Parabolic equations in oscillating thin domains

MARCONE C. PEREIRA[1]

Instituto de Matemática e Estatística
Universidade de São Paulo
São Paulo – Brazil

ECUACIONES DIFERENCIALES Y PROBABILIDAD
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Bahía Blanca, Argentina

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Here we analyze the semilinear parabolic equation

$$(\mathcal{Q}^\epsilon) \qquad \begin{cases} w_t^\epsilon - \Delta w^\epsilon + w^\epsilon = f(w^\epsilon) & \text{in } R^\epsilon, \\ \partial_{N^\epsilon} w^\epsilon = 0 & \text{on } \partial R^\epsilon, \end{cases} \quad t > 0$$

in a thin domain R^ϵ

$$R^\epsilon = \{(x,y) \in \mathbb{R}^2 \ : \ x \in (0,1), \ -\epsilon\, b_\epsilon(x) < y < \epsilon\, G_\epsilon(x)\}.$$

- b_ϵ and G_ϵ are uniformly bounded, smooth and positive in $(0,1)$.
- $f \in \mathcal{C}^2(\mathbb{R})$ is a dissipative nonlinearity: $\limsup_{|s|\to\infty} \frac{f(s)}{s} < 0$.

Under these conditions $(\mathcal{Q}^\epsilon)$ defines a **nonlinear semigroup**

$$T_\epsilon(t) : H^1(R^\epsilon) \mapsto H^1(R^\epsilon)$$

which is **gradient** and has a compact **global attractor**

$$\mathcal{A}_\epsilon \subset H^1(R^\epsilon) \quad \text{for each } \epsilon > 0.$$

We recall that:

1. A *nonlinear semigroup* is a map $T(t) : X \mapsto X$, $t \geq 0$, X being a complete metric space, which satisfies

   i) $T(0) = I$.
   ii) $T(t+s) = T(t)T(s)$, t and $s \geq 0$.
   iii) $T(t)x$ is a continuous function in (t, x).

2. T defines a *gradient system* if

   a) Each bounded positive orbit is precompact.
   b) It possesses a Lyapunov function $V : X \mapsto \mathbb{R}$ such that

      1. V is bounded below.
      2. $V(x) \to \infty$ as $\|x\| \to \infty$.
      3. $V(T(t)x)$ is not increasing in t for each x.
      4. If $V(T(t)x) = V(x)$ for all t, then x is an equilibrium point.

3. An attractor is a **maximal compact invariant set** which attracts all bounded sets of the phase space. It contains all the asymptotic dynamics of the system, and all global bounded solutions lie in the attractor.

- Here we are interested in **the behavior** of the nonlinear semigroup T_ϵ and the attractors $\mathcal{A}_\epsilon$ as $\epsilon \to 0$.
- Since we are in a thin domain situation we would like to get a **1D-parabolic equation** in order to *approximate* $(\mathcal{Q}^\epsilon)$.

Driven by 1D-equations

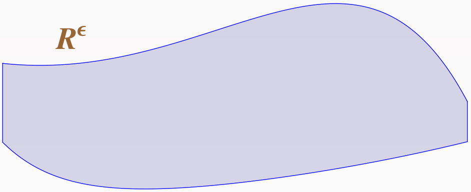
The dynamics of one-dimensional parabolic equations is much better understood than that ones in high dimensional euclidean spaces.[a]

[a] J. K. Hale, Math. Surveys Monograph (1998). P. Polacik, Handbook on Dynamical Systems (2002).

- As we will see, the **profile and dependence on** ϵ play an important role here.

