Estratégias Evolucionariamente Estable para Juegos Simétricos de tres o más Jugadores

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Introduction

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- More specifically, each player is "pre-programmed" to some behavior formally a strategy in the game and one assumes that some evolutionary selection process operates over time on the population distribution of behaviors.

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- We extend the concept of *ESS* for **symmetric games for** $n \geq 3$ **players.**

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- $(N, \mathbf{S}, \boldsymbol{\pi})$ $n-$person game or game
- $\Phi = \prod_{i=1}^{n} \triangle(S_i)$ mixed strategies $\boldsymbol{\sigma} = (\sigma_1, ..., \sigma_n) \in \Phi$

$$\triangle(S_i) = \left\{ \sigma_i \in \mathbb{R}^l : \sigma_i(s_i) \geq 0, \quad \sum_{s_i \in S_i} \sigma_i(s_i) = 1 \right\}$$

Support

- The **support** of $\boldsymbol{\mu} = (\mu_1, ..., \mu_n) \in \Phi$ is $C(\mu) = (C_1(\mu_1), ..., C_n(\mu_n))$, where

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- $\boldsymbol{\mu} \in \Phi$ is said to be **completely mixed strategy** if for all $i \in N$, $C_i(\mu_i) = S_i$.

Notations

- $\boldsymbol{\sigma} = (\sigma_1, ..., \sigma_n)$, $\boldsymbol{\mu} = (\mu_1, ..., \mu_n) \in \Phi$ and t, k natural numbers, we denote by

$$\left(\boldsymbol{\sigma}_{-[t,k]}, \boldsymbol{\mu}\right) = \begin{cases} (\sigma_1, ..., \sigma_{t-1}, \mu_t, ..., \mu_k, \sigma_{k+1}, ..., \sigma_n) & \text{if } t \leq k \\ (\sigma_1, ...\sigma_n) & \text{if } t > k \end{cases}$$

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Best response and Nash equilibrium

- The **best response** correspondence for player $i \in N$, is $B_i : [\Delta S]^{n-1} \to \Delta S_i$

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- σ is a **Nash equilibrium** if and only if $\sigma \in B(\sigma)$

Symmetric Game

Definition

$(N, \mathbf{S}, \pi)$ is a $n-$players **symmetric game** if for all player $i, j \in N$, $S_i = S_j = S$ and

$$\pi_i\left(s_1, ..., s_i, ..., s_j, ..., s_n\right) = \pi_j\left(s_1, ..., s_{i-1}, s_j, s_{i+1}, ..., s_{j-1}, s_i, s_{j+1}, ..., s_n\right)$$

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Remark

If $N = 2$ this definition coincides with the standard definition of symmetric games, ie., $S_1 = S_2$ and

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- N=2. $\pi_1(s_1, s_2) = \pi_2(s_2, s_1)$

	1	2
1	1,1	3,2
2	2,3	4,4

or $B = A^T$

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	1	2
1	1,1,1	3,2,3
2	2,3,3	4,4,5

1

	1	2
1	3,3,2	5,4,4
2	4,5,4	6,6,6

2

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2

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$3 = \pi_1(1, 1, 2) = \pi_3(2, 1, 1) = \pi_2(2, 1, 1) = \pi_1(1, 2, 1) = 3.$

Symmetric Game

- The Three-Player Prisoner's Dilemma

	C	D
C	7,7,7	3,9,3
D	9,3,3	5,5,0

C

	C	D
C	3,3,9	0,5,5
D	5,0,5	6,6,6

D

Symmetric Nash Equilibrium

- Let $(N, \mathbf{S}, \boldsymbol{\pi})$ be , $\boldsymbol{\sigma} = (\sigma_1, ..., \sigma_n)$ is a **symmetric strategy** if for all $i, j \in N$, $\sigma_i = \sigma_j$, i,e., $\boldsymbol{\sigma} = (\sigma, ..., \sigma)$.

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- Let (N, S, π) be a symmetric game, $\boldsymbol{\sigma} = (\sigma, ..., \sigma)$ is a **symmetric Nash equilibrium** if σ is a Nash equilibrium, i.e.,for all $i \in N$, $\mu \in \Delta(S_i)$

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Lemma

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- **Proof.** $\beta^* : \Delta S \to \Delta S$ given by

$$\beta^*(\boldsymbol{\sigma}_{-i}) = \{\mu \in \Delta S : \pi(\mu, \boldsymbol{\sigma}_{-i}) \geq \pi(\mu', \boldsymbol{\sigma}_{-i}), \ \forall \ \mu' \in \Delta S\}.$$

Satisfies all conditions of Kakutani's Fixed Point Theorem and, hence, it has a fixed point.

