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ON $\mathscr{B}$-OPERATOR DERIVATIVES ON NON AMENABLE NUCLEAR BANACH ALGEBRAS

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ABSTRACT. We review recent advances and some problems related to our research about bounded derivations on non amenable nuclear Banach algebras.

Let $\mathfrak{X}$ be an infinite dimensional complex Banach space. By $\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*$ we will denote the completion of the algebraic tensor product of $\mathfrak{X}$ and $\mathfrak{X}^*$ with respect to the projective cross norm $\|\circ\|_\pi$. Thus $\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*$ becomes a Banach algebra by means of the product so that $(x\otimes x^*)(y\otimes y^*) = \langle y,x^*\rangle (x\otimes y^*)$ if $x,y\in\mathfrak{X}$, $x^*,y^*\in\mathfrak{X}^*$. Let $\mathscr{N}_{\mathfrak{X}^*}(\mathfrak{X})$ be the subclass of nuclear operators of $\mathscr{B}(\mathfrak{X})$. All $T\in\mathscr{N}_{\mathfrak{X}^*}(\mathfrak{X})$ can be writen as $Tx=\sum_{n=1}^{\infty}\langle x,y_n^*\rangle y_n$ if $x\in\mathfrak{X}$, with $\{y_n\}_{n=1}^{\infty}\subseteq\mathfrak{X}$, $\{y_n^*\}_{n=1}^{\infty}\subseteq\mathfrak{X}^*$ and $\sum_{n=1}^{\infty}\|y_n\|\|y_n^*\|<\infty$. The infimum of these series taking over all such representations of T furnish a norm $\|T\|_{\mathscr{N}_{\mathfrak{X}^*}(\mathfrak{X})}$ for T so that $\left(\mathscr{N}_{\mathfrak{X}^*}(\mathfrak{X}),\|\circ\|_{\mathscr{N}_{\mathfrak{X}^*}(\mathfrak{X})}\right)$ becomes a Banach algebra.

Amenable Banach algebras were introduced and studied by B. E. Johnson in his definitive monograph [5]. Particularly, the notion of amenability is closely related with questions concerning to bounded derivations on Banach algebras. Briefly, a Banach algebra $\mathscr{U}$ is called *amenable* if its first Hochschild cohomology group $H^1(\mathscr{U},X^*)$ with coefficients in the dual of any Banach $\mathscr{U}$-bimodule X is trivial. If this is the case any derivation $D:\mathscr{U}\to X^*$ is *inner*, i.e. there exists $\lambda\in X^*$ so that $D(a)=\lambda\cdot a-a\cdot\lambda$ if $a\in\mathscr{U}$. Indeed, $\mathscr{U}$ is called *super-amenable* when the first cohomology group of $\mathscr{U}$ with coefficients in any Banach $\mathscr{U}$-bimodule is trivial.

Theorem 1. *(cf.* [8]*, Th. 4.3.5, p. 98) The following assertions are equivalent*

i $\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*$ *is super-amenable.*
ii $\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*$ *is amenable.*
iii $\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*$ *has a bounded approximate identity.*
iv $\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*$ *has a bounded left approximate identity.*
v $\mathscr{N}_{\mathfrak{X}^*}(\mathfrak{X})$ *has a bounded left approximate identity.*
vi $\dim(\mathfrak{X})=\dim(\mathfrak{X}^*)<\infty$.

Consequently, the study of bounded derivations on $\mathscr{N}_{\mathfrak{X}^*}(\mathfrak{X})$ has its own interest as well as the determination of their structure and properties. Fortunately, there is an isometric isomorphism of Banach algebras between $\mathscr{N}_{\mathfrak{X}^*}(\mathfrak{X})$ and $\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*$ (cf. [8], Th. C.1.5). This fact allowed us to improve previous researches done in the frame of Banach algebras of Hilbert-Schmidt type (cf. [1], [2]). The class of bounded derivations $\mathscr{D}(\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*)$ on $\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*$ is a Banach subspace of $\mathscr{B}(\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*)$.

