Gibbs measures on permutations over $\mathbb{Z}^d$ with random multiplicities

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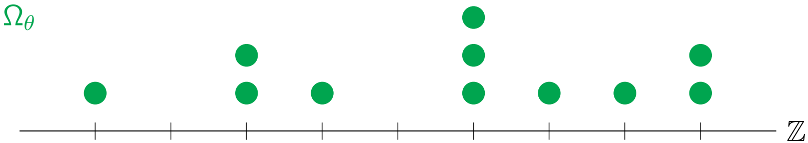
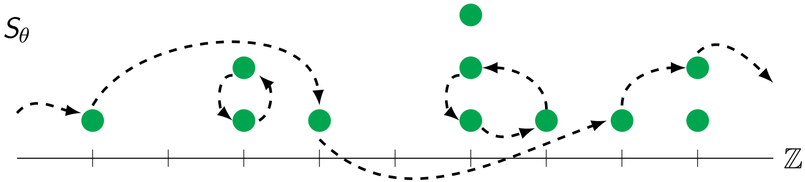


- Let S_θ be the set of permutations of Ω_θ.



- If $\theta(x) = 1$ for all $x \in \mathbb{Z}^d$, we recover the lattice $\mathbb{Z}^d$.
- In general, we are interested in the case when $\{\theta(x)\}_{x \in \mathbb{Z}^d}$ is an i.i.d sequence.

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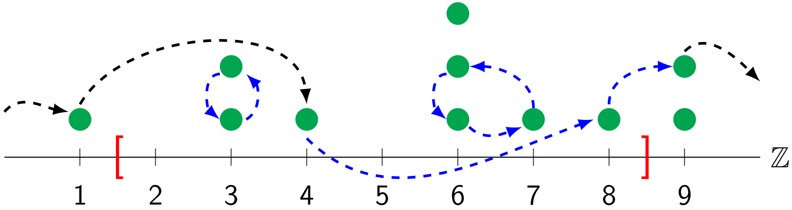

- Only the jumps in blue contribute to the Hamiltonian at Λ, so $H_\Lambda(\sigma) = 19$.

The permutations that are compatible with boundary condition $\xi \in S_\theta$ are given by:

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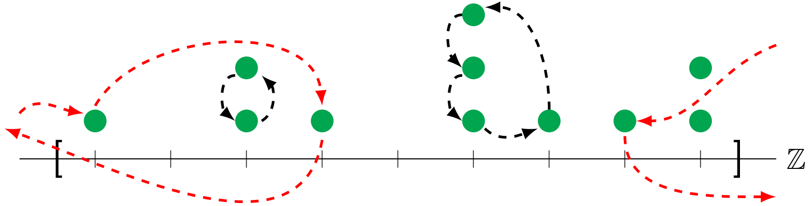

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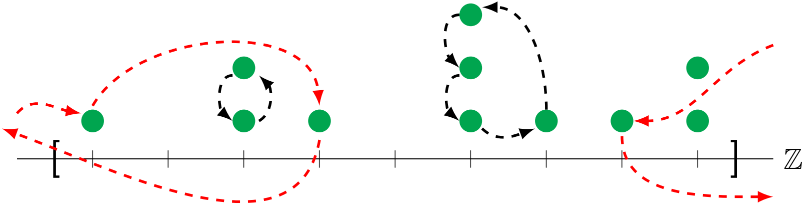

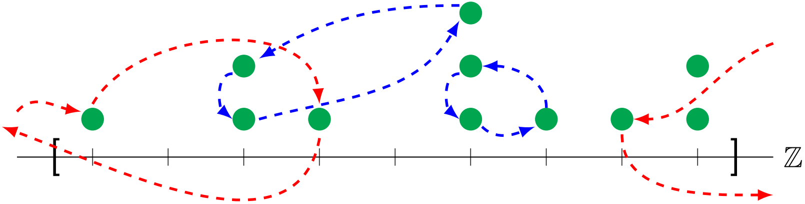

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- A probability measure μ on S_{θ} is a Gibbs measure when for any Borel set A and any $\Lambda \subset \mathbb{Z}^d$ finite, we have

$$\mu(A) = \int G^{\xi}_{\theta,\Lambda,\alpha}(A) \,\mathrm{d}\mu(\xi) .$$

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3. uniqueness in some sense.

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- $d \geq 3$ and $\alpha < \alpha_c$: the scaling limit of the sorted size of cycles converges to the Poisson-Dirichlet distribution, as in the case of uniform permutations[1].

[1] Schramm, Oded; *Compositions of random transpositions*, Israel J. Math., 2005.

Related results

Case: $\Omega_\theta = \mathbb{Z}^d$.

- Biskup and Richthammer[1]: $d = 1$, existence and uniqueness. All cycles are finite for all $\alpha > 0$.
- Armendáriz, Ferrari, Groisman and Leonardi[2]: $d \geq 1$ and α large, existence and uniqueness over finite cycle permutations.

[1] *Gibbs measures on permutations over one-dimensional discrete*, Ann. Appl. Probab., 2015.

[2] *Finite cycle Gibbs measures on permutations of $\mathbb{Z}^d$*, J. Stat. Phys., 2015.

Random multiplicities

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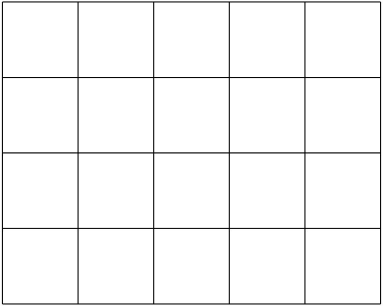

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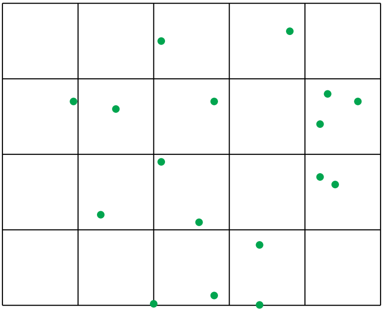

Results

Let $\theta = \{\theta(x)\}_{x \in \mathbb{Z}^d}$ be an i.i.d. sequence of Poisson random variables with mean ρ.

