Port-Dirac Systems and Interconnection

Hernán Cendra, UNS and CONICET

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Dirac structures on a vector space.
Let V be a n-dimensional vector space and V^* be its dual space. Define the (non-degenerate) symmetric pairing $\ll \cdot, \cdot \gg$ on $V \oplus V^*$ by

$$\ll (v_1, \alpha_1), (v_2, \alpha_2) \gg = \langle \alpha_1, v_2 \rangle + \langle \alpha_2, v_1 \rangle ,$$

for $(v_1, \alpha_1), (v_2, \alpha_2) \in V \oplus V^*$, where $\langle \cdot, \cdot \rangle$ is the natural pairing between V^* and V. A *(linear) Dirac structure on* V is a subspace $D \subset V \oplus V^*$ such that $D = D^\perp$, where $D^\perp$ is the orthogonal subspace of D relative to the pairing $\ll \cdot, \cdot \gg$. Note that according to the definition the condition $D = D^\perp$ implies that $\langle \alpha, v \rangle = 0$ for each $(v, \alpha) \in D$.
A vector subspace $D \subset V \oplus V^*$ is a Dirac structure on V if and only if it is maximally isotropic with respect to the symmetric pairing $\ll \cdot, \cdot \gg$, or equivalently, $\dim D = n$ and $\ll (v_1, \alpha_1), (v_2, \alpha_2) \gg = 0$ for all (v_1, α_1), (v_2, α_2) in D.

We have the following basic examples of Dirac structures.

(a) Let F be a subspace of V and $F^\circ = \{\alpha \in V^* \langle \alpha, v\rangle = 0 \text{ for all } v \in F\}$. Then $D_V = F \oplus F^\circ$ is a Dirac structure on V.

(b) On a presymplectic vector space (V, ω), a Dirac structure is given by

$$D_\omega = \{(v, \alpha) \in V \oplus V^* \mid \alpha = \omega^\flat(v)\}$$

(c) A bivector $\pi : V^* \times V^* \to \mathbb{R}$ defines the Dirac structure

$$D_\pi = \{(v, \alpha) \in V \oplus V^* \mid v = \pi^\sharp(\alpha)\}.$$

Lemma. Let D be a Dirac structure on V. Define the subspace $F_D \subset V$ to be the projection of D on V. Define the 2-form ω_D on F_D by $\omega_D(u, v) = \alpha(v)$ where $u \oplus \alpha \in D$. Then ω_D is a skew form on F_D. Conversely, given a vector space V, a subspace $F \subset V$ and a skew form ω on F,

$$D_{(F,\omega)} = \{u \oplus \alpha \mid u \in F,\ \alpha(v) = \omega(u, v) \text{ for all } v \in F\}$$

is the only Dirac structure D on V such that $F_D = F$ and $\omega_D = \omega$. In other words, a Dirac structure D on V is uniquely determined by a subspace $F_D \subset V$ and a 2-form ω_D on it.

When a linear map $L\colon V \to W$ between vector spaces is given and we have a Dirac structure D_W on W, it is possible to induce a Dirac structure D_V on V, the *pull-back of D_W by L*, denoted by $D_V = \mathcal{B}L(D_W)$ and defined as follows.

$$\mathcal{B}L(D_W) = \{(v, L^* w^*) \in V \oplus V^* \mid v \in V,\ w^* \in W^*,\ (Lv, w^*) \in D_W\}.$$

In a similar way, if we have a Dirac structure D_V on V, we can construct a Dirac structure $D_W = \mathcal{F}L(D_V)$ on W, called the *push-forward of D_V by L*, as follows.

$$\mathcal{F}L(D_V) = \{(Lv, w^*) \in W \oplus W^* \mid v \in V,\ w^* \in W^*,\ (v, L^* w^*) \in D_V\}.$$

The composition rules, $\mathcal{F}(L_1 \circ L_2) = \mathcal{F}(L_1) \circ \mathcal{F}(L_2)$ and $\mathcal{B}(L_1 \circ L_2) = \mathcal{B}(L_2) \circ \mathcal{B}(L_1)$, hold.

