Asymptotic Direction for Random Walk in Strong Mixing Random Environment

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Outline

What is an environment?

Random walk in a random environment in dimension d is a canonical Markov chain $(X)_{n\geq 0}$ with state space in $\mathbb{Z}^d$ where the transition probabilities to nearest neighbors are random.

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- An environment ω is an element of the set $\Omega := (\mathcal{P}_\kappa)^{\mathbb{Z}^d}$, we use the notation $\omega(x, e)$, where $x \in \mathbb{Z}^d,\ e \in \mathbb{Z}^d, \mid e \mid = 1$ to mean the x coordinate evaluated at e. Further, let $\mathbb{P}$ be the law of the environment ω, which is a p.m. on $\mathcal{W}$ the canonical σ-algebra on Ω.

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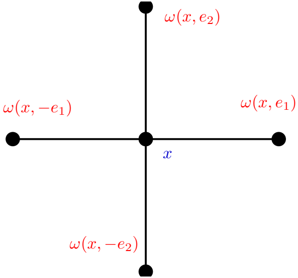

Definitions of Quenched and Annealed Laws

- Let ω be an environment. For $x \in \mathbb{Z}^d$ we define the quenched law $P_{x,\omega}$ as the law of the Markov chain $(X_n)_{n\geq 0}$ with state space $\mathbb{Z}^d$, satisfying

$$P_{x,\omega}[X_0 = x] = 1$$

and stationary transition probabilities

$$P_{x,\omega}[X_{n+1} = X_n + e \mid X_n] = \omega(X_n, e), \ \ |e| = 1$$

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- Define the annealed law P_x via

$$P_x := \mathbb{P} \otimes P_{x,\omega}$$

The Cone $C(x, l, \alpha)$

Let $l \in \mathbb{S}^{d-1} \cap \mathbb{Q}^d$ and R a rotation on $\mathbb{R}^d$ such that $R(e_1) = l$. Define for fixed small $\alpha > 0$, $(2d-1)$-directions for integers $j \in [2, d]$.

$$l_{+j} = l + \alpha R(e_j) / \mid l + \alpha R(e_j) \mid$$
$$l_{-j} = l - \alpha R(e_j) / \mid l_{-j} = l - \alpha R(e_j) \mid$$

Then for $x \in \mathbb{R}^d$ the cone $C(x, l, \alpha)$ is the set

$$\{z \in \mathbb{Z}^d, (z - x) \cdot l_j \geq 0 \text{ for } j \in [2, d]\}$$

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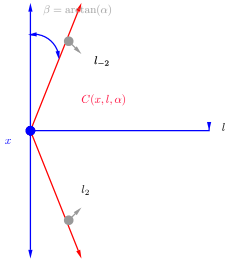

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  1. $\mathbb{P}$ is stationary.
  2. For any sets A and B and $r > 0$, where $A \in \sigma\{\omega(y,\cdot), y \cdot l \leq 0\}, \mathbb{P}(A) > 0$ and $B \in \sigma\{\omega(y,\cdot), y \in C(rl, l, \alpha)\}$ there exists a function $\phi : [0,\infty) \to [0,\infty)$ with $\lim_{t\to\infty} \phi(t) = 0$, such that
  $$|\mathbb{P}(A \cap B)/\mathbb{P}(A) - \mathbb{P}(B)| \leq \phi(r \mid l \mid_1)$$

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- On the other hand, we say that $\mathbb{P}$ has a product structure or the random environment is i.i.d. if there exists some fixed law μ on $\mathcal{P}_\kappa$ so that $\mathbb{P} = \mu^{\otimes \mathbb{Z}^d}$.

Definitions of Asymptotic Behaviors for RWRE

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- we say that there exist an asymptotic direction $\hat{v} \neq 0$ for the RWRE if P_0- almost surely

$$\lim X_n / \mid X_n \mid = \hat{v}.$$

Simenhaus Theorem

Conjecture, under $d \geq 2$

When the random environment is i.i.d. it is conjectured that any RWRE which is transient in direction l is ballistic in that direction.

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Important example

We have got an example of RWRE when the random environment is strong mixing on cones which is transient in a neighborhood of l but is not ballistic in any direction.

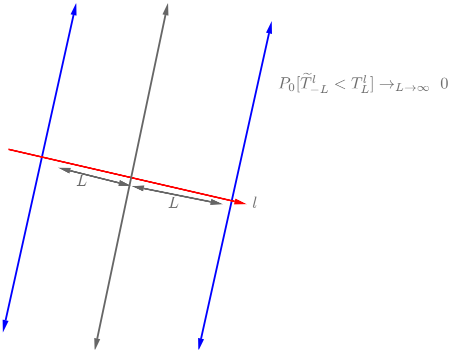
Direction of our research

Main purpose

Our purpose is to obtain mild conditions on the walk in order to get asymptotic laws. We want to build a bridge between i.i.d. renormalization techniques and strong mixing . A first step in the bridge construction being the result given here.