Examples of thin domains

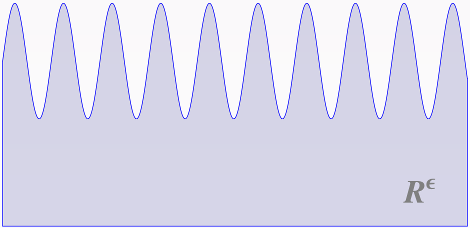
Standard thin domain



$$b_\epsilon(x) = b(x) \quad \text{and} \quad G_\epsilon(x) = g(x)$$

- *Parabolic problems*: J. K. Hale, G. Raugel JMPA (1992); M. Prizzi, K. P. Rybakowski JDE (2001); T. Elsken TMNA (2005); R. P. Silva Monatshefte fur Mathematik (2016).
- *Nonlocal problems*: J. D. Rossi, M. C. Pereira Submitted.

Resonant and oscillating thin domain

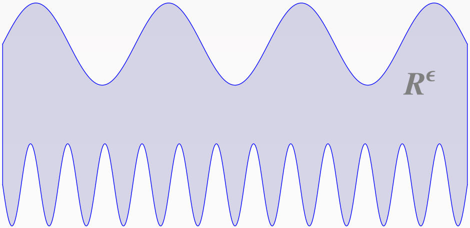


$$b_\epsilon(x) \equiv 0 \quad \text{and} \quad G_\epsilon(x) = g(x/\epsilon)$$

- *Parabolic problems*: A. N. Carvalho, J. M. Arrieta, M. C. Pereira, R. P. Silva Nonlinear Anal. (2011).
- *Elliptic problems*: T. A. Mel'nyk, A. V. Popov J. Math. Sci. (2009).

Examples of thin domains

Thin domain with double oscillatory behavior

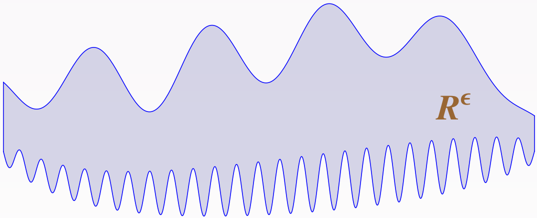


$$b_\epsilon(x) = b(x/\epsilon^\beta) \text{ with } \beta > 1 \quad \text{and} \quad G_\epsilon(x) = g(x/\epsilon)$$

- *Parabolic problems*: M. C. Pereira AMPA (2015).
- *Elliptic problems*: J. M. Arrieta, M. Villanueva-Pesquera MMAS (2014).

The class of thin domains

Variable profile and double oscillatory behavior



$$b_\epsilon(x) = m(x) + n(x)h(x/\epsilon^\beta) \qquad G_\epsilon(x) = j(x) + k(x)g(x/\epsilon^\gamma)$$

h and g, l_h and l_g-periodic functions

β and γ positive constants.

- ⋆ *Variable period and more*: J. M. Arrieta, M. Villanueva-Pesquera SIAM (2016) and PhD Thesis UCM (2016).
- ⋆ *Reaction terms concentrated on boundary*: S. Barros, M. C. Pereira JMAA (2016).

One approach is perform the change

$$x_1 = x, \quad x_2 = y/\epsilon,$$

which stretches R^ϵ in the y-direction by a factor $1/\epsilon$ transforming into

$$\Omega^\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 \ : \ x_1 \in (0,1) \text{ and } -b_\epsilon(x_1) < x_2 < G_\epsilon(x_1)\}.$$

Thus $(\mathcal{Q}^\epsilon)$ becomes

$$(\mathcal{P}^\epsilon) \qquad \begin{cases} u_t^\epsilon - \dfrac{\partial^2 u^\epsilon}{\partial {x_1}^2} - \dfrac{1}{\epsilon^2}\dfrac{\partial^2 u^\epsilon}{\partial {x_2}^2} + u^\epsilon = f(u^\epsilon) & \text{in } \Omega^\epsilon \\ \dfrac{\partial u^\epsilon}{\partial x_1} N_1^\epsilon + \dfrac{1}{\epsilon^2}\dfrac{\partial u^\epsilon}{\partial x_2} N_2^\epsilon = 0 & \text{on } \partial\Omega^\epsilon \end{cases} \quad t > 0,$$

where $N^\epsilon = (N_1^\epsilon, N_2^\epsilon)$ is the outward normal to the boundary of Ω^ϵ.

