The Kesten-Stigum theorem in L^2

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  - ¿What can we say about those which survive until time $t \gg 1$?

The maximal eigenvalue of $\mathcal{L}_c$

Let $\mathcal{L}_c$ be the generator of ABMD(c) defined by the formula

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Goal. Understand ν.

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- If $r > \lambda$ then the dynamics is supercritical, i.e. for all $x > 0$

$$P(N_t^{(x)} \nrightarrow 0) > 0$$

where $N_t^{(x)}$ denotes the amount of particles above 0 at time t.

Kesten's theorem

- Kesten stated that the following result holds:

  Theorem. If $r > \lambda$ then for all $x > 0$ there exists a r.v. $W_\infty^{(x)}$ such that:

  i. For any Borel set $A \subseteq \mathbb{R}_{\geq 0}$

  $$\frac{N_t^{(x)}(A)}{\mathbb{E}(N_t^{(x)})} \xrightarrow{a.s.} \nu(A) \cdot W_\infty^{(x)},$$

  where $N_t^{(x)}(A)$ is the amount of particles in A at time t.

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  ii. $W_\infty^{(x)} > 0$ a.s. on the event of non-absorption $\{N_t^{(x)} \nrightarrow 0\}$.

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- No other proof has been obtained since.

Some previous results - Part I

Let X be a Markov process on some state space J such that:

1. X has an absorbing set ∂J.
2. If $T^{(x)}$ denotes the hitting time of ∂J, then

$$P(T_0^{(x)} > t) \asymp \varphi(x) \cdot p(t) \cdot e^{-\lambda t}$$

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To abbreviate, we shall say that X satisfies the **usual conditions** whenever it fulfills this description.

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- ABMD is not R-positive. In fact, it is R-transitive.

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Furthermore, if $W_\infty^{(x)}$ denotes the L^2-limit of $(W_t^{(x)})_{t\geq 0}$ then:

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Furthermore, if $W_\infty^{(x)}$ denotes the L^2-limit of $(W_t^{(x)})_{t\geq 0}$ then:

iv. We have

$$\mathbb{E}_x(W_\infty) = 1 \qquad \text{and} \qquad \mathbb{E}_x(W_\infty^2) = \sigma^{(x)}.$$

Our main results (II)

Theorem II. If $r > 2\lambda$ then for any Borel set $A \subseteq \mathbb{R}_{\geq 0}$ one has that

$$\frac{N_t^{(x)}(A)}{\mathbb{E}(N_t^{(x)})} \xrightarrow{L^2} \nu(A) \cdot W_\infty^{(x)}.$$

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Theorem III. $W_\infty^{(x)} > 0$ on the event of non-absorption.
In particular, for any Borel set $A \subseteq \mathbb{R}_{\geq 0}$ one has that

$$\frac{N_t^{(x)}(A)}{N_t^{(x)}} \longrightarrow \nu(A)$$

in probability and in L^p for every $p \geq 1$, on the event $\{N_t^{(x)} \nrightarrow 0\}$.

Structure of proof

The proof consists of three parts:

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II. Show that if $r > \lambda$ and $\sigma^{(x)} < +\infty$ then

$$\lim_{t\to+\infty} \left\| \frac{N_t^{(x)}(A)}{\mathbb{E}(N_t^{(x)})} - \nu(A) \cdot W_t^{(x)} \right\|_{L^2} = 0$$

for all $A \in \mathcal{B}(\mathbb{R}_{\geq 0})$.

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III. Show that if $r > 2\lambda$ then $P(W_\infty^{(x)} = 0) = P(N_t^{(x)} \to 0)$.

Sketch of proof : Part I

Note that for any $t, h \geq 0$

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We do this with the help of the many-to-few lemma.

The many-to-few lemma

Lemma. For any pair of measurable functions $f, g : \mathbb{R}_{\geq 0} \to \mathbb{R}$

$$\mathbb{E}_x\left(\sum_{u \in N_t} f(u)\right) = e^{rt}\mathbb{E}(f(X_t)).$$

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where (X, X') is a 2-spine with splitting time $E \sim \varepsilon(2r)$.

Sketch of proof : Part I (continued)

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$$P_x(T_0 > t) = xe^{cx} t^{-\frac{3}{2}} e^{-\lambda t}(1 + \varepsilon(x,t))$$

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ii. One has to deal with the random error terms $\varepsilon(X_s, t)$.

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- We show that this is indeed the case for B-ABMD.

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