The *Forward-Dirac category* denoted $\mathcal{FD}$ has objects (U, D_U), where D_U is a Dirac structure on U, and morphisms $f^{\mathcal{F}} : (U, D_U) \to (V, D_V)$ where $f : U \to V$ is linear and satifies $\mathcal{F}f(D_U) = D_V$. The composition rule $(g \circ f)^{\mathcal{F}} = g^{\mathcal{F}} \circ f^{\mathcal{F}}$ holds. In a similar way we define the *Backward-Dirac category* denoted $\mathcal{BD}$ whose objects are the same as the objects of $\mathcal{FD}$ while the morphisms $f^{B} : (U, D_U) \to (V, D_V)$ satisfy $\mathcal{B}f(D_V) = D_U$. The composition rule $(g \circ f)^{\mathcal{B}} = g^{\mathcal{B}} \circ f^{\mathcal{B}}$ holds.
The map $(U, D_U) \to (U^*, (D_U)^\circ)$ where $(D_U)^\circ$ is the annhilator of D_U in $U^* \oplus U^{**} = (U \oplus U^*)^*$ is a functor from $\mathcal{FD}$ into $\mathcal{BD}$ which establishes an isomorphism of categories between its image as a subcategory of $\mathcal{BD}$ and $\mathcal{FD}$. An entirely similar statement holds interchanging the roles of $\mathcal{BD}$ and $\mathcal{FD}$.

A Dirac structure D on a manifold M, is a vector subbundle of the Whitney sum $D \subset TM \oplus T^*M$ such that for each $x \in M$, $D_x \subset T_xM \oplus T_x^*M$ is a Dirac structure on the vector space T_xM at each point $x \in M$. A *Dirac manifold* is a manifold M with a Dirac structure D on M.

A Dirac structure on M yields a distribution $F_{D_x} \subset T_xM$ whose dimension is not necessarily constant, carrying a presymplectic form $\omega_D(x)\colon F_{D_x} \times F_{D_x} \to \mathbb{R}$ for all $x \in M$.

The fundamental integrability condition that the space of sections of D is closed under the *Courant bracket* unifies the Jacobi identity for a bivector and the closedness of a 2-form.

The integrability condition will not be used in this talk.

The following result describes a situation which is useful in many examples.

Theorem Let M be a manifold, ω be a 2-form on M and F be a regular distribution on M. Define the skew-symmetric bilinear form ω_F on F by restricting ω to $F \times F$. For each $x \in M$, define

$$D_{\omega_F}(x) = \{(v_x, \alpha_x) \in T_xM \oplus T_x^*M \;|v_x \in F_x,\ \alpha_x(u_x) = \omega_F(x)(v_x, u_x) \text{ for all } \ u_x \in F_x\}.$$

Then $D_{\omega_F} \subset TM \oplus T^*M$ is a Dirac structure on M. In fact, it is the only Dirac structure D on M satisfying $F_x = F_{D_x}$ and $\omega_F(x) = \omega_D(x)$ for all $x \in M$.
As usual, we have used the terminology *regular distribution* to mean that F has constant rank. Examples of this Theorem are the case $\omega = 0$, then $D_{\omega_F} = F \oplus F^\circ \subset TM \oplus T^*M$, and the case $F = TM$, then D_ω is the graph of ω.

We now turn to the definition of backward and forward Dirac structures for manifolds in the cases of interest for this work. Let us assume that we have a smooth map $f\colon M \to N$ between two manifolds M and N, and that $D_N \subset TN \oplus T^*N$ is a Dirac structure. At each point $x \in M$, one can use the backward of the map $T_m f$ to construct a subspace on $T_m M \oplus T_m^* M$. When this construction, carried out pointwise for all $x \in M$ results in a new Dirac structure D_M on M, we will say that D_M is the backward of D_N by the map Tf, and we will write $D_M = (Tf)\,(D_N)$. It should be noted that, in general, defining D_M in this way one does not get a smooth a subbundle of $TM \oplus T^*M$. One can use the following sufficient conditions.