- Ω^ϵ is no longer a thin domain but can oscillate.
- By the factor $1/\epsilon^2$ is expected that solutions become homogeneous in x_2-direction. Hence the limiting solution will not depend on x_2 setting a 1D-limiting problem.

We rewrite problem $(\mathcal{P}^\epsilon)$ in an abstract form

$$\begin{cases} u_t^\epsilon + L_\epsilon u^\epsilon = \hat{f}_\epsilon(u^\epsilon) \\ u^\epsilon(0) = u_0^\epsilon \in Z_\epsilon^\alpha \end{cases}.$$

- $L_\epsilon : \mathcal{D}(L_\epsilon) \subset L^2(\Omega^\epsilon) \mapsto L^2(\Omega^\epsilon)$ is self adjoint, positive linear operator with compact resolvent

$$L_\epsilon u = -\frac{\partial^2 u}{\partial {x_1}^2} - \frac{1}{\epsilon^2}\frac{\partial^2 u}{\partial {x_2}^2} + u,$$

$$\mathcal{D}(L_\epsilon) = \left\{ u \in H^2(\Omega^\epsilon) \, : \, \partial_{x_1} u \, N_1^\epsilon + \frac{1}{\epsilon^2}\partial_{x_1} u \, N_2^\epsilon = 0 \text{ on } \partial\Omega^\epsilon \right\}.$$

- Z_ϵ^α is the fractional power scale from L_ϵ with $0 \leqslant \alpha \leqslant 1$.

$$Z_\epsilon^1 = \mathcal{D}(L_\epsilon), \quad Z_\epsilon^{1/2} = H^1(\Omega^\epsilon) \quad \text{and} \quad Z_\epsilon^0 = L^2(\Omega^\epsilon) := Z_\epsilon.$$

- $\hat{f}_\epsilon : Z_\epsilon^\alpha \mapsto Z_\epsilon : u^\epsilon \to f(u^\epsilon)$ is the Nemitskĭi operator.

- Under the growth and dissipative conditions $(\mathcal{P}^\epsilon)$ define a nonlinear semigroups for all $0 \leq \alpha \leq 1/2$ and $t > 0$

$$\{T_\epsilon(t) \,:\, t \geq 0\} \quad \text{in} \quad Z_\epsilon^\alpha.$$

- These dynamical systems are gradient and possess a family of compact global attractors

$$\{\mathscr{A}_\epsilon \subset Z_\epsilon^\alpha \,:\, \epsilon \in (0,1]\}$$

which lie in more regular spaces, namely $L^\infty(\Omega^\epsilon)$.

Getting the limit problem.[a]

[a] Arrieta, Carvalho and Lozada-Cruz, JDE (2006) and (2009).

⋆ First we study the family of resolvent operators

$$L_\epsilon^{-1} : L^2(\Omega^\epsilon) \mapsto L^2(\Omega^\epsilon).$$

We pass to the limit in the elliptic problem

$$\begin{cases} -\dfrac{\partial^2 u^\epsilon}{\partial {x_1}^2} - \dfrac{1}{\epsilon^2}\dfrac{\partial^2 u^\epsilon}{\partial {x_2}^2} + u^\epsilon = f^\epsilon & \text{in } \Omega^\epsilon \\ \dfrac{\partial u^\epsilon}{\partial x_1} N_1^\epsilon + \dfrac{1}{\epsilon^2}\dfrac{\partial u^\epsilon}{\partial x_2} N_2^\epsilon = 0 & \text{on } \partial\Omega^\epsilon \end{cases}$$

assuming $\|f^\epsilon\|_{L^2(\Omega^\epsilon)} \leq C$ for some C independent of ϵ to obtain

the **limit equation**, and then, the **limit operator** L_0.

Since the spaces can depend on ϵ
we need an approach to compare them.