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Example 2. *Let* $\upsilon \in \mathfrak{X}\widehat{\otimes}\mathfrak{X}^*$, $\Delta_\upsilon(\alpha) = \upsilon \cdot \alpha - \alpha \cdot \upsilon$, $\alpha \in \mathfrak{X}\widehat{\otimes}\mathfrak{X}^*$. *Therefore* $\Delta_\upsilon \in \mathscr{D}(\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*)$ *is the inner derivation defined by* υ. *In general, it is known that every bounded derivation on the uniform Banach algebra of bounded operators* $\mathscr{B}(\mathfrak{X})$ *is inner (cf.* [6]*).*

Problem 3. *What is the precise norm of* Δ_υ*?- This problem could be hard. For instance, let* $\mathfrak{X}$ *be a Hilbert space,* $T \in \mathscr{B}(\mathfrak{X})$, Δ_T *be the inner derivation induced by* T *on* $\mathscr{B}(\mathfrak{X})$. *Then J. G. Stampfli showed that* $\|\Delta_T\| = 2\,\mathrm{dist}(T, \mathbb{C}\cdot \mathrm{Id}_{\mathfrak{X}})$ *(cf.* [11]*). B. E. Johnson noted that the above formula is no longer true in the general case. If* $\mathfrak{X}$ *is a uniformly convex Banach space the validity of Stampfli´s formula is a necessary and sufficient condition in order that* $\mathfrak{X}$ *be a Hilbert space (see* [4] *and* [7]*).*

Example 4. *Given* $T \in \mathscr{B}(\mathfrak{X})$ *there is a unique* $\delta_T \in \mathscr{D}(\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*)$ *so that*

$$\delta_T(x \otimes x^*) = T(x) \otimes x^* - x \otimes T^*(x^*)$$

for all basic tensor $x \otimes x^* \in \mathfrak{X}\widehat{\otimes}\mathfrak{X}^*$. *It is said that* δ_T *is the* $\mathscr{B}$*-derivation supported by* T.

Problem 5. *Let* $\delta : \mathscr{B}(\mathfrak{X}) \to \mathscr{D}(\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*)$, $\delta(T) = \delta_T$ *if* $T \in \mathscr{B}(\mathfrak{X})$. *Then* δ *is a linear bounded operator so that*

$$\delta(S \circ T) \triangleq [\delta(S), \delta(T)] = \delta(S) \circ \delta(T) - \delta(T) \circ \delta(S)$$

if $S, T \in \mathscr{B}(\mathfrak{X})$. *It would be relevant to evaluate* $\|\delta\|$.

Lemma 6. $\ker(\delta) = \mathbb{C}\cdot \mathrm{Id}_{\mathfrak{X}}$.

Proof. Let $T \in \mathscr{B}(\mathfrak{X})$ so that $\delta_T = 0$ and let $\lambda \in \sigma(T)$. If λ belongs to the compression spectrum of T let $x^* \in \mathfrak{X}^* - \{0\}$ so that $x^*|_{\mathrm{R}(T - \lambda\,\mathrm{Id}_{\mathfrak{X}})} \equiv 0$. For all $x \in \mathfrak{X}$ we have

$$\langle x, T^*(x^*)\rangle = \langle T(x), x^*\rangle = \langle \lambda x, x^*\rangle = \langle x, \lambda x^*\rangle,$$

i.e. $(T^* - \lambda\,\mathrm{Id}_{\mathfrak{X}^*})(x^*) = 0$. Moreover, since

$$(T(x) - \lambda x) \otimes x^* = x \otimes (T^*(x^*) - \lambda x^*) = 0,$$

the projective norm is a cross-norm and $x^* \neq 0$ then $T = \lambda\,\mathrm{Id}_{\mathfrak{X}}$. If $\lambda \in \sigma_{ap}(T)$ we choose a sequence $\{y_n\}_{n=1}^\infty$ of unit vectors of $\mathfrak{X}$ so that $T(y_n) - \lambda y_n \to 0$. If $y^* \in \mathfrak{X}^*$ then

$$\begin{aligned} 0 &= \lim_{n\to\infty} \|(T(y_n) - \lambda y_n) \otimes y^*\|_\pi \\ &= \lim_{n\to\infty} \|y_n \otimes T^*(y^*) - \lambda y^*\|_\pi = \|T^*(y^*) - \lambda y^*\|. \end{aligned}$$

