Una Versión Combinatoria de la Inversa de Drazin para Árboles

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Motivation of Our Work

Given a tree T with singular adjacency matrix $A(T)$

- Assign a **pseudoinverse matrix** to $A(T)$
- Find **information** about T given by the pseudoinverse

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Motivation

Introduction

Drazin Inverse of a Matrix

The Combinatorial Drazin Inverse of a Tree

$RA = AR$

$ARA = A$

$RAR = R$

In this work we give a tool to answer the following question:

How many maximum matchings in a given a tree T and a given $e \in E(T)$, have the edge e as a member?

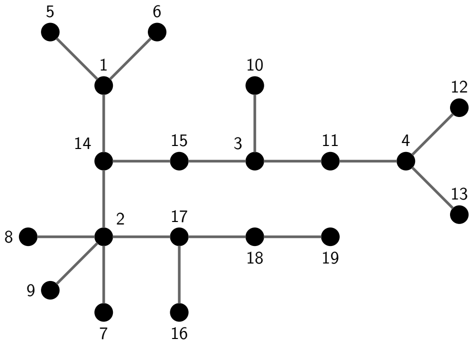
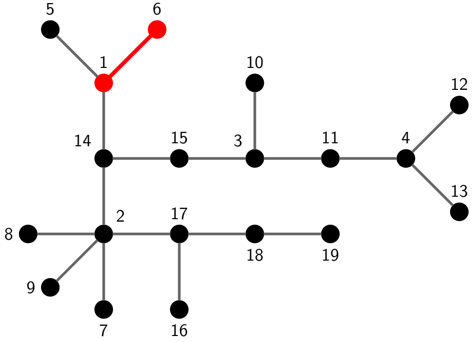


Figure: A tree T



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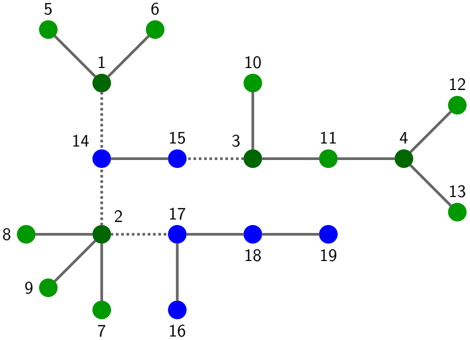
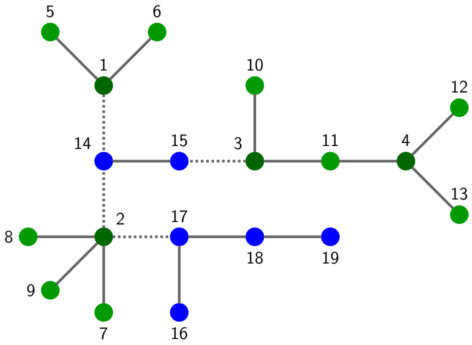


Figure: T has 30 maximum matchings



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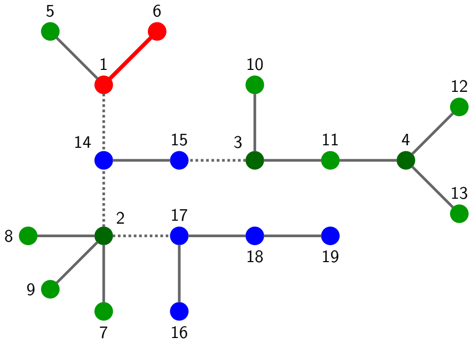
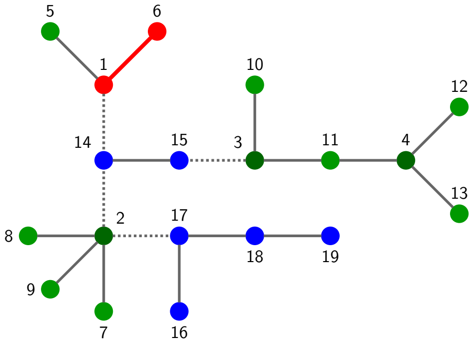