I) If $T_x f$ is surjective for each $x \in M$, then $D_M = \mathcal{B}(Tf)\,(D_N)$ is a Dirac structure on M.

II) If $i_M \colon M \hookrightarrow N$ is a submanifold, then $D_M = \mathcal{B}(Ti_M)\,(D_N)$ is a Dirac structure if $D_N \cap TM^\circ$ has constant rank (the clean-intersection condition).

III) If D_N is given by the graph of a 2-form ω, then $D_M = \mathcal{B}(Tf)\,(D_N)$ is a Dirac structure on M.

Let now $f\colon M \to N$ be a smooth surjective submersion, and a Dirac structure D_M on M be given. When we aim at defining the push-forward, we first need to ask for Tf-invariance of D_M, meaning that

$$\mathcal{F}(T_x f)(D_M(x)) = \mathcal{F}(T_{x'} f)(D_M(x')), \qquad \text{whenever } f(x) = f(x').$$

The sufficient condition we will use to ensure that $\mathcal{F}(Tf)(D_M)$ defines a Dirac structure is the following:

IV) Let $f\colon M \to N$ be a surjective submersion and D_M be a Dirac structure on M. If D_M is Tf-invariant and $\ker(Tf) \cap D_M$ has constant rank, then $D_N = \mathcal{F}(Tf)\,(D_M)$ defines a forward structure.

A **Dirac system** is an differential relation of the type

$$(x, \dot{x}) \oplus d\mathcal{E}(x) \in D,$$

where D is a Dirac structure on M and $\mathcal{E}$ is a given function on M called the *Energy*.

Nonholonomic mechanics Dirac dynamical systems in the not necessarily integrable case may be viewed as a synthesis and a generalization of nonholonomic mechanics, as we will show next.

Define a Dirac structure $D_\Delta \subseteq TM \oplus T^*M$ on $M = TQ \oplus T^*Q$ associated to a given distribution $\Delta \subseteq TQ$ on a manifold Q by the local expression

$$\begin{aligned} D_\Delta(q, v, p) = \{(q, v, p, &\dot{q}, \dot{v}, \dot{p}, \alpha, \gamma, \beta) \mid \\ &\dot{q} \in \Delta(q),\ \alpha + \dot{p} \in \Delta^\circ(q),\ \beta = \dot{q},\ \gamma = 0\} \end{aligned}$$

which has an intrinsic meaning.

It is straightforward to check that for $\mathcal{E}(q, v, p) = pv - \mathcal{L}(q, p)$ the Dirac system

$$(x, \dot{x}) \oplus d\mathcal{E}(x) \in D_\Delta,$$

where $x = (q, v, p)$, is equivalent to the Lagrange d'Alembert equations,

$$\begin{aligned} \dot{p} - \frac{\partial \mathcal{L}}{\partial q} &\in \Delta^\circ \\ \dot{q} &= v \\ p &= \frac{\partial \mathcal{L}}{\partial v} \\ \dot{q} &\in \Delta, \end{aligned}$$

An LC circuit can be approached as a nonholonomic system as described before, where the charge space is a vector space $E = Q$, then $M = TE \oplus T^*E$ and defining $\bar{D}_\Delta$ and $\mathcal{E}$ as before with the Lagrangian

$$\mathcal{L}(q, v) = \frac{1}{2} \sum_{i=1}^{n} L_i v_i^2 - \frac{1}{2} \sum_{i=1}^{n} \frac{1}{C_i} q_i^2 .$$

Define the linear maps $\varphi \colon E \to E^*$ and $\psi \colon E \to E^*$ by

$$\varphi(v) = \frac{\partial \mathcal{L}}{\partial v} = (L_1 v_1, \ldots, L_n v_n) ,$$
$$\psi(q) = \frac{\partial \mathcal{L}}{\partial q} = -(q_1/C_1, \ldots, q_n/C_n) .$$

The evolution equations for an LC circuit become

$$\begin{aligned} \dot{p} - \psi(q) &\in \Delta^\circ \\ \dot{q} &= v \\ p &= \varphi(v) \\ \dot{q} &\in \Delta. \end{aligned}$$