Here we introduce the following operators

i) $E_\epsilon : Z_0 = L^2(0,1) \mapsto Z_\epsilon = L^2(\Omega^\epsilon)$

$$(E_\epsilon u)(x_1, x_2) = u(x_1), \quad (x_1, x_2) \in \Omega^\epsilon.$$

ii) $M_\epsilon : Z_\epsilon \mapsto Z_0$

$$(M_\epsilon u^\epsilon)(x) = \frac{1}{p_\epsilon(x)} \int_{-b_\epsilon(x)}^{G_\epsilon(x)} u^\epsilon(x, s)\, ds, \quad x \in (0,1)$$

where

$$p_\epsilon = b_\epsilon + G_\epsilon \to p \quad \text{weakly}^* \text{ in } L^\infty(0,1)$$

as $\epsilon \to 0$.

Indeed, if we have a family of adjoint, positive linear operators

$$\{L_\epsilon^{-1}\}_{\epsilon\in[0,1]}$$

with compact resolvent satisfying

$$\|L_\epsilon^{-1} - E_\epsilon L_0^{-1} M_\epsilon\|_{\mathcal{L}(Z_\epsilon)} \leq \nu(\epsilon) \quad \forall \epsilon \in (0, \epsilon_0)$$

for some $\epsilon_0 > 0$ with $\nu(\epsilon) \to 0$ as $\epsilon \to 0$ we get

I) Upper and lower semicontinuity of eigenvalues and eigenfunctions of L_ϵ at $\epsilon = 0$.

II) Continuity of the semigroup: for some $0 \leqslant \alpha < 1/2$ and $\omega \in (0,1)$, there exists $\nu_\alpha(\epsilon) \overset{\epsilon\to 0}{\longrightarrow} 0$, such that

$$\|e^{-L_\epsilon t} - E_\epsilon e^{-L_0 t} M_\epsilon\|_{\mathcal{L}(Z_\epsilon, Z_\epsilon^\alpha)} \leqslant \nu_\alpha(\epsilon) e^{-\omega t} t^{\alpha-1}$$

for all $t > 0$.

III) Continuity of the nonlinear semigroup in bounded intervals by the variation of constants formula

$$T_\epsilon(t)u_0^\epsilon = e^{-L_\epsilon t}u_0^\epsilon + \int_0^t e^{-L_\epsilon(t-s)}\hat{f}_\epsilon(T_\epsilon(s)u_0^\epsilon)\, ds.$$

IV) Upper semicontinuity of attractors at $\epsilon = 0$ in Z_ϵ^α

$$\sup_{\varphi^\epsilon \in \mathscr{A}_\epsilon} \left[\inf_{\varphi \in \mathscr{A}_0} \left\{ \|\varphi^\epsilon - E_\epsilon \varphi\|_{Z_\epsilon^\alpha} \right\} \right] \to 0, \text{ as } \epsilon \to 0$$

also as a consequence of the uniformly bounds given by *Arrieta, Carvalho, Rodríguez-Bernal, Comm. Part. Diff. Eq. (2000).*

Note that our *abstract limit problem* is

$$\begin{cases} u_t + L_0 u = \hat{f}_0(u) \\ u(0) = u_0 \in Z_0^\alpha \end{cases}.$$

Until here, we need to identify the limit operator L_0 in such way that

$$\|L_\epsilon^{-1} - E_\epsilon L_0^{-1} M_\epsilon\|_{\mathcal{L}(Z_\epsilon)} \leq \nu(\epsilon) \quad \forall \epsilon \in (0, \epsilon_0).$$

- We remember that $Z_\epsilon = L^2(\Omega^\epsilon)$.

Let us analyze the elliptic problem.