Figure: The edge $\{1, 6\}$ is in 15 maximum matchings



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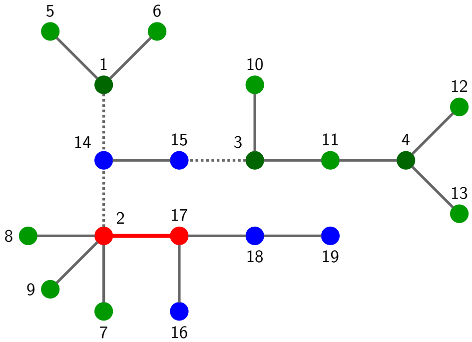
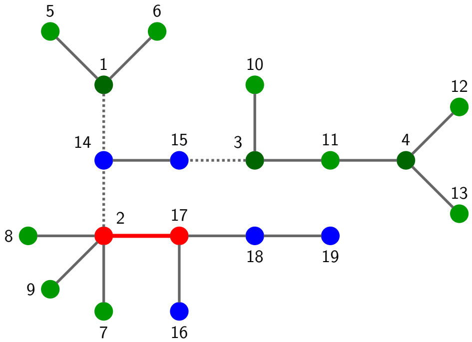


Figure: The edge $\{2, 17\}$ is not in any maximum matching



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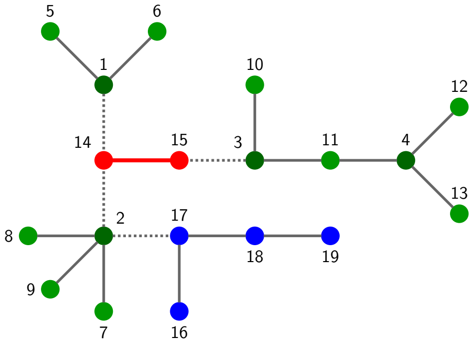


Figure: The edge $\{14, 15\}$ is in every maximum matching

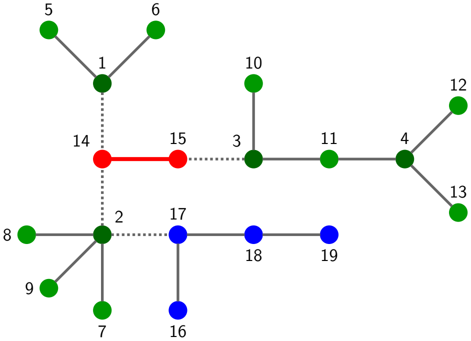


Figure: The edge $\{14, 15\}$ is in every maximum matching

In general we answer

> *In how many maximum matchings of T, a given tree T and a path P between two vertices of T, the path P is co-augmenting in these matchings?*

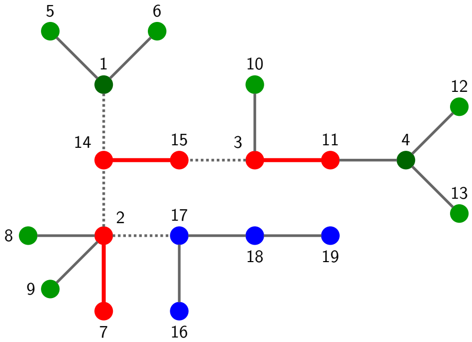
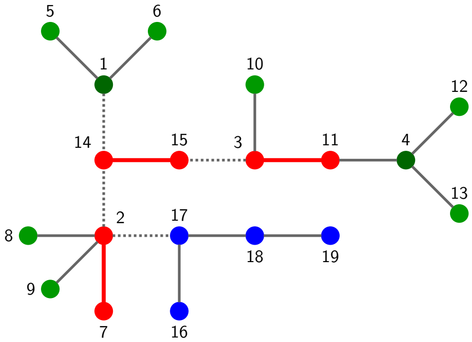


Figure: The path from 7 to 11 is co-augmenting in 4 maximum matchings



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Motivation

Introduction

Drazin Inverse of a Matrix

The Combinatorial Drazin Inverse of a Tree

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The Index of a Matrix

The smallest positive integer k for which the equation

$$\mathbb{R}^n = R(A^k) \bigoplus N(A^k),$$

holds is called the **index** of A. The Range-Nullspace Decomposition Theorem guarantees that this positive integer k there exists.