Evolutionary stable strategy (ESS) [N=2]

Definition

Let Γ be a $2-$symmetric game, a strategy $\sigma \in \Delta S$ is an **evolutionary stable strategy** (ESS) if for all $\mu \in \Delta S$, and $\mu \neq \sigma$ there exists $\varepsilon_{\mu} \in (0,1)$ such that for all $\varepsilon \in \left(0, \varepsilon_{\mu}\right)$ we have

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Example

Hawk-Dove Game

	H	D
H	$\frac{v-c}{2}, \frac{v-c}{2}$	$v, 0$
D	$0, v$	$\frac{v}{2}, \frac{v}{2}$

	H	D
H	-1	4
D	0	2

$\sigma = (2/3, 1/3)$ is (unique) **symmetric Nash equilibrium** and **ESS**.

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$\sigma = (2/3, 1/3)$ is (unique) **symmetric Nash equilibrium** and **ESS**. $\pi(\sigma, \mu) > \pi(\mu, \mu) \Leftrightarrow \pi(\sigma - \mu, \mu) > 0$,
$\pi(\sigma - \mu, \mu) = \frac{1}{3}(2 - 3\mu_1)^2 > 0$ for all $\mu \neq \sigma$, $\mu = (\mu_1, 1 - \mu_1)$.

Evolutionary stable strategy (ESS) [N=3]

Definition

Let Γ be a 3−symmetric game, a strategy $\sigma \in \Delta S$ is an **evolutionary stable strategy** (ESS) if for all $\mu \in \Delta S$, and $\mu \neq \sigma$ there exists $\varepsilon_{\mu} \in (0,1)$ such that for all $\varepsilon \in \left(0, \varepsilon_{\mu}\right)$ we have

$$\pi(\sigma, (1-\varepsilon)\,\sigma + \varepsilon\mu, (1-\varepsilon)\,\sigma + \varepsilon\mu) > \pi(\mu, (1-\varepsilon)\,\sigma + \varepsilon\mu, (1-\varepsilon)\,\sigma + \varepsilon\mu)$$

or

$$\pi(\sigma, \mu(\varepsilon), \mu(\varepsilon)) > \pi(\mu, \mu(\varepsilon), \mu(\varepsilon)),$$

where $\mu(\varepsilon) = (1-\varepsilon)\,\sigma + \varepsilon\mu$.

Evolutionary stable strategy (ESS) [N=3]

Proposition (N=3)

Let Γ be, a strategy $\sigma \in \Delta S$ is an ESS **if and only if** *for all $\mu \in \Delta S$, and $\mu \neq \sigma$*

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3. *if* $\pi(\sigma, \sigma, \sigma) = \pi(\mu, \sigma, \sigma)$ *and* $\pi(\sigma, \mu, \sigma) = \pi(\mu, \mu, \sigma)$ *then* $\pi(\sigma, \mu, \mu) > \pi(\mu, \mu, \mu)$.

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$$\pi(\sigma, \mu(\epsilon), \dots, \mu(\epsilon)) > \pi(\mu, \mu(\epsilon), \dots, \mu(\epsilon))$$

where $\mu(\epsilon) = (1 - \epsilon)\sigma + \epsilon\mu$.

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2. *For all* $1 \leq k < n - 1$. *If for all* j, *such that,* $1 \leq j \leq k$,

$$\pi\left((\boldsymbol{\sigma}_{-[1,j]}, \boldsymbol{\mu})\right) = \pi\left((\boldsymbol{\sigma}_{-[2,j]}, \boldsymbol{\mu})\right)$$

then, $\pi\left((\boldsymbol{\sigma}_{-[2,k+1]}, \boldsymbol{\mu})\right) \geq \pi_1\left((\boldsymbol{\sigma}_{-[1,k+1]}, \boldsymbol{\mu})\right)$.

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then $\pi\left((\boldsymbol{\sigma}_{-[2,n]}, \boldsymbol{\mu})\right) > \pi\left((\boldsymbol{\sigma}_{-[1,n]}, \boldsymbol{\mu})\right) = \pi(\boldsymbol{\mu})$.

Results for N=3

Lemma

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Lemma (N=2)

If $\boldsymbol{\sigma} = (\sigma, \sigma)$ and $\boldsymbol{\mu} = (\mu, \mu)$ are symmetric Nash equilibrium $(\sigma \neq \mu)$, and $C(\mu) \subseteq B(\boldsymbol{\sigma}_{-1})$ then σ is not an ESS.