Its variational formulation is find $u^\epsilon \in H^1(\Omega^\epsilon)$ such that

$$\int_{\Omega^\epsilon} \Big\{ \frac{\partial u^\epsilon}{\partial x_1}\frac{\partial \varphi}{\partial x_1} + \frac{1}{\epsilon^2}\frac{\partial u^\epsilon}{\partial x_2}\frac{\partial \varphi}{\partial x_2} + u^\epsilon \varphi \Big\} dx_1 dx_2 = \int_{\Omega^\epsilon} f^\epsilon \varphi dx_1 dx_2, \ \forall \varphi \in H^1(\Omega^\epsilon).$$

Taking $\varphi = u^\epsilon$ we get

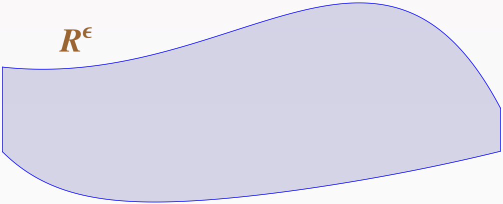
$$\Big\| \frac{\partial u^\epsilon}{\partial x_1} \Big\|^2_{L^2(\Omega^\epsilon)} + \frac{1}{\epsilon^2} \Big\| \frac{\partial u^\epsilon}{\partial x_2} \Big\|^2_{L^2(\Omega^\epsilon)} + \|u^\epsilon\|^2_{L^2(\Omega^\epsilon)} \leq \|f^\epsilon\|_{L^2(\Omega^\epsilon)} \|u^\epsilon\|_{L^2(\Omega^\epsilon)}.$$

Since $\|f^\epsilon\|_{L^2(\Omega^\epsilon)} \leq C$ we have for $\epsilon \in (0,1]$

- $\|u^\epsilon\|_{H^1(\Omega^\epsilon)}$ uniformly bounded and
- $\Big\| \frac{\partial u^\epsilon}{\partial x_2} \Big\|_{L^2(\Omega^\epsilon)} \leq \epsilon\, C.$

⋆ *The dependence on ϵ plays an important role here.*

Hale and Raugel, J. Math. Pures et Appl. (1992).



$$b_\epsilon(x) = b(x) \quad \text{and} \quad G_\epsilon(x) = g(x).$$

Strong convergence to the limit problem

$$\begin{cases} -\dfrac{1}{c(x)}(c(x)u_x(x))_x + u(x) = f(x) \quad x \in (0,1) \\ u_x(0) = u_x(1) = 0, \end{cases}$$

$$c(x) = g(x) + b(x).$$

Passing to the limit

- Here $\Omega^\epsilon = \Omega$ does not depende on ϵ

$$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in (0,1) \text{ and } -b(x_1) < x_2 < g(x_1)\}.$$

- $\|f^\epsilon\|_{L^2(\Omega)} \le C$ we have $\|u^\epsilon\|_{H^1(\Omega)}$ uniformly bounded, thus,

$$u^\epsilon \to u_0 \quad \text{weakly in } H^1(\Omega)$$

for some $u_0 \in H^1(\Omega)$.

- Since $\left\| \frac{\partial u^\epsilon}{\partial x_2} \right\|_{L^2(\Omega)} \le \epsilon\, C$ we have $u_0(x_1, x_2) = u_0(x_1)$

$$\int_\Omega u_0 \frac{\partial \varphi}{\partial x_2}\, dx_1 dx_2 = \lim_{\epsilon \to 0} \int_\Omega u^\epsilon \frac{\partial \varphi}{\partial x_2}\, dx_1 dx_2$$
$$= -\lim_{\epsilon \to 0} \int_\Omega \frac{\partial u^\epsilon}{\partial x_2} \varphi\, dx_1 dx_2 = 0.$$

