\left\langle D^2u\mathcal{B}_2(x)\frac{Du}{|Du|}, \mathcal{B}_2(x)\frac{Du}{|Du|}\right\rangle\right]$

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Either $\forall\ \phi \in C^2_{loc}(\Omega)$ such that $u - \phi$ has a local minimum at x_0 and $|D\phi(x_0)| \neq 0$ holds

$$F\left(x_0, D\phi(x_0), D^2\phi(x_0)\right) \leq g(x_0, \phi(x_0)) \quad (resp. \ \geq g(x_0, \phi(x_0)))$$

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Or there exists $B(x_0, \varepsilon)$ where $u = K$ and holds

$$g(x, K) \geq 0 \ \ \forall \ \ x \in B(x_0, \varepsilon) \quad (resp.\ g(x, K) \leq 0)$$

Historic overview about regularity theory

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Fully nonlinear setting

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- [Berindelli-Demengel and Imbert-Silvestre] $C^{1,\alpha}$ local regularity.

A "fine" family and its invariance scaling properties

Definition

For a fully nonlinear operator F fulfilling (F1)-(F3) we say that $u \in \mathfrak{J}(F, \lambda_0, \mu)(B_1)$ if

- ✓ $F(x, Du, D^2u) = \lambda_0(x) u_+^{\mu}(x) \ll 1$ in B_1 for $0 \leq \mu < \gamma + 1$.
- ✓ $0 \leq u \leq 1$ and $0 < \mathfrak{m} \leq \lambda_0 \leq \mathfrak{M}$ in B_1.
- ✓ $u(0) = 0$.

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Doubling property

We shall adopt the following notation

$$\mathcal{S}_{(r,x_0)}[u] := \sup_{B_r(x_0)} u(x),$$

Moreover, we will omit the center of the ball when $x_0 = 0$.

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Definition

We define for $u \in \mathfrak{J}(F, \lambda_0, \mu)(B_1)$ the following set

$$\mathbb{V}_{\gamma,\mu}[u] := \left\{ j \in \mathbb{N} \cup \{0\};\ \mathcal{S}_{\frac{1}{2^j}}[u] \le \mathfrak{A}(n, \gamma, \mu).\mathcal{S}_{\frac{1}{2^{j+1}}}[u] \right\},$$

which delineates the sharp geometric decay at free boundary points for dead-core solutions in certain dyadic levels.

Growth rate for dead core solutions

Next result assures that all function in $\mathfrak{J}(F, \lambda_0, \mu)(B_1)$ fulfilling the doubling property $\mathbb{V}_{\gamma,\mu}[u]$ has a sharp geometric decay at free boundary points.

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Lemma

There exists a positive constant $\mathfrak{C}_0 = \mathfrak{C}_0(n, \lambda, \Lambda, \gamma, \mu, \mathfrak{M})$ such that

$$\mathcal{S}_{\frac{1}{2^{j+1}}}[u] \leq \mathfrak{C}_0. \left(\frac{1}{2^j}\right)^{\frac{\gamma+2}{\gamma+1-\mu}} \tag{2.1}$$

for all $u \in \mathfrak{J}(F, \lambda_0, \mu)(B_1)$ and $j \in \mathbb{V}_{\gamma,\mu}[u]$.

Growth rate for dead core solutions

Proof: Suppose that the thesis of Lemma fails to hold. Then, for each $k \in \mathbb{N}$ we might find $u_k \in \mathfrak{J}(F, \lambda_0^k, \mu)(B_1)$ and $j_k \in \mathbb{V}_{\gamma,\mu}[u_k]$ such that

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$$\mathcal{S}_{\frac{1}{2^{j_k+1}}}[u_k] \geq \max\left\{k.\left(\frac{1}{2^{j_k}}\right)^{\frac{\gamma+2}{\gamma+1-\mu}}, \mathfrak{A}^{-1}\mathcal{S}_{\frac{1}{2^{j_k}}}[u]\right\}. \qquad (2.2)$$

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Now, define the auxiliary function

$$v_k(x) := \frac{u_k\left(\frac{1}{2^{j_k}}x\right)}{\mathcal{S}_{\frac{1}{2^{j_k+1}}}[u_k]} \quad \text{in} \quad B_1.$$