As above we conclude that $T = \lambda\,\mathrm{Id}_{\mathfrak{X}}$. □

Let us assume that $\mathfrak{X}$ has a bounded shrinking basis $\mathscr{X} = \{x_n\}_{n=1}^\infty$ whose associated sequence of coefficient functionals is $\mathscr{X}^* = \{x_n^*\}_{n=1}^\infty$. Then a basis $\{z_n\}_{n=1}^\infty$ of $\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*$ is induced if we arrange all tensors $x_n \otimes x_m^*$ for $r, s \in \mathbb{N}$ in a right way. For, if $m \in \mathbb{N}$ let $n \in \mathbb{N}$ so that $(n-1)^2 < m \leq n^2$ we write

$$\sigma(m) = \begin{cases} \left(m - (n-1)^2, n\right) & if \quad (n-1)^2 + 1 \leq m \leq (n-1)^2 + n, \\ \left(n, n^2 - m + 1\right) & if \quad (n-1)^2 + n \leq m \leq n^2. \end{cases}$$

Therefore $\sigma : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ becomes a bijective function and it suffices to put $z_n = x_{\sigma_1(n)} \otimes x_{\sigma_2(n)}^*$ (cf. [9], [10]).

Theorem 7. *(cf.* [3]*) If $\delta \in \mathscr{D}(\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*)$ there are unique sequences $\{\mathfrak{h}_n\}_{n\in\mathbb{N}}$ and $\{\mathfrak{y}_u^v\}_{u,v\in\mathbb{N}}$ so that if $u,v \in \mathbb{N}$ then*

$$\delta\left(z_{\sigma^{-1}(u,v)}\right) = (\mathfrak{h}_u - \mathfrak{h}_v)\, z_{\sigma^{-1}(u,v)} + \sum_{n=1}^{\infty}\left(\mathfrak{y}_u^n \cdot z_{\sigma^{-1}(n,v)} - \mathfrak{y}_n^v \cdot z_{\sigma^{-1}(u,n)}\right).$$

We say that $\mathfrak{h} = \mathfrak{h}[\delta]$ and that $\eta = \eta[\delta]$ are the $\mathfrak{h}$ and $\mathfrak{y}$ sequences of δ respectively. Indeed, $\mathfrak{h}[\delta] = \left\{\left\langle \delta(z_{n^2}), z_{n^2}^*\right\rangle\right\}_{n=1}^{\infty}$ and $\eta[\delta] = \left\{\left\langle \delta(z_{n^2}), z_{m^2}^*\right\rangle\right\}_{n,m=1}^{\infty}$. An $\mathscr{X}$-Hadamard bounded derivation on $\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*$ is any derivation with null η sequence. In [3] *it is proved that they constitute a complementary Banach subspace of $\mathscr{D}(\mathfrak{X}\widehat{\otimes}\mathfrak{X}^*)$.*

Problem 8. *Characterize the class of Hadamard derivations intrinsically or independently of any basis.*

Problem 9. *What is the relation between $\mathscr{X}$-Hadamard and $\mathscr{B}$-derivations?- We conjecture that any $\mathscr{X}$-Hadamard derivation is realized as a $\mathscr{B}$-derivation by a multiplier operator of both $\mathfrak{X}$ and $\mathfrak{X}^*$ relative to the basis $\mathscr{X}$ and $\mathscr{X}^*$ respectively. As a consequence of Lemma 6 the corresponding supporting operator must be unique up to a constant multiple of* $\mathrm{Id}_{\mathfrak{X}}$.

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