We pass to the limit in the variational formulation

$$\int_\Omega \left\{ \frac{\partial u^\epsilon}{\partial x_1}\frac{\partial \varphi}{\partial x_1} + \frac{1}{\epsilon^2}\frac{\partial u^\epsilon}{\partial x_2}\frac{\partial \varphi}{\partial x_2} + u^\epsilon \varphi \right\} dx_1 dx_2 = \int_\Omega f^\epsilon \varphi dx_1 dx_2$$

taking $\varphi(x_1, x_2) = \varphi(x_1) \in H^1(\Omega)$

$$\int_\Omega \left\{ \frac{\partial u_0}{\partial x_1}\frac{\partial \varphi}{\partial x_1} + u_0 \varphi \right\} dx_1 dx_2 = \int_0^1 \hat{f} \varphi dx_1$$

where $\hat{f} \in L^2(0,1)$ is the weak limit of

$$\hat{f}^\epsilon(x_1) = \int_{-b(x_1)}^{g(x_1)} f^\epsilon(x_1, x_2)\, dx_2 \qquad x_1 \in (0,1).$$

Since u_0 and φ **do not depend on** x_2

$$\int_0^1 \left(\int_{-b(x_1)}^{g(x_1)} dx_2 \right) \left(\frac{\partial u_0}{\partial x_1} \frac{\partial \varphi}{\partial x_1} + \varphi\, u_0 \right) dx_1$$

$$= \int_0^1 c(x_1) \left\{ \frac{\partial u_0}{\partial x_1} \frac{\partial \varphi}{\partial x_1} + u_0 \varphi \right\} dx_1 = \int_0^1 \hat{f}\, \varphi dx_1.$$

That is, u_0 is solution of

$$\begin{cases} -\dfrac{1}{c(x)}(c(x) u_x(x))_x + u(x) = \dfrac{\hat{f}(x)}{c(x)} & x \in (0,1) \\ u_x(0) = u_x(1) = 0, \end{cases}$$

where

$$c(x) = g(x) + b(x) \qquad \text{and} \qquad f = \frac{\hat{f}(x)}{c(x)}.$$

$$\begin{aligned}
\int_0^1 c \, \frac{du_0}{dx}^2 dx &= \int_\Omega |\nabla u_0|^2 \, dx_1 dx_2 \\
&\leq \liminf_{\epsilon \in (0,1)} \int_\Omega |\nabla u^\epsilon|^2 \, dx_1 dx_2 \leq \limsup_{\epsilon \in (0,1)} \int_\Omega |\nabla u^\epsilon|^2 \, dx_1 dx_2 \\
&\leq \limsup_{\epsilon \in (0,1)} \int_\Omega \left\{ \frac{\partial u^\epsilon}{\partial x_1}^2 + \frac{1}{\epsilon^2} \frac{\partial u^\epsilon}{\partial x_2}^2 \right\} dx_1 dx_2 \\
&\leq -\int_0^1 c \, u_0^2 \, dx + \int_0^1 c \, f \, u_0 \, dx = \int_0^1 c \, \frac{du_0}{dx}^2 dx.
\end{aligned}$$

Hence $\|u^\epsilon\|_{H^1(\Omega)} \to \|u_0\|_{H^1(\Omega)}$, and then

$$u^\epsilon \to u_0 \quad \text{strongly in } H^1(\Omega).$$

In the abstract form

$$u^\epsilon = L_\epsilon^{-1} f^\epsilon \quad \text{and} \quad u_0 = L_0^{-1} f$$

with

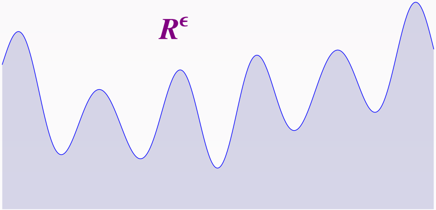
$$L_0 u = -\frac{1}{c(x)}(c(x)\, u_x)_x + u$$

$$\mathcal{D}(L_0) = \left\{ u \in H^2(0,1) \,|\, u'(0) = u'(1) = 0 \right\}.$$

For this case can be proved

$$\| L_\epsilon^{-1} - E_\epsilon L_0^{-1} M_\epsilon \|_{\mathcal{L}(L^2(\Omega), H^1(\Omega))} = O(\epsilon).$$

Arrieta, Ph.D. Thesis, Georgia Tech (1991).



