

Branching Brownian particles with spatial selection and the KPP equation

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joint work with Pablo Groisman y Matthieu Jonckheere.

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Infinitely many travelling waves

$$u(x, t) = w(x - ct), \quad \text{with } c = c(f).$$

But only one has physical meaning, $c = \sqrt{2}$.

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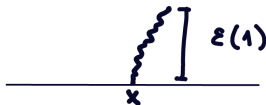
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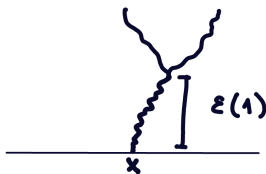
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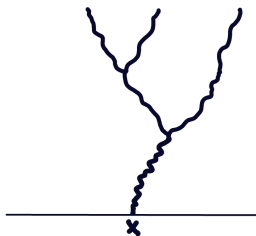
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Selection Principle

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- Brunet, Derrida; 1997,2001,...:
Effect of microscopic noise in front propagation,
mainly numerical and heuristic arguments.

$$v_N - v_{min} \simeq \frac{-K}{\log^2 N}$$

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We are interested in
$$\mathbf{F}_N(\mathbf{x}, \mathbf{t}) = \frac{\sum_{i=1}^N \mathbf{1}_{\xi_t^i \leq \mathbf{x}}}{N}.$$

$$\text{LGN: } (\xi^i)_{i \geq 1}, \quad \xi^i \sim X, i.i.d \quad \Rightarrow \quad F_N(x) \xrightarrow[N \rightarrow \infty]{P} P(X \leq x) \quad \forall x.$$

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Hydrodynamic limit

If $\bar{\xi}_0^N \sim \rho_0^N$ such that $F_N(x, 0) \xrightarrow[N \rightarrow \infty]{P} u_0(x) \quad \forall x$, then

$$F_N(x, t) \xrightarrow[N \rightarrow \infty]{P} u(x, t), \quad \forall x,$$

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$$\begin{aligned} \partial_t u_N &= \overbrace{\partial_{xx} u_N}^{\text{Difussion}} - \overbrace{\frac{N}{N-1} E[F_N(1 - F_N)]}^{\text{jumps}} \\ &= \partial_{xx} u_N - u_N(1 - u_N) + \frac{N}{N-1} V(F_N(x, t)) + \frac{1}{N-1} (u_N(1 - u_N)). \end{aligned}$$

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Variance / Propagation of chaos

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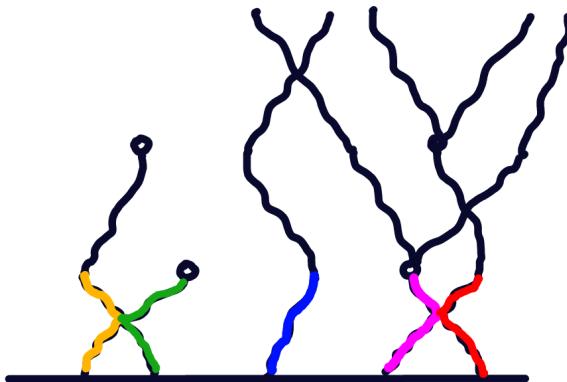
Lemma

$$\sup_{\xi_0^N} \frac{1}{N^2} \left| \sum_{i=1}^N \sum_j P(\xi_t^i \leq x, \xi_t^j \leq x) - P(\xi_t^i \leq x)P(\xi_t^j \leq x) \right| \leq \frac{2e^t}{N}.$$