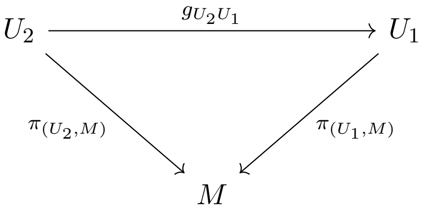
In this simple case one can apply a generalization of the Dirac algorithm to find equations of motion in Hamiltonian form. The last two examples show how the notion of Dirac system encompass nonholonomic systems in mechanics and the theory of LC circuits in an unique formalism.

Closed and open Port-Dirac structures. We will describe the notions of closed forward-port-Dirac structures and also open forward-port-Dirac structures. The dual notions of closed backward-port-Dirac structures and also open backward-port-Dirac structures can be defined essentially by reversing arrows. They are all extensions of the notion of Dirac structures. We will consider two important issues, namely, interconnection and dynamics.
Statements related to *backward* are dual from those related to *forward* and could be obtained one from each other directly by duality or by direct proof.

A *closed forward-port-Dirac structure* is a 5-uple

$$A = (\pi_{(U_1,M)}, \pi_{(U_2,M)}, D_{U_1}, D_{U_2}, g_{U_2,U_1})$$

where $\pi_{(U_i,M)} : U_i \longrightarrow M$ is a vector bundle and D_{U_i} is a Dirac structure on U_i , $i = 1, 2$, $g_{U_2U_1} : U_2 \longrightarrow U_1$ is a vector bundle map over the identity 1_M and the following diagram commutes.



The Dirac structure D_A on U_1 is defined by

$$D_A = \mathcal{F}(\Phi_A)(D_{U_1} \oplus D_{U_2})$$

where $\Phi_A : U_1 \oplus U_2 \longrightarrow U_1$ is given by $\Phi_A(u_1 \oplus u_2) = u_1 + g_{U_2,U_1}(u_2)$. In other words, D_A is the set of all $(u_1, \alpha_1) \in U_1 \oplus U_1^*$ such that there exists $(u_2, \alpha_2) \in U_2 \oplus U_2^*$ such that

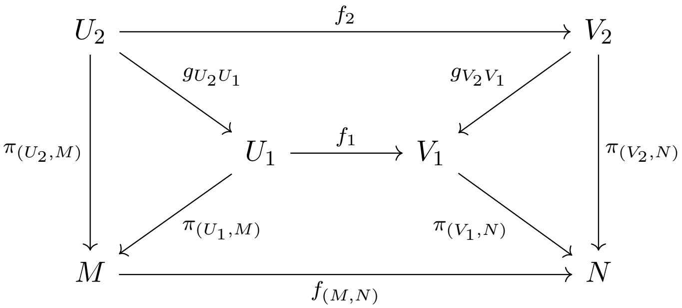
$$\begin{aligned} (u_1 - g_{U_2,U_1}(u_2), \alpha_1) &\in D_{U_1} \\ (u_2, \alpha_2) &\in D_{U_2} \\ g^*_{U_2 U_1} \alpha_1 &= \alpha_2 \end{aligned}$$

Example Consider the particular case in which D_{U_1} is the Dirac structure associated to a bivector π and $D_{U_2} = U_2 \oplus \{0\}$ and assume that $\operatorname{Im} g_{U_2,U_1}$ is a subbundle of U_1. Then (**??**) becomes

$$\begin{aligned} u_1 - g_{U_2,U_1}(u_2), \alpha_1) &= \pi^\sharp \alpha_1 \\ g^*_{U_2 U_1} \alpha_1 &= 0. \end{aligned}$$

We can recognize this equation as being related to some questions in control theory.

Let $B = (\pi_{(V_1,N)}, \pi_{(V_2,N)}, D_{V_1}, D_{V_2}, g_{V_2,V_1})$.
A *forward morphism* $f : A \longrightarrow B$ is determined by two vector bundle morphisms $f_i : U_i \longrightarrow V_i$, $i = 1, 2$, over a surjective submersion $f_{(M,N)} : M \longrightarrow N$, that satisfy $\mathcal{F}(f_i)D_{U_i} = D_{V_i}$, $i = 1, 2$ and the following diagram commutes.