Core-Nilpotent Decomposition

Theorem

If A is an $n \times n$ singular matrix of index k such that $\text{rank}(A^k) = r$, then there exists a nonsingular matrix Q such that

$$Q^{-1}AQ = \begin{bmatrix} C_{r \times r} & O \\ O & N \end{bmatrix}$$

in which C is nonsingular, and N is nilpotent with nilpotency index k.

The Drazin Inverse Matrix

Definition

Given a square matrix A with a core-nilpotent decomposition $(Q,\ C,\ N)$, the matrix

$$D = Q \begin{bmatrix} C^{-1} & O \\ O & O \end{bmatrix} Q^{-1}$$

The matrix D is called the **Drazin Inverse** of A.

The Drazin Inverse Matrix

A characterization result

Theorem (Drazin, 1958)

Let A be any square matrix with $\text{index}(A) = k$ and let D be the Drazin inverse of A. Then D is the only matrix such that

1. $A^{k+1}D = A^k$.
2. $D^2A = D$.
3. $AD = DA$.

The Drazin Inverse Matrix

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The Drazin Inverse Matrix

The Symmetric Case

Theorem

For any symmetric matrix A, there is a unique matrix D such that

1. $AD = DA$.
2. $ADA = A$.
3. $DAD = D$.

What We Do in This Work

- We define a matrix $R(T)$ associated to T.
- The matrix $R(T)$ is defined in a combinatorial way.
- We prove that $R(T)$ fullfills all the three conditions stated in the latter result with A, the adjacency matrix of T.
- Because of the uniqueness part of the Drazin's Theorem, $R(T)$ is the Drazin Inverse of $A(T)$.

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Co-augmenting Path in a Matching

Definition

Given a tree T, a matching M in T and $v, w \in V(T)$, a co-augmenting path P in M, with endpoints at v and w is a path such that the edges of P incident in these vertices belong to M and for every $x \in V(P) \setminus \{u, v\}$ and every $e \in E(P)$ with $x \in e$ exactly one these edges belongs to M.

