On the Characterization of Aggregate Dynamic Preferences

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- Aggregate preferences satisfy dynamic consistency

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- For each agent $i \in N$, intertemporal preferences over nonnegative sequences $\boldsymbol{c}^i := \{c_t^i\}_{t=0}^{\infty}$ represented by

$$w_0^i(\boldsymbol{c}^i) = \sum_{t=0}^{\infty} (\delta^i)^t u(c_t^i) \tag{1}$$

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$$\Pi(k_0) := \{\mathbf{k} \in K^\infty : 0 \leq k_{t+1} \leq f(k_t),\ t \in T\}$$

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- Given k_0, the set of all feasible consumption paths is given by

$$\Omega(k_0) := \left\{\hat{\boldsymbol{c}} \in \ell_+^n : 0 \leq \sum_i c_t^i \leq f(k_t),\ \text{for some } \boldsymbol{k} \in \Pi(k_0),\ t \in T\right\}$$

Pareto Problem with Heterogeneous Discounting

- Utility possibility set $\mathcal{U}(k)$

$$\mathcal{U}(k) := \left\{ z \in \underline{\mathbb{R}}^n : z^i = w_0^i(\boldsymbol{c}^i),\ i = 1, \ldots, n,\ \text{for some } \hat{\boldsymbol{c}} \in \Pi(k) \right\}$$

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- Pareto weights: $\theta := (\theta^1, \ldots, \theta^n)$ in the $(n-1)$-dimensional simplex,

$$\Theta^n := \left\{ \theta \in \mathbb{R}_+^n : \theta^i \geq 0,\ i = 1, \ldots, n;\ \text{and } \textstyle\sum_i \theta^i = 1 \right\}. \tag{2}$$

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- Value function is the support function of the set $\mathcal{U}(k)$

$$V(k, \theta) := \sup_{z \in \mathcal{U}(k)} \textstyle\sum_i \theta^i z^i$$

where V is strictly increasing and strictly concave in k, strictly convex in θ, twice continuously differentiable

Main Aggregation Result

Theorem

There exist maps $U : X \times \Theta^n \to \underline{\mathbb{R}}$, $\mu : \Theta^n \to \mathbb{R}_+$, *and* $F : \Theta^n \to \Theta^n$, *such that the value of the Pareto problem* (PP) *satisfies the following functional equation*

$$V(k,\theta) = \sup_{y \in \Gamma(k)} \Big[U(f(k) - y, \theta) + \mu(\theta) V(y, F(\theta)) \Big], \tag{3}$$

for all (k,θ) *in the interior of* $K \times \Theta^n$.

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- Let $V : K \times \Theta^n$ be the value of the following program

$$\sup_{\hat{c}, y \geq 0,\, z \in \mathcal{U}} \inf_{\tau \in \Theta^n} \sum_{i=1}^{n} \theta^i \left[u(c^i) + \delta^i z^i \right], \tag{PP}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} c^i + y \leq f(k),$$

$$\sum_{i=1}^{n} \tau^i z^i - V(y, \tau) \leq 0.$$

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$$\mathscr{L}(\hat{c}, y, z, \tau, \lambda, \mu | k, \theta) :=$$

$$\sum_i \theta^i [u(c^i) + \delta^i z^i] + \lambda [f(k) - \sum_i c^i - y] - \mu [\sum_i \tau^i z^i - V(\tau, y)],$$

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where $\lambda, \mu \geq 0$ are Lagrange multipliers

- A solution to (PP) is obtained by solving

$$V(k, \theta) = \sup_{(\hat{c}, y, z, \lambda) \in \Phi} \inf_{(\tau, \mu) \in \Psi} \mathscr{L}(\hat{c}, y, z, \tau, \lambda, \mu | k, \theta),$$

where $\Phi := \hat{X} \times \mathbb{R}_+ \times \mathcal{U} \times \mathbb{R}_+$ and $\Psi := \Theta^n \times \mathbb{R}_+$

KKT Optimality Conditions (interior solution)

Given $(k_0, \theta_0) \in K \times \Theta^n$, an optimal path is a sequence

$$\left\{ \left(c_t^i, w_{t+1}^i, \theta_{t+1}^i\right)_{i=1}^n, k_{t+1}, \lambda_t, \mu_t \right\}_{t=0}^\infty$$

that satisfies for each t

$$\begin{aligned}
\theta_t^i u'(c_t^i) &= \lambda_t, && i \in N, \\
w_t^i &= u(c_t^i) + \delta^i w_{t+1}^i, && i \in N, \\
\theta_t^i \delta^i &= \mu_t \theta_{t+1}^i, && i \in N, \\
\textstyle\sum_i \theta_{t+1}^i &= 1, && \\
\textstyle\sum_i c_t^i + k_{t+1} &= f(k_t), && \\
\lambda_t &= \mu_t \lambda_{t+1} f'(k_{t+1}). &&
\end{aligned}$$