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Theorem. (Armendáriz, Ferrari, F.)

Consider $\rho \in (0, 1/2)$. If $\alpha \geq \alpha_*(\rho, d)$, then for almost every realization of θ we have:

- there exists a Gibbs measure μ_θ.
- μ_θ concentrates on permutations whose decomposition has only finite cycles.
  It is the unique Gibbs measure with this property.
- μ_θ can be obtained as a weak limit of $(G^{\mathrm{id}}_{\theta,\Lambda})_{\Lambda\Subset\mathbb{Z}^d}$.

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A finite cycle permutation σ can be represented as: $\eta \in \{0,1\}^{\Gamma_\theta}$ such that $\eta(\gamma) = \mathbf{1}\{\gamma \in \sigma\}$.

η is called the gas of cycles of σ.

Gas of cycles

Using the gas of cycles representation we can write the set of compatible permutations with the identity b.c. as:

$$S_{\theta,\Lambda}^{\mathrm{id}} = \{\eta \in \{0,1\}^{\Gamma_\theta} : \eta(\gamma)\eta(\gamma') = 0 \text{ if } \{\gamma\} \cap \{\gamma'\} \neq \emptyset \ \forall \gamma, \gamma' \in \Gamma_{\theta,\Lambda}\},$$

and the specification at volume Λ:

$$G_{\theta,\Lambda}^{\mathrm{id}}(\sigma) = \frac{1}{Z_{\theta,\Lambda,\alpha}^{\mathrm{id}}} \prod_{\gamma \in \Gamma_{\theta,\Lambda}} \left(e^{-\alpha H(\gamma)}\right)^{\eta(\gamma)} \mathbf{1}\{\eta \in S_{\theta,\Lambda}^{\mathrm{id}}\}.$$

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Lemma.

On $\mathbb{N}_0^{\Gamma_\theta}$ consider the following Birth and Death process:

- A cycle γ is born at rate $e^{-\alpha H(\gamma)}$, independently of others.
- A cycle γ (that is alive) dies at rate 1, also independently of others.

Then ν_θ is invariant for this dynamics.

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Lemma.

Let $\Lambda \subset \mathbb{Z}^d$ finite. On $\{0,1\}^{\Gamma_{\theta,\Lambda}}$ consider the following dynamics:

- A cycle γ tries to appear at rate $e^{-\alpha H(\gamma)}$ but it is effectively added if γ is compatible with all cycles already present.
- A cycle γ (that is alive) is removed at rate 1, also independently of others.

Then $G^{\mathrm{id}}_{\theta,\Lambda}$ is invariant for this dynamics.

Note that the dynamics is well defined since the space state $\{0,1\}^{\Gamma_{\theta,\Lambda}}$ is finite.

Domination by a Poisson measure

Lemma.

Let $\Lambda \subset \mathbb{Z}^d$ finite. Then, ν_θ stochastically dominates $G^{\mathrm{id}}_{\theta,\Lambda}$, i.e., there exists a construction such that

$$\eta^{\Lambda}(\gamma) \le \eta^{o}(\gamma) \qquad \text{almost surely } \forall\, \gamma \in \Gamma_\theta,$$

when η^{Λ} is sampled with $G^{\mathrm{id}}_{\theta,\Lambda}$ and η^{o} is sampled with ν_θ.

Idea of proof

Let $\mathcal{N}$ a Poisson process on $\Gamma_{\theta,\Lambda} \times \mathbb{R} \times \mathbb{R}^+$ with rate measure $e^{-\alpha H(\gamma)} \times \mathrm{d}t \times e^{-s}\mathrm{d}s$.

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- Define $\eta_t^o = (\eta_t^o(\gamma) \colon \gamma \in \Gamma_\theta)$ as:

$$\eta_t^o(\gamma) = \sum_{(\gamma, t', s') \in \mathcal{N}} \mathbf{1}\{t' \leq t < t + s'\}.$$

- η_t^o has distribution ν_θ for all $t \in \mathbb{R}$.

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We can define the process η_t^Λ as:

$$\eta_t^\Lambda(\gamma) = \sum_{(\gamma,t',s')\in\mathcal{N}} \mathbf{1}\{t' \le t < t'+s'\}\,\mathbf{1}\{(\gamma,t',s') \in \mathcal{K}^\Lambda\}\,\mathbf{1}\{\gamma \in \Gamma_{\theta,\Lambda}\}\,.$$

The problem is, how to decide if (γ, t', s') is in $\mathcal{K}^\Lambda$?

Idea of proof

For $(\gamma, t', s') \in \mathcal{N}$ denote by $A_\Lambda^{(\gamma,t',s')}$ the set of marks in $\mathcal{N}$ that were born in the past and may influence the birth of (γ, t', s').

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So, η_t^Λ is constructed in such a way that:

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Indeed, $A_\Lambda^{(\gamma,t',s')}$ is finite for all $(\gamma, t', s') \in \mathcal{N}$ almost surely.

Existence and uniqueness

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For the uniqueness suppose that μ and μ' are Gibbs measures. We show that:

- Almost surely w.r.t. $\mu \otimes \mu'$ there exist a sequence of finite sets $\Delta_j \nearrow \mathbb{Z}^d$ such that each Δ_j is a separating set.

Thanks for your attention!