Co-augmenting Path in a Matching

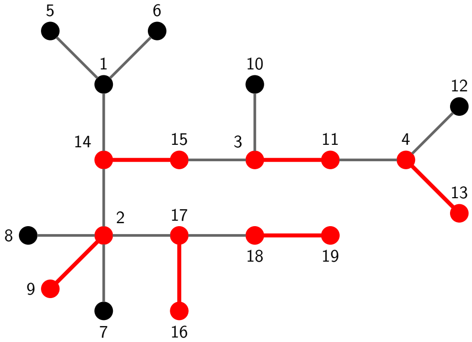


Figure: A matching M

Co-augmenting Path in a Matching

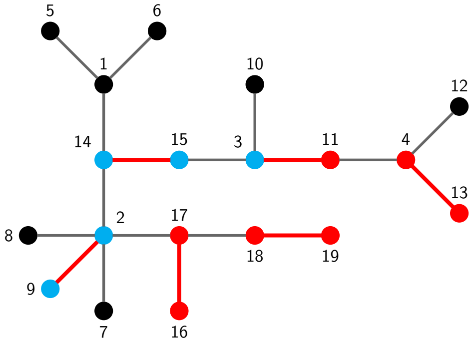


Figure: An alternating path in M

Co-augmenting Path in a Matching

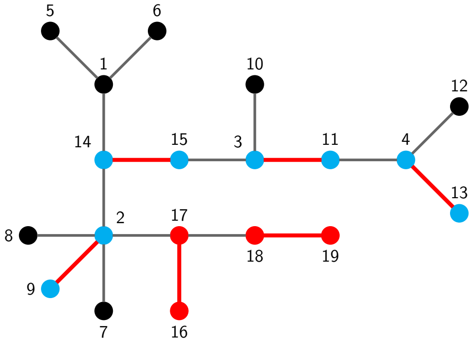


Figure: An alternating path that is co-augmenting in M

Some Notation

The Set of Maximum Matchings of T

$$\mathcal{M}(T) := \{M \mid M \text{ is a maximum matching of } T\}$$

Some Notation

The Size of $\mathcal{M}(T)$

$$m(T) := |\mathcal{M}(T)|$$

Some Notation

The Number of Maximum Matchings in which a Path is Co-augmenting

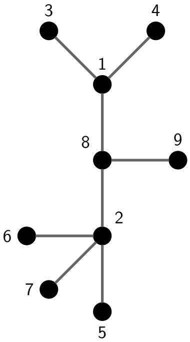
$$m(T, i, j) := |\{M \in \mathcal{M}(T) \mid iP_T j \text{ is co-augmenting in } M\}|$$

The Combinatorial Drazin Inverse Matrix

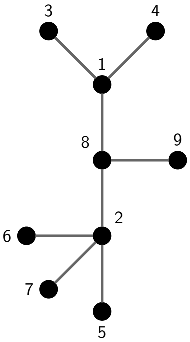
Definition

Given a tree T of order n, we define the n by n matrix $[r_{ij}]$ as:

$$r_{ij} := \begin{cases} (-1)^{\lfloor \frac{d(i,j)}{2} \rfloor} \dfrac{m(T,i,j)}{m(T)} & : d(i,j) \text{ is odd} \\ 0 & : \text{ otherwise} \end{cases}$$



$$R(T) = \frac{1}{6}\begin{bmatrix} 0 & 0 & 3 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 & 2 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 \\ 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & -3 & -3 & -2 & -2 & -2 & 6 & 0 \end{bmatrix}$$



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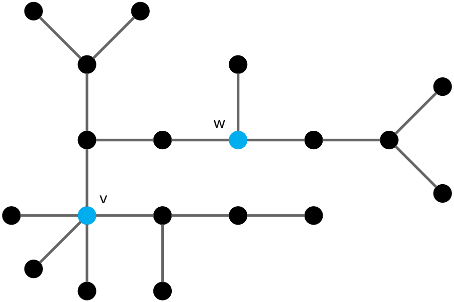
The Neighborhood of v Away from w

Definition

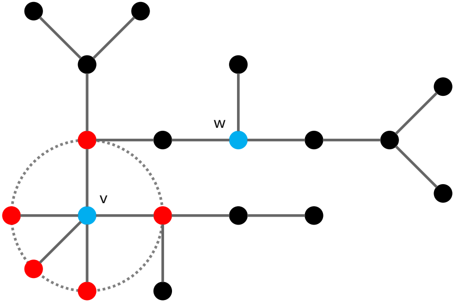
Given $v, w \in V(T)$

$$N(v \leftarrow w) := \{x \in N(v) \mid d(x, v) = d(v, w) + 1\}$$

The Neighborhood of v Away from w



The Neighborhood of v Away from w



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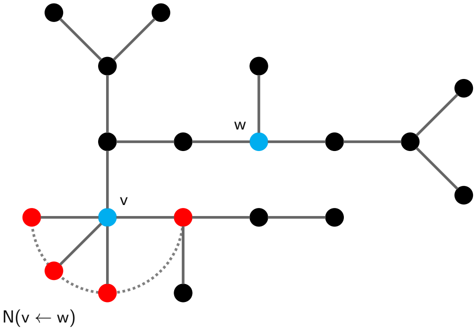


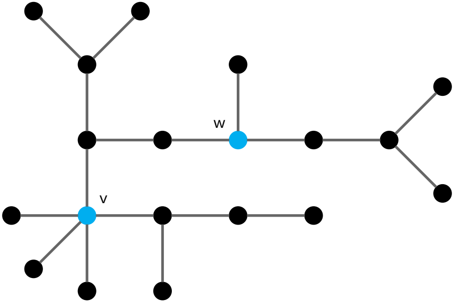
The Neighbor of v Closest to w

Definition

Given $v, w \in V(T)$, the vertex $u(v \to w)$ is the vertex of $N(v)$ such that

$$d(w, u(v \to w)) = d(v, w) - 1$$

The Neighbor of v Closest to w



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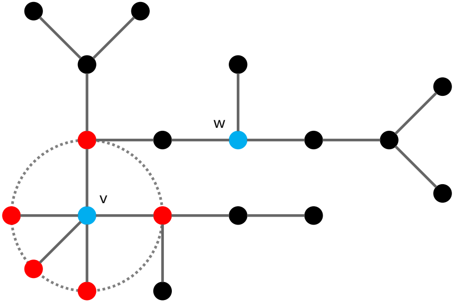