$$G_\epsilon(x) = m(x) + n(x)g(x/\epsilon^\gamma) \quad \text{and}$$
$$b_\epsilon(x) \equiv 0 \quad \text{with} \quad 0 < \gamma < 1.$$

The limit problem is

$$\begin{cases} -\dfrac{1}{r(x)}\left(\dfrac{1}{s(x)}u_x(x)\right)_x + u(x) = f(x) \quad x \in (0,1) \\ u_x(0) = u_x(1) = 0 \end{cases}$$

where

i) $G_\epsilon(x) \rightharpoonup r(x), \quad w - L^2(0,1)$

ii) $\frac{1}{G_\epsilon(x)} \rightharpoonup s(x), \quad w - L^2(0,1).$

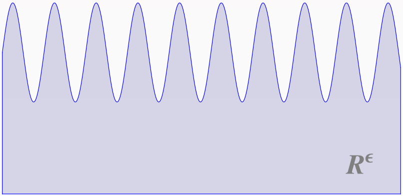
As observed by Bensoussan, Lions and Papanicolaou (1978)
H^1**-strong convergence is actually false**.

Now let us consider the case

$$b_\epsilon(x) \equiv 0 \quad \text{and} \quad G_\epsilon(x) = g(x/\epsilon),\ \gamma = 1$$

$$\Omega^\epsilon = \{(x, y) \in \mathbb{R}^2 \ : \ 0 < x < 1, 0 < y < g(x/\epsilon)\}$$

where $g : \mathbb{R} \mapsto \mathbb{R}$ is a smooth periodic function with period L.



In order to do that we need the following ingredients:

- The Multiple Scale method.[2]
- Extension operators P_ϵ.
- Oscillatory test functions method of Tartar.[3]

[2] Bensoussan, Lions, Papanicolaou, (1978).

[3] D. Cioranescu and J. Paulin, (1998).

We obtain:

A. Weak convergence with $P_\epsilon : H^1(\Omega^\epsilon) \mapsto H^1(\Omega)$

$$P_\epsilon u^\epsilon \to u_0 \quad \text{weakly in } H^1(\Omega) \quad \text{as } \epsilon \to 0.$$

where u_0 satisfies the *homogenized equation* given by

$$\begin{cases} -q\, u''(x) + u(x) = f(x), \quad x \in (0,1) \\ u'(0) = u'(1) = 0 \end{cases}$$

with

$$q = \frac{1}{|Y^*|} \int_{Y^*} \left\{ 1 - \frac{\partial X}{\partial y}(y,z) \right\} dydz$$

$$Y^* = \{(y,z) \,:\, y \in (0,L),\, 0 < z < g(y)\}$$

and X is given by

$$\begin{cases} -\Delta_{y,z} X = 0 \quad \text{in } Y^* \\ \partial_N X = N_1 \quad \text{on } B \\ X \quad L-\text{periodic in } y \end{cases}$$

where B is the upper and lower boundary of ∂Y^*.

B. Using **corrector approach** we also get strong convergence in $H^1(R^\epsilon)$ with an appropriated norm:

$$\epsilon^{-1/2} \| w^\epsilon - w_0 - \epsilon w_1 \|_{H^1(R^\epsilon)} \leq C \, \epsilon^{1/2}, \quad \text{for } \epsilon \approx 0$$

where w_0 is the solution of the homogenized equation and

$$w_1(x_1, x_2) = -X(x_1/\epsilon, x_2/\epsilon) \frac{dw_0}{dx}(x_1) \quad \text{for } (x_1, x_2) \in R^\epsilon.$$

is the *first order corrector*.[4]

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[4] M. Pereira and R. Silva, DCDS (2013).