Growth rate for dead core solutions

Hence, v_k fulfils

- ✓ $0 \leq v_k(x) \leq \dfrac{\mathcal{S}_{\frac{1}{2^{j_k}}}[u_k]}{\mathcal{S}_{\frac{1}{2^{j_k+1}}}[u_k]} \leq \mathfrak{A}$ in B_1 and $v_k(0) = 0$.
- ✓ $\mathcal{S}_{\frac{1}{2}}[v_k] = 1$
- ✓ $F_k\left(x, Dv_k, D^2 v_k\right) = \hat{\lambda}_0^k(x)(v_k)_+^{\mu}(x)$ in B_1 in the viscosity sense,

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Growth rate for dead core solutions

where

$$F_k(x, \overrightarrow{p}, M) := F\left(\frac{1}{2^{j_k}}x, \overrightarrow{p}, M\right)$$

and

$$\hat{\lambda}_0^k(x) := \frac{1}{2^{(\gamma+2)j_k}} \frac{1}{\mathcal{S}^{\gamma+1-\mu}_{\frac{1}{2^{j_k+1}}}[u_k]} \lambda_0^k\left(\frac{1}{2^{j_k}}x\right)$$

Therefore,

$$\left\|\hat{\lambda}_0^k(x)(v_k)_+^{\mu}(x)\right\|_{L^\infty(B_1)} \leq \mathfrak{A}^{\mu}.\mathfrak{M}.\left(\frac{1}{k}\right)^{\gamma+1-\mu} \to 0 \quad \text{as} \quad k \to \infty.$$

Growth rate for dead core solutions

The previous sentences together with standard compactness arguments for fully nonlinear elliptic equations imply that, up to a subsequence, $v_k \to v$ local uniformly in $\overline{B_{\frac{3}{4}}}$ and $F_k \to F_0$.

Growth rate for dead core solutions

The previous sentences together with standard compactness arguments for fully nonlinear elliptic equations imply that, up to a subsequence, $v_k \to v$ local uniformly in $\overline{B_{\frac{3}{4}}}$ and $F_k \to F_0$. Furthermore, by stability results we have that:

- ✓ $F_0(Dv, D^2v) = 0 \quad \text{in} \quad B_{\frac{3}{4}}$.
- ✓ $0 \leq v \leq \mathfrak{A} \quad \text{in} \quad B_1$ and $v(0) = 0$.
- ✓ $\mathcal{S}_{\frac{1}{2}}[v] = 1$.

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- ✓ $F_0(Dv, D^2v) = 0 \quad \text{in} \quad B_{\frac{3}{4}}$.
- ✓ $0 \le v \le \mathfrak{A} \quad \text{in} \quad B_1$ and $v(0) = 0$.
- ✓ $\mathcal{S}_{\frac{1}{2}}[v] = 1$.

According to Strong Minimum Principle (or Harnack inequality) we have that $v \equiv 0$ in $B_{\frac{3}{4}}$, which clearly yields a contradiction. □

Growth rate for dead core solutions

The previous result yields an estimate for all level $0 < r \ll 1$

Theorem

There exists a positive constant $\mathfrak{C} = \mathfrak{C}(n, \lambda, \Lambda, \gamma, \mu, \mathfrak{M})$ such that for all $u \in \mathfrak{J}(F, \lambda_0, \mu)(B_1)$

$$u(x) \leq \mathfrak{C}.|x|^{\frac{\gamma+2}{\gamma+1-\mu}} \quad \forall\ x \in B_{\frac{1}{2}} \tag{2.3}$$

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Growth rate for dead core solutions

Proof: We claim that

$$\mathcal{S}_{\frac{1}{2^j}}[u] \le \mathfrak{C}_0.\left(\frac{1}{2^{j-1}}\right)^{\frac{\gamma+2}{\gamma+1-\mu}} \quad \forall\, j \in \mathbb{N}, \tag{2.4}$$

WLOG suppose $\mathfrak{C}_0 \ge 1$, then (2.4) holds for $j = 0$.

Suppose now that (2.4) holds for some $j \in \mathbb{N}$ and let us verify the $(j+1)^{\text{th}}$ step of induction.

In fact, if $j \in \mathbb{V}_{\gamma,\mu}[u]$ then the result holds directly by Lemma 3.