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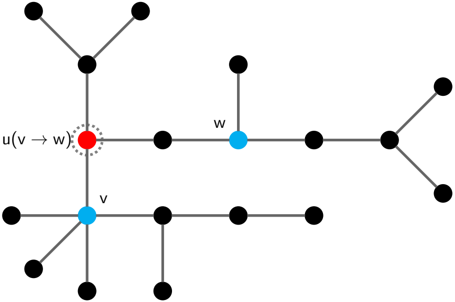


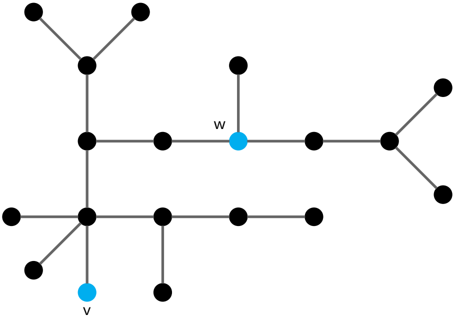
The Flower of v with Respect to w

Definition

For any tree T, and any $v, w \in V(T)$, the *vw*-**flower** in T is

$$F_T(v, w) := m(T) \sum_{x \sim w} r_{vx}$$

The Flower of v with Respect to w



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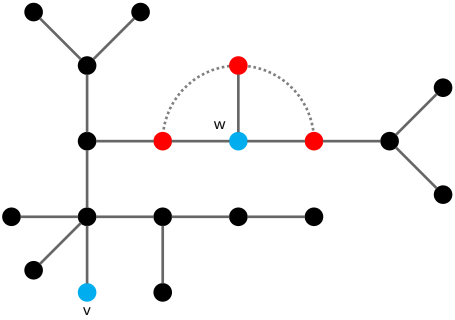

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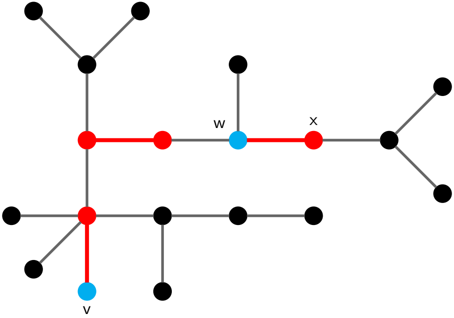

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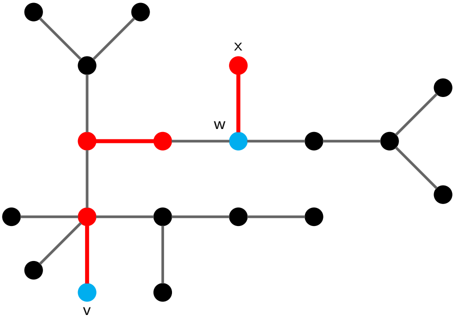

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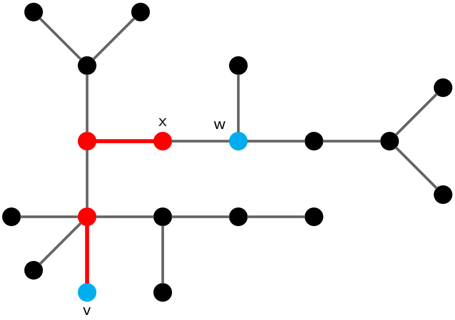