Motivation

Transience in direction l implies



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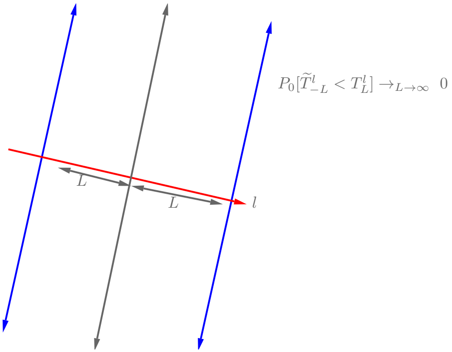


The functional control of these probabilities has been successful so as to prove ballistic behavior in the i.i.d. environment case.

Polynomial conditional criteria

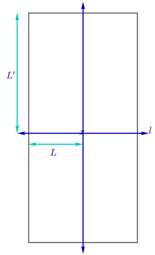
- For each $A \subset \mathbb{Z}^d$ we define

$$\partial A := \{z \in \mathbb{Z}^d : z \notin A, \text{ there exists some } y \in A \text{ such that } |y - z| = 1\}.$$

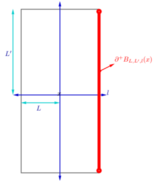
- Define also the stopping time

$$T_A := \inf\{n \geq 0 : X_n \notin A\}.$$

- Given $L, L' > 0$, $x \in \mathbb{Z}^d$ and $l \in \mathbb{S}^{d-1}$ we define the boxes $B_{L,L',l}(x) :=$



- Define the *positive boundary* of $B_{L,L',l}(x)$, denoted by $\partial^+ B_{L,L',l}(x)$, as



Polynomial Conditional Criteria

- Define also the half-space

$$H_{x,l} := \{y \in \mathbb{Z}^d : y \cdot l < x \cdot l\},$$

- And the corresponding σ-algebra of the environment on that half-space

$$\mathcal{H}_{x,l} := \sigma(\omega(y) : y \in H_{x,l}).$$

Polynomial Conditional Criteria

For $M \geq 1$, we say that the *non-effective polynomial* conditional criteria $(PC)_{M,c}|l$ is satisfied if there exists some $c > 0$ so that for $y \in H_{0,l}$ one has that

$$\lim_{L\to\infty} L^M \sup P_0 \left[X_{T_{B_{L,cL,l}(0)}} \notin \partial^+ B_{L,cL,l}(0), T_{B_{L,cL,l}(0)} < T_{H_{y,l}} | \mathcal{H}_{y,l} \right] = 0, \tag{1}$$

where the supremum is taken over all possible environments to the left of $y \cdot l$.

Main Result

Theorem (Ramírez, G.)

Let $l \in \mathbb{S}^{d-1} \cap \mathbb{Q}^d$, $M > 6d$, $c > 0$ *and* $0 < \alpha \leq \min\{\frac{1}{9}, \frac{1}{2c+1}\}$. *Consider a d- dimensional random walk in a random environment with stationary law satisfying the the uniform ellipticity condition* $(UE)|l$*, the cone mixing condition* $(CM)_{\alpha,\phi}|l$ *and the non-effective polynomial condition* $(PC)_{M,c}|l$*. Then, there exists a deterministic* $\hat{v} \in \mathbb{S}^{d-1}$ *such that* P_0*-a.s. one has that*

$$\lim_{n\to\infty} \frac{X_n}{|X_n|} = \hat{v}.$$

Proof of Theorem

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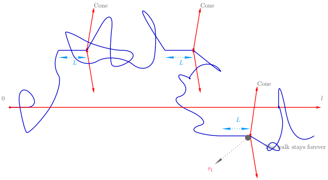
2. For $L > 0$ fixed but large, there exist a random time sequence $(\tau_i(L))_{i \geq 1}$ so that τ_1



and for $i \geq 2$, we define $\tau_i = \tau_1 \circ \theta_{\tau_{i-1}}$. They are finite thanks to Step 1.

Proof of the Theorem

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4. Using Step 1, we use the Comets and Zeitouni coupling decomposition, i.e. for $n \geq 2$

$$\kappa^L(X_{\tau_n} - X_{\tau_{n-1}}) = \tilde{X}_n + Y_n$$

where $(\tilde{X}_n)_{n\geq 2}$ is i.i.d. sequence and $\tilde{X}_2$ distributes as $\kappa^L X_{\tau_1}$ under $P_0[\cdot \mid D' = \infty]$. Then with the help of step 3 we get rid the fluctuation made by Y_n.

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5. The previous step and standard arguments prove the Theorem.

Known results in mixing random environment

Strong law under stronger assumptions

Comets and Zeitouni proved ballistic behavior when the environment is cone mixing, however under a strong assumption of integrability for the regeneration times.

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Thanks