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The inverse problem of the calculus of variations: An Introduction

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The *inverse problem of the calculus of variations* is the problem of finding conditions, ensuring that a given system of (ordinary or partial) differential equations coincides with the system of Euler-Lagrange equations of an integral variational functional. Its origin, dated 1886, is connected with the names of Sonin and Helmholtz; a newer modified version of the inverse problem for systems of ordinary second order equations, using *variational integrating factors*, was presented by Douglas in 1944. Since then the problem, lying on the border of the calculus of variations, mathematical analysis of differential equations, differential geometry, and topology of manifolds was studied by many authors. However, in its generality it still belongs to mathematical problems that wait for a complete solution. The aim of this lecture series is to give an introduction to the local and global inverse problem.

First we consider the variationality problem for systems of differential equations. We derive the Helmholtz variationality conditions and find integrability conditions for the Douglas's problem.

The global inverse problem is formulated within the global variational theory, extending the classical calculus of variations from Euclidean spaces to smooth manifolds. The problem is to find conditions when a system of equations on a manifold, which is locally variational, admits a global Lagrangian. We introduce underlying variational concepts in terms of differential forms, and study the theory of variational sequences, in which one arrow represents the Euler-Lagrange mapping of the calculus of variations. The sequence relates properties of the Euler-Lagrange mapping with the De Rham cohomology of the underlying manifold.

In these lectures we do not consider the inverse problem for vector fields on tangent bundle, which is related with the Douglas's problem.

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The following list includes selected titles, in which further references can be found. It is restricted to sources of mathematical character. Reference [1] contains relatively complete bibliography.

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