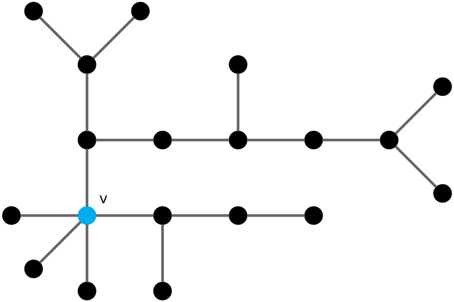
The Set of Edges Incident to a Vertex v

Definition

Given $v \in V(T)$

$$E_v(T) := \{e \in E(T) \mid v \in e\}$$

The Set of Edges Incident to a Vertex v



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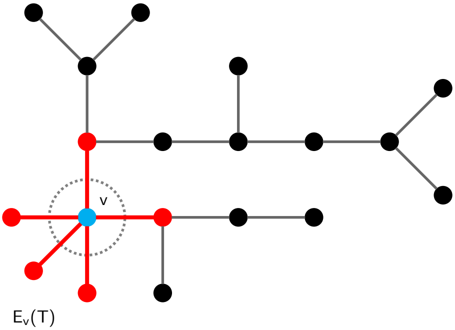


Some properties of Flowers

Lemma

For any tree T and any $i \in V(T)$

$$F_T(i,i) = |\{M \in \mathcal{M}(T) : M \cap E_i(T) \neq \varnothing\}|$$

The Set of Matchings in Which a Path is Co-augmenting

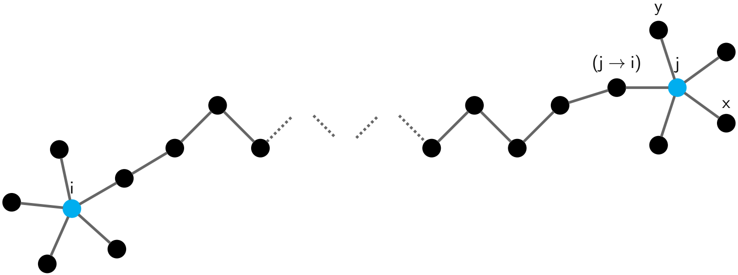
Definition

For any tree T and any $i, j \in V(T)$, if $i \neq j$ and $d(i, j)$ is even, then

$$\mathcal{M}_{i,j}(T) := \{M \in \mathcal{M}(T) \mid M \cap E_j(T) = \varnothing, \text{and}$$

$$iP_T u(j \to i) \text{ is co-augmenting}\}$$

The Set of Matchings in Which a Path is Co-augmenting



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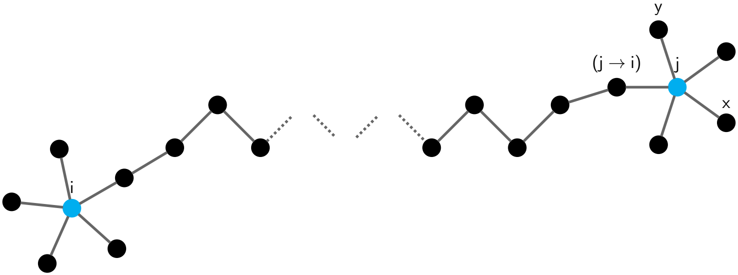
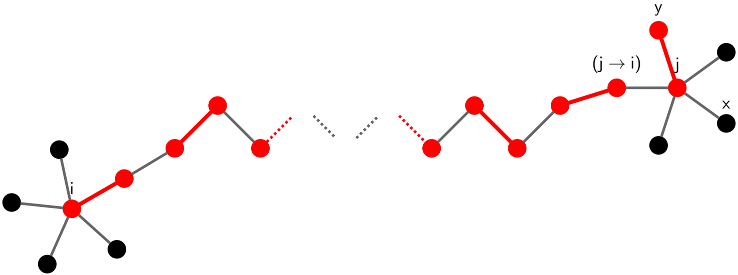
$$F_T(i,j) = (-1)^{\lfloor \frac{d(i,u(j \to i))}{2} \rfloor} |\mathcal{M}_{i,j}(T)|$$

Proof. Let $x, y \in N(j \leftarrow i)$



Proof.