Lemma (N=3)

If $\boldsymbol{\sigma} = (\sigma, \sigma, \sigma)$ and $\boldsymbol{\mu} = (\mu, \mu, \mu)$ are symmetric Nash equilibrium $(\sigma \neq \mu)$, (μ, μ, σ) is a Nash equilibrium, and $C(\mu) \subseteq B(\boldsymbol{\sigma}_{-1})$, then σ is not an ESS.

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Example (False, if (μ, μ, σ) is not a NE) Consider

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$\sigma = (1, 0)$ and $\mu = (1/2, 1/2)$ are symmetric NE. (μ, μ, σ) is not a NE, and $C(\mu) \subseteq B(\boldsymbol{\sigma}_{-1})$ and σ is an ESS.

Results for N=3

Lemma (Van Damme (1999), N=2)

If σ is an ESS and $\boldsymbol{\mu}$ is a symmetric Nash equilibrium with $C(\mu) \subseteq B(\sigma, \sigma)$, the $\sigma = \mu$

Example (False N=3)

Consider the following symmetric game:

0	1
0	0

	1	0
	0	2

$\sigma = (1, 0)$ is an *ESS* and $\mu = (0, 1)$ is a symmetric NE with $C(\mu) \subseteq B(\sigma, \sigma, \sigma)$ and $\sigma \neq \mu$.

Results for N=3

Proposition (N=2)

Let Γ *be a symmetric game, with* $|S| = 2$*, and* $a_{11} \neq a_{21}$ *or* $a_{12} \neq a_{22}$*. Then* Γ *has an ESS.*

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Let Γ be a symmetric game, with $|S| = 2$, and $a_{11} \neq a_{21}$ or $a_{12} \neq a_{22}$. Then Γ has an ESS.

Proposition (N=3)

Let Γ be a $3-$symmetric game, with $|S| = 2$ and $(\sigma, ..., \sigma)$ a Nash equilibrium. σ is an ESS if and only if $a_{111} > a_{211}$ or $a_{122} < a_{222}$ or

$$\sum_{i_3=1}^{2} \left(a_{1,1,i_3} - a_{2,1,i_3} + a_{2,2,i_3} - a_{1,2,i_3}\right) \sigma\left(i_3\right) < 0$$

and

$$\sum_{i_2=1}^{2} \left(a_{1,i_2,1} - a_{2,i_2,1} + a_{2,i_2,2} - a_{1,i_2,2}\right) \sigma\left(i_2\right) < 0.$$

The ESS set is finite

Theorem

The set of ESS is finite (but possibly zero)

The ESS set is finite

Remark Let $\boldsymbol{\sigma}$ be a symmetric NE profile and let $\{\boldsymbol{\sigma}^{\varepsilon}\}_{\varepsilon\downarrow 0}$ be a sequence of symmetric NE profiles such that $\boldsymbol{\sigma}^{\varepsilon} \to \boldsymbol{\sigma}$, when $\epsilon \downarrow 0$. There exist ε_0, such that for all $\varepsilon < \varepsilon_0$ $C(\sigma) \subseteq C(\sigma^{\varepsilon})$

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$$\begin{array}{ll} \pi(\sigma, \sigma^{\varepsilon}, ..., \sigma^{\varepsilon}) = \pi(\sigma^{\varepsilon}, \sigma^{\varepsilon}, ..., \sigma^{\varepsilon}) & C(\sigma) \subseteq B(\boldsymbol{\sigma}^{\varepsilon}), \\ \pi(\sigma^{\varepsilon}, \sigma, ..., \sigma) = \pi(\sigma, \sigma, ..., \sigma) & C(\sigma^{\varepsilon}) \subseteq B(\boldsymbol{\sigma}). \end{array}$$

The ESS set is finite

Lemma (1)

Let σ^{ε} be a sequence of mixed strategy such that $\sigma^{\varepsilon} \longrightarrow \sigma$, when $\varepsilon \longrightarrow 0$, being $\sigma^{\varepsilon} \neq \sigma$, we define

$\delta(\varepsilon) = \max_j \left\{ \frac{\left|\sigma_j - \sigma_j^{\varepsilon}\right|}{\sigma_j} : \sigma_j > 0 \right\}$. *Then*

1. $\delta(\varepsilon) \longrightarrow 0$, *when* $\varepsilon \longrightarrow 0$, *and*
2. $\mu^{\varepsilon} = \frac{\sigma^{\varepsilon} - (1 - \delta(\varepsilon))\sigma}{\delta(\varepsilon)} \in \Delta(S)$ *i.e.,*
   $\sigma^{\varepsilon} = (1 - \delta(\varepsilon))\sigma + \delta(\varepsilon)\mu^{\varepsilon}$.