Growth rate for dead core solutions

On the other hand, if $j \notin \mathbb{V}_{\gamma,\mu}[u]$, then using the induction hypothesis

$$\mathcal{S}_{\frac{1}{2^{j+1}}}[u] \leq \left(\frac{1}{2}\right)^{\frac{\gamma+2}{\gamma+1-\mu}} .\mathcal{S}_{\frac{1}{2^j}}[u] \leq \mathfrak{C}_0. \left(\frac{1}{2}\right)^{\frac{\gamma+2}{\gamma+1-\mu}} \left(\frac{1}{2^{j-1}}\right)^{\frac{\gamma+2}{\gamma+1-\mu}} = \mathfrak{C}_1. \left(\frac{1}{2^j}\right)^{\gamma}$$

Therefore, (2.4) holds for all $j \in \mathbb{N}$.

Finally, for $r \in (0,1)$ let $j \in \mathbb{N}$ the greatest integer such that $\frac{1}{2^{j+1}} \leq r < \frac{1}{2^j}$. Then,

$$\mathcal{S}_r[u] \leq \mathcal{S}_{\frac{1}{2^j}}[u] \leq \mathfrak{C}_0. \left(\frac{1}{2^{j-1}}\right)^{\frac{\gamma+2}{\gamma+1-\mu}} \leq \mathfrak{C}(n,\lambda,\Lambda,\gamma,\mu,\mathfrak{M}).r^{\frac{\gamma+2}{\gamma+1-\mu}} \qquad \square$$

Growth rate for dead core solutions

By using Theorem 4 we can prove a similar growth rate for the gradient of functions $u \in \mathfrak{J}(F, \lambda_0, \mu)(B_1)$.

Lemma

There exists a positive constant $\mathfrak{C}_1 = \mathfrak{C}_1(n, \lambda, \Lambda, \gamma, \mu, \mathfrak{M})$ such that for all $u \in \mathfrak{J}(F, \lambda_0, \mu)(B_1)$

$$|Du(x)| \leq \mathfrak{C}_1 . |x|^{\frac{1+\mu}{\gamma+1-\mu}} \quad \forall\ x \in B_{\frac{1}{2}},$$

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Non-degeneracy and its consequences

The next result gives precisely the growth rate at which non-negative viscosity solutions leave their dead core sets.

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Theorem (Non-degeneracy)

Let u be a nonnegative, bounded viscosity solution to (1.1) in B_1 and let $x_0 \in \overline{\{u > 0\}} \cap B_{\frac{1}{2}}$ be a generic point in the closure of the non-coincidence set. Then for any $0 < r < \frac{1}{2}$, there holds

$$\sup_{B_r(x_0)} u(x) \geq c_0(n, \mathfrak{m}, \gamma, \mu).r^{\frac{2+\gamma}{\gamma+1-\mu}}. \tag{3.1}$$

Non-degeneracy and its consequences

Proof: Let us define the scaled functions

$$u_r(x) := \frac{u(x_0 + rx)}{r^{\frac{\gamma+2}{\gamma+1-\mu}}}.$$

Now, let us introduce the comparison function

$$\Psi(x) := \left[\mathfrak{m}.\frac{(\gamma+1-\mu)^{\gamma+2}}{n(\mu+1)(\gamma+2)^{\gamma+1}}\right]^{\frac{1}{\gamma+1-\mu}} |x-x_0|^{\frac{\gamma+2}{\gamma+1-\mu}}$$

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Non-degeneracy and its consequences

Straightforward calculus shows that

$$\mathcal{G}(x, D\Psi, D^2\Psi) - \hat{\lambda}_0(x).\Psi^{\mu}(x) \le \mathcal{G}(x, Du_r, D^2u_r) - \hat{\lambda}_0(x).(u_r)_+^{\mu}(x),$$

where

$$\mathcal{G}(x, \overrightarrow{p}, M) := F(x_0 + rx, \overrightarrow{p}, M) \quad \text{and} \quad \hat{\lambda}_0(x) := \lambda_0(x_0 + rx)$$

Non-degeneracy and its consequences

Finally, if $u_r \leq \Psi$ on the whole boundary of B_1, then the Comparison Principle would imply that

$$u_r \leq \Psi \quad \text{in} \quad B_1,$$

which clearly contradicts the assumption that $u_r(0) > 0$. Therefore, there exists a point $Y \in \partial B_1$ such that

$$u_r(Y) > \Psi(Y) = \left[\mathfrak{m}.\frac{(\gamma+1-\mu)^{\gamma+2}}{n\,(\mu+1)\,(\gamma+2)^{\gamma+1}}\right]^{\frac{1}{\gamma+1-\mu}}$$