Idea: Embed our process in N independent BBM

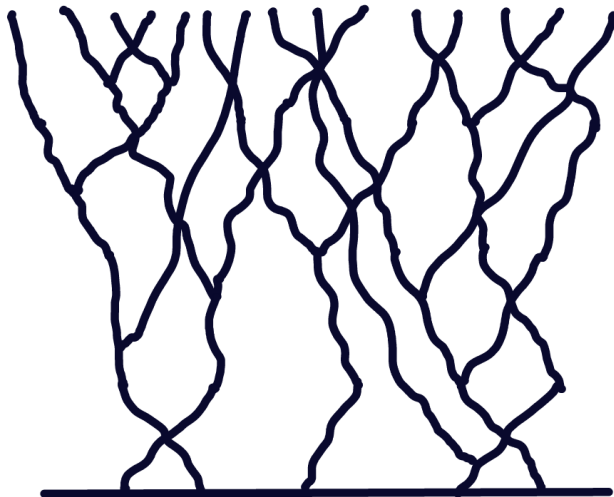
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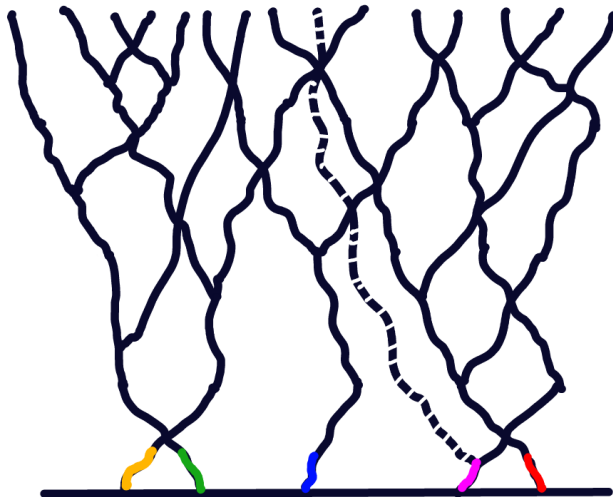
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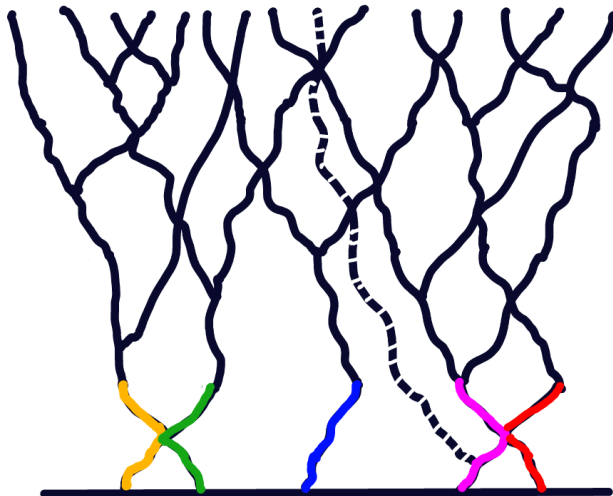
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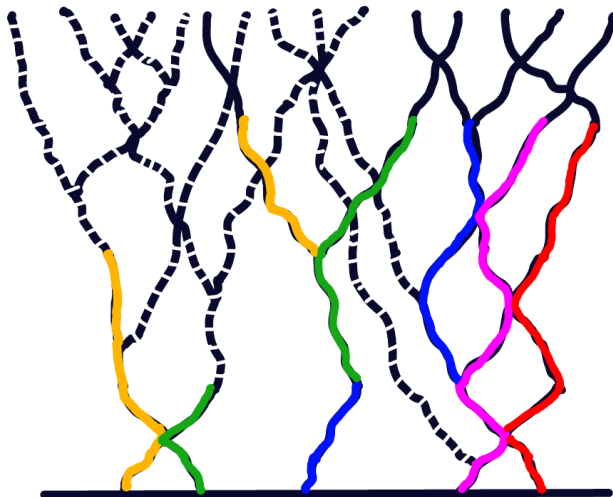
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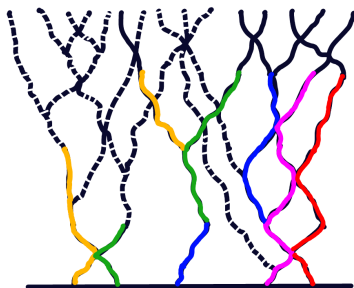
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Maximum of the BBM

Let $M_t := \max\{X_t^1, \dots, X_t^{N_t}\}$. Then

$$\lim_{t \rightarrow \infty} \frac{M_t}{t} = \sqrt{2}, \quad \text{c.s.}$$

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$$\stackrel{(2)}{\Rightarrow} c_2 = v_N$$

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$$\lim_{t \rightarrow \infty} \lim_{b \rightarrow \infty} \int_{m(t)-b}^{m(t)+b} u^0(x, t)(1 - u^0(x, t)) dx = \sqrt{2}$$

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Gracias por su atención!