Then the closed forward-port-Dirac structures form a category and one can prove that if $f : A \longrightarrow B$, $f = (f_1, f_2)$, is a forward morphism, then $\mathcal{F}(f_1)D_A = D_B$. This implies that the assignment $A \mapsto (U_1, D_A)$ and $f \mapsto \mathcal{F}(f_1)$ is a functor from the category of closed-forward-port-Dirac structures to the category $\mathcal{FD}$.

Open forward-port-Dirac structures. We will call *open forward-port-Dirac structurs* objects obtained by replacing in the definition of a closed forward-Dirac system A the Dirac structure D_{U_2} by the coisotropic structure $U_2 \oplus U_2^*$, and we will denote them

$$A = (\pi_{(U_1,M)}, \pi_{(U_2,M)}, D_{U_1}, U_2 \oplus U_2^*, g_{U_2,U_1}).$$

Note that the vector bundle $U_1 \oplus U_2$ carries the coisotropic structure $\Sigma = D_{U_1} \oplus (U_2 \oplus U_2^*)$. The coisotropic structure Σ_A on U_1 is defined by

$$\Sigma_A = \mathcal{F}(\Phi_A)(D_{U_1} \oplus (U_2 \oplus U_2^*)).$$

We can deduce that D_A is the set of all $(u_1, \alpha_1) \in U_1 \oplus U_1^*$ such that there exists $(u_2, \alpha_2) \in U_2 \oplus U_2^*$ such that

$$\begin{aligned} (u_1 - g_{U_2,U_1}(u_2), \alpha_1) &\in D_{U_1} \\ g^*_{U_2 U_1}\alpha_1 &= \alpha_2, \end{aligned}$$

Since $(u_2, \alpha_2) \in U_2 \oplus U_2^*$ are arbitrary the previous equations are in a sense equivalent to

$$(u_1 - g_{U_2,U_1}(u_2), \alpha_1) \in D_{U_1}.$$

Dynamics on closed or open forward-port-Dirac structures Let us consider a closed or open forward-port-Dirac structure A where $U_1 = TM$. Then by definition the dynamics is given by the Dirac system $(x, \dot{x}) \oplus dE(x) \in D_A$ or the coisotropic system $(x, \dot{x}) \oplus dE(x) \in \Sigma_A$, respectively, where $E : M \to \mathbb{R}$ is an energy function.
In the case of closed forward-port-Dirac system equation we obtain

$$\begin{aligned} ((x, \dot{x}) - g_{U_2,U_1}(u_2), dE(x)) &\in D_{U_1} \\ (u_2, \alpha_2) &\in D_{U_2} \\ g^*_{U_2,U_1} dE(x) &= \alpha_2. \end{aligned}$$

Particularly interesting is the case in which D_{U_1} is the graph of a Poisson bracket π on M and D_{U_2} is the trivial fiberwise presymplectic structure $U_2 \oplus \{0\}$ on U_2, which gives

$$(x,\dot{x}) = \pi^\sharp dE(x) + g_{U_2,U_1}(u_2)$$
$$g^*_{U_2,U_1} dE(x) = 0,$$

If $g_{U_2,U_1}(u_2) = 0$ then one obtains Hamilton's equations.
In the case of an open forward-port-Dirac system one obtains

$$((x,\dot{x}) - g_{U_2,U_1}(u_2), \alpha_1) \in D_{U_1},$$

and if D_{U_1} is given by a Poisson structure $\pi^\sharp$ one obtains

$$(x,\dot{x}) = \pi^\sharp dE(x) + g_{U_2,U_1}(u_2)$$

which is simply a system with control parameters u_2.

