$\forall\exists!$-classes and algebraic functions

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Given a first order language $\mathcal{L}$, a $\forall\exists!$-sentence in $\mathcal{L}$ is a sentence of the form $\forall x_1, ..., x_n \exists! y_1, ..., y_m O(\overline{x}, \overline{y})$, where O is a quantifier-free $\mathcal{L}$-formula, and $\exists!$ stands for "there exist unique". A $\forall\exists!$ *class* is the class of all models of a set of $\forall\exists!$-sentences. A class of models $\mathcal{K}$ has the *intersection property (i.p.)* if given $\mathbf{A} \in \mathcal{K}$ and $\mathbf{A} \supseteq \mathbf{A}_i \in \mathcal{K}$, $i \in I$, such that $\bigcap_{i \in I} A_i \neq \emptyset$, then $\bigcap_{i \in I} \mathbf{A}_i \in \mathcal{K}$. It is easy to see that every $\forall\exists!$ class has the i.p.. C.C. Chang conjectured in [1] that an elementary class $\mathcal{K}$ has the i.p. if and only if it is a $\forall\exists!$ class. In the paper [2] M. O. Rabin disproves Chang's conjecture, and also characterizes the elementary classes with i.p. as certain $\forall\exists$ classes. It turns out that $\forall\exists!$ classes have a further property not necessarily true of every elementary class with i.p.: a class $\mathcal{K}$ is *closed under fixed point submodels* if for every $\mathbf{A} \in \mathcal{K}$ and γ an automorphism of $\mathbf{A}$ the submodel with universe $\mathrm{Fix}(\gamma) = \{a \in A : \gamma(a)\}$ is in $\mathcal{K}$ (whenever the set $\mathrm{Fix}(\gamma)$ is non-empty). In the case that the elementary class $\mathcal{K}$ is formed (up to isomorphism) by a finite number of finite models, we prove that this additional closure condition is enough to ensure that $\mathcal{K}$ is a $\forall\exists!$ class.

Theorem 1 *Let $\mathcal{K}$ be a finite set of finite $\mathcal{L}$-models. Then $\mathbb{I}(\mathcal{K})$ is finitely axiomatizable by $\forall\exists!$-sentences if and only if $\mathcal{K}$ has the i.p. and $\mathcal{K}$ is closed under fixed point submodels.*

Given a $\forall\exists!$-sentence $\varphi = \forall\overline{x}\exists!\overline{y}O(\overline{x}, \overline{y})$ and a model $\mathbf{A}$ of φ we can implicitly define m functions $[\varphi]_i^{\mathbf{A}} : A^n \to A$, $1 \leq i \leq m$, by $([\varphi]_1^{\mathbf{A}}(\overline{a}), ..., [\varphi]_n^{\mathbf{A}}(\overline{a}))$ = the unique $\overline{b}$ such that $\mathbf{A} \vDash O(\overline{a}, \overline{b})$. The sentence φ is called an *equational function definition sentence* (*EFD-sentence*) if O is a conjunction of term equalities. A function f is *algebraic on* $\mathbf{A}$ provided there is an EFD-sentence φ such that $f = [\varphi]_i^{\mathbf{A}}$. An interesting problem is: given a model $\mathbf{A}$ characterize all algebraic functions on $\mathbf{A}$. In our talk we show how to apply Theorem 1 to solve this problem for certain kinds of models $\mathbf{A}$.

[1] C.C. Chang, *Unions of chains of models and direct products of models*, Summer Institute for, Symbolic Logic, Cornell University (1957), 141–143.

[2] M.O. Rabin, *Classes of models and sets of sentences with the intersection property*, Annales de la Faculté des sciences de l'Université de Clermont (1962), no. 7, 39–53.