- Let $x, y \in N(v \leftarrow w)$, then
    - sign of r_{ix} = sign of r_{iy}
    - sign of r_{ix} = - sign of $r_{i\ u(j \rightarrow i)}$
- Let $M \in \mathcal{M}(T)$ and $x, y \in N(v \leftarrow w)$,
    - iPx is co-augmenting in M then iPy is not co-augmenting in M

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  - iPx is co-augmenting in M then iPy is not co-augmenting in M

- If $M \in \mathcal{M}_{i,j}(T)$ is computed in r_{ix} then M is computed in $r_{i,u(j\to i)}$
- Let $m = \deg(j)$, $x_m = u(j \to i)$ and $M_k \in \mathcal{M}_{i,j}(T)$

	r_{ix_1}	r_{ix_2}	r_{ix_3}	$\cdots$	$r_{ix_{m-1}}$	r_{ix_m}
M_1	$+$	0	0	$\cdots$	0	$-$
M_2	0	$+$	0	$\cdots$	0	$-$
M_3	$+$	0	0	$\cdots$	0	$-$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
M_t	0	0	0	$\cdots$	$+$	$-$
M_{t+1}	0	0	0	$\cdots$	0	$-$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$M_{m(T)}$	0	0	0	$\cdots$	0	$-$

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M_3	$+$	0	0	$\cdots$	0	$-$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
M_t	0	0	0	$\cdots$	$+$	$-$
M_{t+1}	0	0	0	$\cdots$	0	$-$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$M_{m(T)}$	0	0	0	$\cdots$	0	$-$

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M_2	0	$+$	0	$\cdots$	0	$-$
M_3	$+$	0	0	$\cdots$	0	$-$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
M_t	0	0	0	$\cdots$	$+$	$-$
M_{t+1}	0	0	0	$\cdots$	0	$-$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$M_{m(T)}$	0	0	0	$\cdots$	0	$-$

Lemma

For any $i, j \in V(T)$ such that $i \neq j$, $F_T(i,j) = F_T(j,i)$.

Proof. For any $i, j \in V(T)$ such that $i \neq j$

$$d(i, u(j \to i)) = d(j, u(i \to j))$$

therefore, by the previous lemma, is enough to prove

$$|\mathcal{M}_{i,j}(T)| = |\mathcal{M}_{j,i}(T)|$$

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Lemma

For any $i, j \in V(T)$ such that $i \neq j$, $F_T(i,j) = F_T(j,i)$.

Proof. For any $i, j \in V(T)$ such that $i \neq j$

$$d(i, u(j \to i)) = d(j, u(i \to j))$$

therefore, by the previous lemma, is enough to prove

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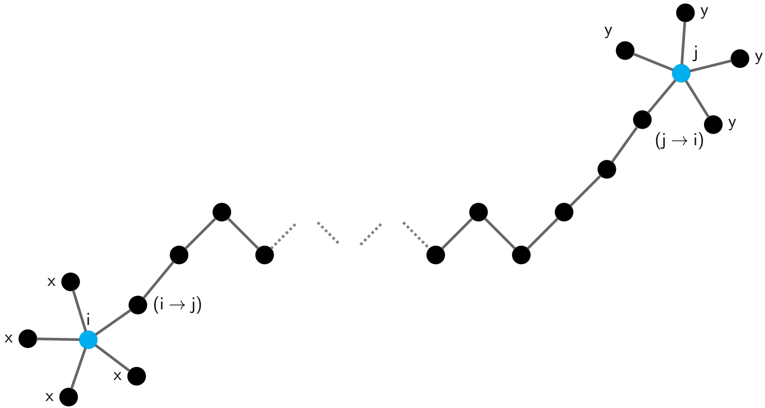