Dynamics

- The Lagrange multiplier μ_t can be interpreted as an aggregate discount factor which is time-dependent

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- Aggregate instantaneous utility function $U(c_t, \theta_t)$ obtained from Pareto optimal allocations and feasibility constraints
- The Lagrange multiplier λ_t is the marginal utility of aggregate consumption

$$\lambda_t = \frac{\partial U(c_t, \theta_t)}{\partial c_t} \qquad\qquad t = 0, 1, \ldots$$

Aggregate Preferences

- Aggregate utility index defined as $W_t := \sum_i \theta^i_t w^i_t$, then

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$$\sum_{i=1}^{n} z_i(c_t, \theta_t) = c_t, \quad \text{and} \quad U(c_t, \theta_t) = \sum_{i=1}^{n} \theta_t^i \, u\left(z_i(c_t, \theta_t)\right).$$

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- Separability of aggregate instantaneous utility function

$$U(c_t, \theta_t) = G(c_t) H(\theta_t)$$

Separability

Definition

A utility function $u : \mathbb{R}_+ \to \mathbb{R}$ satisfies linear absolute tolerance to consumption fluctuation (LATCF) if

$$u(c_t^i) = \frac{\gamma}{1-\gamma}\left[\left(\phi + \frac{\rho}{\gamma}\, c_t^i\right)^{1-\gamma} - 1\right], \qquad 0 < \gamma < +\infty,\ \gamma \neq 1$$

with $\phi + (\rho/\gamma)\, c \geq 0$, $0 < \rho < +\infty$, and $\phi \in \mathbb{R}$.

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Proposition

If each agent has an instantaneous utility function in the LATCF class, then the sharing rule is linear in aggregate consumption c_t, i.e.,

$$z_i(c_t, \theta_t) = a_i(\theta_t)\, c_t + b_i(\theta_t),$$

where $a_i(\theta_t) \geq 0$, $\sum_i a_i(\theta_t) = 1$ and $\sum_i b_i(\theta_t) = 0$, for all θ_t and for all t.

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- A sequence $\pi = (\pi_t)$ of transition probabilities $\pi_t : \Gamma_t \times S_t \to \mathcal{P}_t$
- Prior distribution $\pi_0 := (\pi_0^1, \ldots, \pi_0^n) \in \Delta^n$

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- Posterior distribution: Bayesian sequential update

$$\pi^s_{t+1} = \frac{\pi^s_t \delta^s}{\sum_s \pi^s_t \delta^s}, \qquad s \in S$$

where $\pi^s_{t+1} := P(X_{t+1} = s | X_t = s)$, for each $s \in S$.

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- "Before uncertainty is resolved"

$$V_0 = \pi^1_0 \left[u(c^1_0) + \hat{\nu}_0 \, z^1_1 \right] + \cdots + \pi^n_0 \left[u(c^n_0) + \hat{\nu}_0 \, z^n_1 \right]$$

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- "After uncertainty is resolved"

$$V^s_\pi = u(c^s_0) + \nu_0 \sum_{s \in S} \pi^s_1 z^s_1 = u(c^s_0) + \nu_0 \sum_{s \in S} \pi^s_1 \left[u(c^s_1) + \hat{\nu}_1 \, z^s_2 \right]$$

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$$V = \pi_0^1 \, u(c_0^1) + \nu_0 \, \pi_1^1 \, u(c_1^1) + \nu_0 \, \nu_1 \, \pi_2^1 \, u(c_2^1) + \cdots$$
$$+ \cdots +$$
$$+ \pi_0^n \, u(c_0^n) + \nu_0 \, \pi_1^n \, u(c_1^n) + \nu_0 \, \nu_1 \, \pi_2^n \, u(c_2^n) + \cdots$$

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& + \cdots + \\
& + \pi_0^n \, u(c_0^n) + \nu_0 \, \pi_1^n \, u(c_1^n) + \nu_0 \, \nu_1 \, \pi_2^n \, u(c_2^n) + \cdots
\end{aligned}
$$

- Stochastic recursive preferences with Bayesian updating imply that optimal choices of (c_t^i, z_{t+1}^i), $i = 1, \ldots, n$, $t = 0, 1, \ldots$ are dynamically consistent

References I

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