For this case setting

$$u^\epsilon = L_\epsilon^{-1} f^\epsilon \quad \text{and} \quad u_0 = L_0^{-1} f$$

with

$$L_0 u = -q\, u_{xx} + u$$
$$\mathcal{D}(L_0) = \{u \in H^2(0,1) \,|\, u'(0) = u'(1) = 0\}$$

we get

$$\|L_\epsilon^{-1} - E_\epsilon L_0^{-1} M_\epsilon\|_{L^2(\Omega^\epsilon)} \to 0 \quad \text{as } \epsilon \to 0.$$

- Note that it is enough to guarantee continuity of the nonlinear semigroup in bounded time, as well as the upper semicontinuity of the attractors.

Other elliptic cases with oscillating boundary.

a) $b_\epsilon \equiv 0$ and $\gamma = 1$.

- T. Mel'nik and A. Popov, J. Math. Scien. (2009).
- J. Arrieta, A. Carvalho, M. Pereira, R. Silva, N. Anal. (2011).
- J. Arrieta and M. Pereira, J.M.P.Appl. (2011).
- J. Arrieta and Villanueva-Pesquera, SIAM J. Math. Anal. (2016). (Variable period.)

b) $b_\epsilon \equiv 0$ and $\gamma > 1$.

- N. Ansini and A. Braides, J.d'Anal. Math. (2001).
- J. Arrieta and M. Pereira, JMAA (2013).

c) $\beta = 1$ and $\gamma > 1$.

- J. Arrieta and M. Villanueva-Pesquera, MMAS (2014).
- M. Pereira, Ann. Mat. P. Appl. (2015).

d) $\beta < 1$ and $\gamma > 1$; β and $\gamma > 1$; β and $\gamma < 1$.

- M. Villanueva-Pesquera, PhD Thesis UCM (2016).

Lower semicontinuity of the attractors:

$$\sup_{\varphi\in\mathscr{A}_0}\left[\inf_{\varphi^\epsilon\in\mathscr{A}_\epsilon}\left\{\|\varphi^\epsilon - E_\epsilon\varphi\|_{Z^\alpha_\epsilon}\right\}\right]\to 0,\ \text{as } \epsilon\to 0.$$

According to

[J. K. Hale and G. Raugel, Ann. Mat. Pura Appl. (1989)]

If the limiting equation is gradient, has a finite number of equilibria, all of them hyperbolic, the perturbed nonlinear semigroups vary continuously, the sets of equilibria have fixed finite cardinality and vary continuously with the parameter, and the local unstable manifolds of the perturbed problems are lower semicontinuous, then the family of attractors behaves lower semicontinuously.[a]

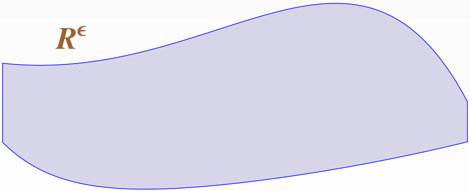
[a] See also Arrieta, Carvalho, Langa, Rodríguez-Bernal, J. of Dyn. Syst. and P. Diff. Eq. (2012).

a) Without oscillatory boundary.

- J. Hale and G. Raugel, JMPA (1992).
- J. Arrieta and E. Santamaría, PhD Thesis UCM (2013).
  Here they prove

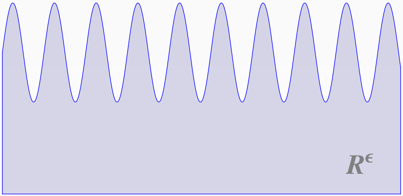
$$\mathrm{dist}_{H^1(\cdot)}(\mathscr{A}_0, \mathscr{A}_\epsilon) \leq C\,\epsilon\,|\ln(\epsilon)|$$

  where dist_H is the symmetric distance of Hausdorff.



b) With oscillatory boundary: $b \equiv 0$ and $G_\epsilon(x) = g(x/\epsilon)$.

- J. Arrieta, A. Carvalho, M. Pereira, R. Silva, Non. Analysis (2011).



The other cases and estimate to the attractors are still being investigated.

'Apesar de você,
Amanhã há de ser
outro dia.'

Chico Buarque[5]

THANK YOU.

[5] Pela democracia.