The ESS set is finite

Lemma (2)

Let $\boldsymbol{\sigma}$ *be ESS and let* $\boldsymbol{\sigma}^{\varepsilon}$ *be a sequence of ESS such that* $\boldsymbol{\sigma}^{\varepsilon} \to \boldsymbol{\sigma}$*, when* $\varepsilon \downarrow 0$*, where, for all* $\epsilon > 0$ *we have that* $\boldsymbol{\sigma}^{\varepsilon} \neq \boldsymbol{\sigma}$*. If* $\pi\left(\sigma^{\varepsilon}, \sigma, \sigma^{\varepsilon}, \ldots, \sigma^{\varepsilon}\right) \geq \pi\left(\sigma, \sigma, \sigma^{\varepsilon}, \ldots, \sigma^{\varepsilon}\right)$ *and there exists* $k' \geq 0$*, such that for all* $0 \leq k \leq k'$*,*

$$\pi\left(\mu^{\varepsilon}, \sigma, \underbrace{\sigma, \ldots, \sigma}_{(n-2-k)-times}, \underbrace{\mu^{\varepsilon}, \ldots, \mu^{\varepsilon}}_{k-times}\right) = \pi\left(\sigma, \sigma, \underbrace{\sigma, \ldots, \sigma}_{(n-2-k)-times}, \underbrace{\mu^{\varepsilon}, \ldots, \mu^{\varepsilon}}_{k-times}\right),$$

where μ^{ε} *and* $\delta(\varepsilon)$ *are as in Lemma (1) , then*

$$\pi\left(\mu^{\varepsilon}, \sigma, \underbrace{\sigma, \ldots, \sigma}_{(n-3-k')-times}, \underbrace{\mu^{\varepsilon}, \ldots, \mu^{\varepsilon}}_{(k'+1)-times}\right) \geq \pi\left(\sigma, \sigma, \underbrace{\sigma, \ldots, \sigma}_{(n-3-k')-times}, \underbrace{\mu^{\varepsilon}, \ldots,}_{(k'+1)-ti}\right.$$

Proof. The ESS set is finite

If not. $\sigma^{\varepsilon} \to \sigma$ ESS + Remark

$$\pi(\sigma,\sigma,...,\sigma) \overset{*}{=} \pi(\sigma^{\varepsilon},\sigma,...,\sigma)$$
$$\pi(\sigma^{\varepsilon},\sigma^{\varepsilon},...,\sigma^{\varepsilon}) = \pi(\sigma,\sigma^{\varepsilon},...,\sigma^{\varepsilon})$$

(*) + Lemma (1)+ $\sigma^{\varepsilon} = (1-\delta(\varepsilon))\sigma + \delta(\varepsilon)\mu^{\varepsilon}$

$$\pi(\sigma,\sigma,...,\sigma) = \pi(\mu^{\varepsilon},\sigma,...,\sigma)$$

$k' := \max_t \quad \{$for all $k,\ 0 \le k \le t,$

$$\pi(\mu^{\varepsilon},\sigma,\underbrace{\sigma,...,\sigma}_{(n-2-k)},\underbrace{\mu^{\varepsilon},...,\mu^{\varepsilon}}_{k}) = \pi(\sigma,\sigma,\underbrace{\sigma,...,\sigma}_{(n-2-k)},\underbrace{\mu^{\varepsilon},...,\mu^{\varepsilon}}_{k})$$

Lemma (2)+ k',

$$\pi\left(\mu^{\varepsilon},\sigma,\underbrace{\sigma,...,\sigma}_{(k'+1)-times},\underbrace{\mu^{\varepsilon},...,\mu^{\varepsilon}}_{(n-3-k')-times}\right) > \pi\left(\sigma,\sigma,\underbrace{\sigma,...,\sigma}_{(k'+1)-times},\underbrace{\mu^{\varepsilon},...,\mu^{\varepsilon}}_{(n-3-k')-ti}\right.$$

contradicting that $\sigma \in ESS$.

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  - individuals are n-paired random,
  - each individuals engages in exactly one contest,
  - the payoff (fitness, expected number of offspring) to an s_1-strategist as a result of a contest with $n-1-$individuals is $\pi(s_1, s_2, ..., s_n)$

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- The corresponding dynamics for the population shares x_i or replicator dynamics

$$\dot{x}_i = [\pi(e_i, \mathbf{x}_{-1}) - \pi(\mathbf{x})] x_i = \pi(e_i - x_i, \mathbf{x}_{-1}) x_i$$

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Theorem

A population state is asymptotically stable in the replicator dynamics if and only if the corresponding mixed strategy is evolutionarily stable.

Muchas gracias!!