and scaling back we finish the proof of the Theorem. $\square$

Non-degeneracy and its consequences

Corollary (**Uniform positive density)**

Let u be a nonnegative, bounded viscosity solution to (1.1) in B_1 and $x_0 \in \partial\{u > 0\} \cap B_{\frac{1}{2}}$ a free boundary point. Then for any $0 < \rho < \frac{1}{2}$,

$$\mathcal{L}^n(B_\rho(x_0) \cap \{u > 0\}) \geq \theta.\rho^n.$$

Corollary (**Porosity of the free boundary)**

There exists a constant $0 < \varsigma = \varsigma(n, \lambda, \Lambda, \gamma, \mu) \leq 1$ such that

$$\mathcal{H}^{n-\varsigma}\left(\partial\{u > 0\} \cap B_{\frac{1}{2}}\right) < \infty.$$

Global analysis results

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Theorem (Improved regularity along free boundary)

Let u be a nonnegative and bounded viscosity solution to

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and consider $x_0 \in \partial\{u > 0\} \setminus \partial\Omega$ a free boundary point. Then for $r_0 := \min\left\{\frac{1}{2}, \frac{dist(x_0, \partial\Omega)}{2}\right\}$ and any $x \in B_{r_0}(x_0) \cap \{u > 0\}$ there holds

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$$u(x) \leq C(n, \lambda, \Lambda, \gamma, \mu, \|\lambda_0\|_{L^\infty(\Omega)}, dist(x_0, \partial\Omega)).\|u\|_{L^\infty(\Omega)}.|x - x_0|^{\frac{\gamma+2}{\gamma+1-\mu}}.$$

Liouville type results

Theorem (**Liouville type theorem I)**

Let $u \in C^0(\mathbb{R}^n)$ be a viscosity solution to (1.1) such that u(0) = 0.

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Liouville type results

Theorem (Liouville type theorem I)

Let $u \in C^0(\mathbb{R}^n)$ be a viscosity solution to (1.1) such that u(0) = 0. If $\limsup_{|x|\to\infty} \frac{u(x)}{|x|^{\frac{\gamma+2}{\gamma+1-\mu}}} = 0$, *then* $u \equiv 0$.

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Liouville type results

Proof: Let $(r_j)_{j\in\mathbb{N}}$ such that $r_j \to \infty$ as $j \to \infty$.

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$$u_{r_j}(x) := \frac{u(r_j x)}{r_j^{\frac{\gamma+2}{\gamma+1-\mu}}}.$$

Thus,

$$\mathfrak{F}\left(r_j x, Du_{r_j}, D^2 u_{r_j}\right) = \lambda_0(r_j x).u_+^{\mu}(r_j x) \text{ in } B_1 \quad \text{and} \quad u_{r_j}(0) = 0.$$

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We claim that

$$\|u_{r_j}\|_{L^\infty(B_1)} = \text{o}(1).$$

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We claim that

$$\|u_{r_j}\|_{L^\infty(B_1)} = \text{o}(1).$$

Consider for each $j \in \mathbb{N}$, $x_j \in \overline{B_1}$ such that $u_{r_j}(x_j) = \sup\limits_{B_1} u_{r_j}(x)$,

Dead core type problems

we must analyse two possibilities:

1. If $\lim_{j\to\infty} |r_j x_j| = \infty$, then by our assumptions, we have

$$u_{r_j}(x_j) \le \frac{u(r_j x_j)}{|r_j x_j|^{\frac{\gamma+2}{\gamma+1-\mu}}} \to 0, \quad \text{as } j \to \infty.$$

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2. Otherwise, if $(r_j x_j)_{j\in\mathbb{N}}$ remains bounded, we obtain the same previous conclusion for $u_{r_j}(x_j)$, since u is a continuous function.

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we must analyse two possibilities:

1. If $\lim\limits_{j\to\infty} |r_j x_j| = \infty$, then by our assumptions, we have

$$u_{r_j}(x_j) \leq \frac{u(r_j x_j)}{|r_j x_j|^{\frac{\gamma+2}{\gamma+1-\mu}}} \to 0, \quad \text{as } j \to \infty.$$

2. Otherwise, if $(r_j x_j)_{j\in\mathbb{N}}$ remains bounded, we obtain the same previous conclusion for $u_{r_j}(x_j)$, since u is a continuous function.