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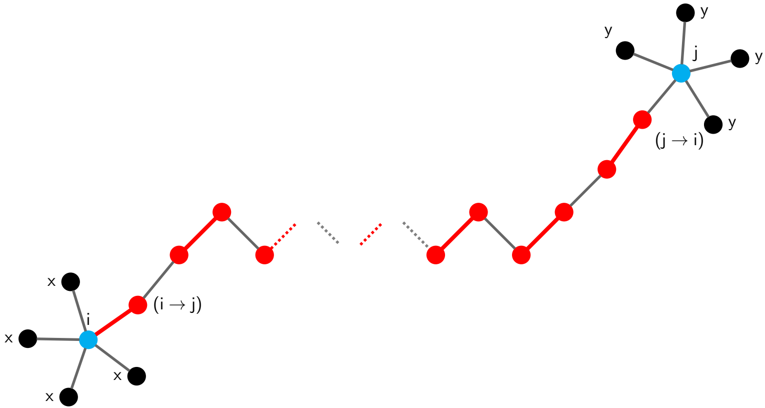

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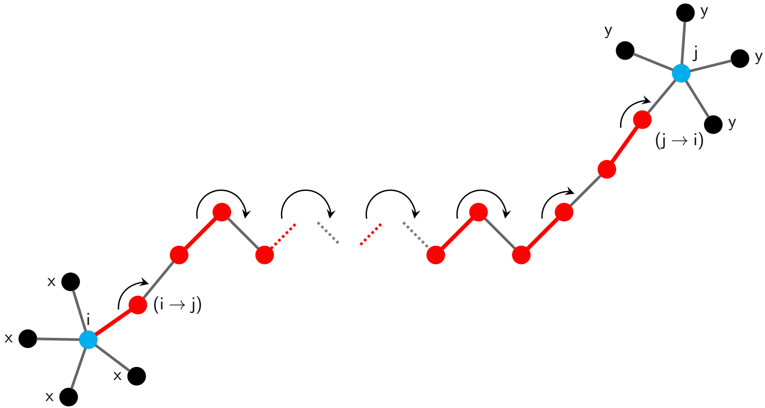

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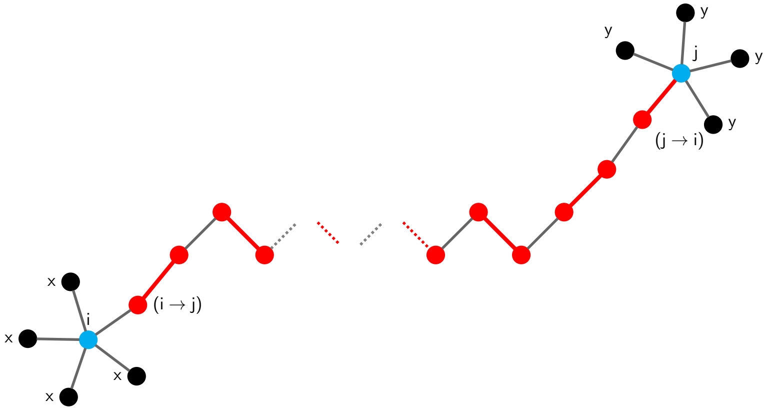

In summary

Let $M \in \mathcal{M}_{i,j}(T)$, then

- $iPu(j \to i)$ is co-augmenting in M
- $M \cap E_i(T) = \{\{i, u(i \to j)\}\}$
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We obtain a matching $\tilde{M} \in \mathcal{M}_{j,i}(T)$ as

$$M \setminus (M \cap iPu(j \to i))$$

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Theorem

For any tree T,

$$AR = RA$$

Proof.

$$
\begin{aligned}
(A(T)R(T))_{ij} &= \sum_{v \sim i} r_{vj} \\
&= \frac{1}{m(T)} F_T(i,j) \\
&= \frac{1}{m(T)} F_T(j,i) \\
&= \sum_{v \sim j} r_{vi} \\
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Motivation

Introduction

Drazin Inverse of a Matrix

The Combinatorial Drazin Inverse of a Tree

$RA = AR$

$ARA = A$

$RAR = R$

Theorem

For any tree T,

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For any tree T,

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Idea of proof:

- Consider R and RAR as operators.
- Construct a base for $N(A(T))$ introducing the notion of basic S-tree.
- Prove that $N(A(T)) = N(R(T))$.
- Prove that $RAR = R$ by showing that $RARb = Rb$ for every b in a base B of $\mathbb{R}^n$.