Interconnection of a closed forward port-Dirac structure. Let

$$A = \left(\pi_{(U_1,M)}, \pi_{(U_2,M)}, D_{U_1}, D_{U_2}, g_{U_2U_1}\right)$$

be a given closed forward port-Dirac structure and let $D'_{U_i} \subset U_i \oplus U_i^*$ be given Dirac structures on U_i, $i = 1, 2$, called the *interconnecting* Dirac structures. We define the backward interconnected Dirac structures

$$\mathcal{I}_{\mathcal{B}}(D_{U_i}, D'_{U_i}) = \mathcal{B}(d_{U_i})(D_{U_i} \times D'_{U_i}), \qquad i = 1, 2,$$

where $d_{U_i} : U_i \to U_i \times U_i$, $i = 1, 2$, is the fiberwise diagonal map. Note that for any Dirac structure D we have: $\mathcal{I}_{\mathcal{B}}(D_1, D_1) = D_1$. The backward interconnected closed forward port-Dirac structure is, by definition the following.

$$\mathcal{I}_{\mathcal{B}}(A, D'_{U_1}, D'_{U_2}) = \Big(\pi_{(U_1,M)}, \pi_{(U_2,M)}, \mathcal{I}_{\mathcal{B}}(D_{U_1}, D'_{U_1}), \mathcal{I}_{\mathcal{B}}(D_{U_2}, D'_{U_2}), g_{U_2U_1}\Big).$$

Interconnection of two given closed forward port-Dirac structures. Let

$$A = \left(\pi_{(U_1,M)}, \pi_{(U_2,M)}, D_{U_1}, D_{U_2}, g_{U_2U_1}\right)$$
$$B = \left(\pi_{(V_1,M)}, \pi_{(V_2,M)}, D_{V_1}, D_{V_2}, g_{V_2V_1}\right)$$

be given. The product structure is by definition

$$A \times B = \left(\pi_{(W_1,K)}, \pi_{(W_2,K)}, D_{W_1}, D_{W_2}, g_{W_2W_1}\right)$$

with $K = M \times N$, $W_i = U_i \times V_i$, $D_{W_i} = D_{U_i} \times D_{V_i}$, $g_{W_2W_1} = g_{U_2U_1} \times g_{V_2V_1}$.
We can now interconnect the structures in the way described before, that is, given interconnecting Dirac structures D'_{W_i}, $i = 1, 2$, we have the interconnected structure:

$$\mathcal{I}_{\mathcal{B}}(A \times B, D'_{W_1}, D'_{W_2}).$$

Interconnection of open forward port-Dirac systems.
Some backward interconnection of open forward port-Dirac systems can also be described by a similar formalism. Consider an open system

$$A = \left(\pi_{(U_1,M)}, \pi_{(U_2,M)}, D_{U_1}, U_2 \oplus U_2^*, g_{U_2U_1}\right).$$

Given interconnecting structures $D'_{U_i} \subset U_i \oplus U_i^*$, $i = 1, 2$, we define the backward interconnected system of A via D'_{U_i} as follows.

$$\mathcal{I}_{\mathcal{B}}(A, D'_{U_1}, D'_{U_2}) = \left(\pi_{(U_1,M)}, \pi_{(U_2,M)}, \mathcal{I}_{\mathcal{B}}(D_{U_1}, D'_{U_1}), D'_{U_2}, g_{U_2U_1}\right).$$

Note that $D'_{U_2} = \mathcal{I}_{\mathcal{B}}(U_2 \oplus U_2^*, D'_{U_2})$, defined with the same formalism used for closed forward port-Dirac systems but with D'_{U_2} replaced by $U_2 \oplus U_2^*$. If we choose $D'_{U_1} = D_{U_1}$ we get

$$\mathcal{I}_{\mathcal{B}}(A, D_{U_1}, D'_{U_2}) = \left(\pi_{(U_1,M)}, \pi_{(U_2,M)}, D_{U_1}, D'_{U_2}, g_{U_2U_1}\right),$$

which corresponds to the notion of closing the ports.

More general interconnections. Many more general types of interconnections can be defined by changing in the previous definitios *backward* by *forward* and also considering some other interconnecting Dirac structures. As a whole they cover a class of interconnections of interest in physical systems.

THANKS FOR YOUR ATTENTION!