By applying Theorem 1 we get

$$u_{r_j}(x) \leq \mathrm{o}(1) \cdot |x|^{\frac{\gamma+2}{\gamma+1-\mu}} \quad \text{in} \quad B_{\frac{r_0}{2}}, \;\; r_0 \ll 1. \tag{4.1}$$

Dead core type problems

Suppose that there exists a $z_0 \in \mathbb{R}^n \setminus \{0\}$ such that $u(z_0) > 0$.

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Finally, for $r_j \gg \frac{2}{r_0}|z_0|$, we obtain

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Dead core type problems

$$\frac{u(z_0)}{|z_0|^{\frac{\gamma+2}{\gamma+1-\mu}}} \leq \sup_{B_{|z_0|}} \frac{u(x)}{|x|^{\frac{\gamma+2}{\gamma+1-\mu}}} \leq \sup_{B_{\frac{r_0 r_j}{2}}} \frac{u(x)}{|x|^{\frac{\gamma+2}{\gamma+1-\mu}}} \leq \sup_{B_{\frac{r_0}{2}}} \frac{u_{r_j}(x)}{|x|^{\frac{\gamma+2}{\gamma+1-\mu}}} \leq \frac{u(z_0)}{7|z_0|^{\frac{\gamma+2}{\gamma+1-\mu}}}.$$

Dead core type problems

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This contradiction proves that $\sup_{\mathbb{R}^n} u(x) = 0$. □

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Liouville type results

Next, we shall study the following boundary value problem

$$\begin{cases} \mathfrak{F}(x, Du, D^2u) = \lambda_0 . u_+^{\mu}(x) & \text{in} \quad B_r(x_0) \\ u(x) = \theta & \text{on} \quad \partial B_r(x_0), \end{cases} \tag{4.2}$$

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where λ_0 and θ are strictly positive constants. Moreover, we assume the following property on $\mathfrak{F}$:

(F4) **[Invariance under rotations]** $\mathfrak{F}$ is an Hessian operator, this is,

$$\mathfrak{F}\left(Ox, O^t\overrightarrow{p}, O^tMO\right) = \mathfrak{F}\left(x, \overrightarrow{p}, M\right)$$

for any $(x, \overrightarrow{p}, M) \in \mathbb{R}^n \times (\mathbb{R}^n/\{0\}) \times Sym(n)$ and $O \in \mathcal{O}(n)$ (matrix orthogonal group).

Liouville type results

By uniqueness of the Dirichlet boundary problem and invariance under $\mathcal{O}(n)$ of $\mathfrak{F}$ viscosity solutions to such a boundary value problem are radially symmetric.

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Liouville type results

By uniqueness of the Dirichlet boundary problem and invariance under $\mathcal{O}(n)$ of $\mathfrak{F}$ viscosity solutions to such a boundary value problem are radially symmetric.
Now, let us treat the corresponding one-dimensional ODE to (4.2)

$$\left\{ \begin{array}{rcll} |v'(t)|^{\gamma}.v''(t) & = & \lambda_0.v^{\mu} & \text{in} \quad (0,T) \\ v(0) & = & 0 & \\ v(T) & = & \theta & \end{array} \right. \tag{4.3}$$

Straightforward calculation shows that $v(t) = \Theta(\lambda_0, \gamma, \mu).t^{\frac{\gamma+2}{\gamma+1-\mu}}$ is a solution to (4.3), where

$$\Theta(\lambda_0, \gamma, \mu) := \left(\lambda_0 \cdot \frac{(\gamma+1-\mu)^{\gamma+2}}{(\gamma+2)^{\gamma+1}(\gamma+1)}\right)^{\frac{1}{\gamma+1-\mu}}$$

and

$$T := \left(\frac{\theta}{\Theta(\lambda_0, \gamma, \mu)}\right)^{\frac{\gamma+1-\mu}{\gamma+2}}.$$

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Liouville type results

Now, fix $x_0 \in \mathbb{R}^n$ and $0 < r_0 < R_0$. Let us assume the *compatibility condition* for dead-core problem, namely $R_0 > T$.

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Liouville type results

Now, fix $x_0 \in \mathbb{R}^n$ and $0 < r_0 < R_0$. Let us assume the *compatibility condition* for dead-core problem, namely $R_0 > T$. For $r_0 = R_0 - T$ the radially symmetric function given by

$$v(x) := \Theta(\lambda_0, \gamma, \mu)\left(|x - x_0| - R_0 + \left(\frac{\theta}{\Theta(\lambda_0, \gamma, \mu)}\right)^{\frac{\gamma+1-\mu}{\gamma+2}}\right)_+^{\frac{\gamma+2}{\gamma+1-\mu}}$$

fulfils (4.2) in the viscosity sense.

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Liouville type results

Finally, let u be a viscosity solution to

$$\mathfrak{F}(x, Du, D^2u) = \lambda_0 . u_+^{\mu}(x), \quad \text{in} \quad \Omega \subset \mathbb{R}^n,$$

and $x_0 \in \Omega$ an interior point.

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$$\mathfrak{F}(x, Du, D^2u) = \lambda_0 . u_+^{\mu}(x), \quad \text{in} \quad \Omega \subset \mathbb{R}^n,$$

and $x_0 \in \Omega$ an interior point. If for some $0 < s < \mathrm{dist}(x_0, \partial\Omega)$, we have

$$\sup_{B_s(x_0)} u < \Theta(\lambda_0, \gamma, \mu).s^{\frac{\gamma+2}{\gamma+1-\mu}},$$

then x_0 must be a plateau point.

Liouville type results

Theorem (Liouville type theorem II)

Let u be a viscosity solution to

$$\mathfrak{F}\left(x, Du, D^2u\right) = \lambda_0 u_+^{\mu}(x) \quad \textit{in} \quad \mathbb{R}^n. \tag{4.4}$$

Then $u \equiv 0$ provided

$$\limsup_{|x|\to\infty} \frac{u(x)}{|x|^{\frac{\gamma+2}{\gamma+1-\mu}}} < \Theta(\lambda_0, \gamma, \mu). \tag{4.5}$$

Liouville type results

For $R_0 > 0$ fixed consider $\omega\colon \overline{B_{R_0}(0)} \to \mathbb{R}$ the unique viscosity solution to

$$\begin{cases} \mathfrak{F}\left(x, D\omega, D^2\omega\right) & = & \lambda_0 \omega_+^{\mu}(x) & \text{in} & B_{R_0}(0) \\ \omega(x) & = & \sup\limits_{\partial B_{R_0}(0)} u(x) & \text{on} & \partial B_{R_0}(0). \end{cases}$$

By Comparison Principle $u \leq \omega$ in $B_{R_0}(0)$.

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By Comparison Principle $u \leq \omega$ in $B_{R_0}(0)$. Moreover, by hypothesis (4.5)

$$\frac{1}{R_0^{\frac{\gamma+2}{\gamma+1-\mu}}} \cdot \sup_{\partial B_{R_0}(0)} u(x) \leq \frac{\mathcal{S}_{R_0}[u]}{R_0^{\frac{\gamma+2}{\gamma+1-\mu}}} \leq \sigma_{R_0} \cdot \Theta(\lambda_0, \gamma, \mu) \qquad (4.6)$$

Liouville type results

$$\omega(x) = \Theta(\lambda_0, \gamma, \mu)\left(|x| - R_0 + \left(\frac{\sup\limits_{\partial B_{R_0}(0)} u(x)}{\Theta(\lambda_0, \gamma, \mu)}\right)^{\frac{\gamma+1-\mu}{\gamma+2}}\right)_+^{\frac{\gamma+2}{\gamma+1-\mu}} \tag{4.7}$$

Liouville type results

$$\omega(x) = \Theta(\lambda_0, \gamma, \mu)\left(|x| - R_0 + \left(\frac{\sup\limits_{\partial B_{R_0}(0)} u(x)}{\Theta(\lambda_0, \gamma, \mu)}\right)^{\frac{\gamma+1-\mu}{\gamma+2}}\right)_+^{\frac{\gamma+2}{\gamma+1-\mu}} \tag{4.7}$$

Therefore, due to sentences (4.6) and (4.7) we can conclude that

$$u(x) \leq \Theta(\lambda_0, \gamma, \mu)\left(|x| - \left(1 - \sigma_{R_0}^{\frac{\gamma+1-\mu}{\gamma+2}}\right) R_0\right)_+^{\frac{\gamma+2}{\gamma+1-\mu}} \to 0 \quad \text{as} \quad R_0 \to \infty.$$

■

Some extensions and final comments

1. Fully nonlinear uniformly elliptic operators.

[Teixeira, Eduardo] Regularity for the fully nonlinear dead-core problem. **Math. Ann.** 364 (2016), no. 3-4, 1121-1134.

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3. Parabolic counterpart:

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3. Parabolic counterpart:

   ✓ p-Laplacian type operators;

   ✓ Fully nonlinear (uniformly parabolic) operators.

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