

A REMARK ON AN APPROXIMATE FUNCTIONAL EQUATION FOR $\zeta(s)$

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ABSTRACT. We derive an approximate functional equation for Riemann zeta function in the critical strip with sharp error term using a combinatorial identity.

1. INTRODUCTION

Perhaps the simplest of all approximate formulas for $\zeta(s)$, the Riemann zeta function, is

$$\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} + O(x^{-\sigma}),$$

which holds uniformly for $0 < \sigma_0 \leq \sigma$, $|t| < \frac{2\pi x}{C}$, where C is a given constant greater than 1 (here, as usual, $s = \sigma + it$). See [3], pg. 77.

In the present note we present an approximate functional equation for $\zeta(s)$ in the critical strip (Theorem 2) which differs from the classical one and depends on a certain combinatorial identity (Lemma 2). Our approximate functional equation has a sharp error term but the main term is combinatorially complicated. We give some evidence that this main term behaves like a jump function.

No use of this functional equation is made in this note.

2. THE APPROXIMATE FUNCTIONAL EQUATION

Our main results are Theorems 1 and 2. Theorem 2 gives an approximate functional equation with sharp error term.

Theorem 1. *If $s = \sigma + it$ with $0 < \sigma < 1$ then*

$$\zeta(s) = \frac{\sin(\pi s)}{(1 - 2^{1-s})\pi} \int_0^\infty x^{-s} \left(\sum_{n=1}^\infty \frac{(2n-2)!}{4^{n-1}(2n+x)\dots(1+x)} \left\{ \frac{3}{2}n + \frac{1}{2}x - \frac{1}{4} \right\} \right) dx. \quad (1)$$

For any real number x , let $[x]$ denote the integer part of x .

Theorem 2. *Assume $N = [c_1 t]$ with $\frac{\pi}{2 \log 4} < c_1$. Then the following formula holds uniformly if $0 < \sigma \leq \sigma_0 < 1$, $t > 0$:*

$$(1 - 2^{1-s})\zeta(s) = \frac{1}{2} \sum_{j=0}^{2N-1} \frac{(-1)^j}{(1+j)^s} a_{j,N} + O(t^{-\sigma} e^{-(c_1 \log 4 - \frac{\pi}{2})t}),$$

where $a_{j,N} = \sum_{n=[\frac{j}{2}]+1}^N \frac{1}{(2n-1)4^{n-1}} \binom{2n-1}{j} (3n - j - \frac{3}{2})$.

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3. PROOFS

We need to recall

Lemma 3. *Let $Q(x)$ be a meromorphic function of x having no poles on the positive real axis and such that $x^a Q(x) \rightarrow 0$ both when $x \rightarrow 0$ and $x \rightarrow \infty$. Also $\int_{C_{\rho_i}} (-z)^{a-1} Q(z) dz \rightarrow 0$ if $i \rightarrow \infty$ where C_{ρ_i} is a sequence of circles (squares) centered at the origin with increasing radii (diameters) tending to infinity. Then*

$$\int_0^\infty x^{a-1} Q(x) dx = \frac{\pi}{\sin(\pi a)} \sum r, \quad (2)$$

where $\sum r$ denotes the sum of the residues of $(-z)^{a-1} Q(z)$, and the residues in $\sum r$ are added according to their distance to the origin. Here $(-z)^{a-1} = e^{(a-1)\log(-z)}$, where $-\pi \leq \text{Arg}(-z) < \pi$.

This lemma is well-known and we refer the reader to [4] pg. 117.

Proof of Theorem 1. We have

$$\sum_{n=1}^\infty \frac{(-1)^{n-1}}{(n+x)} = \sum_{n=1}^\infty \frac{(2n-2)!}{4^{n-1}(2n+x)\dots(1+x)} \left\{ \frac{3}{2}n + \frac{1}{2}x - \frac{1}{4} \right\}, \quad (3)$$

This formula is proved in Lemma 2 below.

Let a be a real number such that $0 < a < \frac{1}{2}$ and let $Q(x) = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{(n+x)}$. It is not difficult to show that $\int_{C_{\rho_i}} (-z)^{a-1} Q(z) dz \rightarrow 0$ if $i \rightarrow \infty$, where C_{ρ_i} is the square centered at zero of side $2i+1$. Also $x^a Q(x) \rightarrow 0$ if $x \rightarrow 0$. Applying Lemma 1 we obtain

$$\int_0^\infty x^{a-1} \left(\sum_{n=1}^\infty \frac{(-1)^{n-1}}{(n+x)} \right) dx = \frac{\pi}{\sin(\pi a)} \sum_{n=1}^\infty (-1)^{n-1} n^{a-1}. \quad (4)$$

Setting $1-a = s = \sigma + it$, using analytic continuation, formula (3) and $\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^s} = (1-2^{1-s})\zeta(s)$ ($\sigma > 0$), we arrive to formula (1). \square

Lemma 4. *The following identity holds*

$$\sum_{n=1}^\infty \frac{(-1)^{n-1}}{(n+x)} = \sum_{n=1}^\infty \frac{(2n-2)!}{4^{n-1}(2n+x)\dots(1+x)} \left\{ \frac{3}{2}n + \frac{1}{2}x - \frac{1}{4} \right\}.$$

Proof. First we have

$$\sum_{k=1}^K \frac{b_1 \dots b_{k-1}}{x(x+a_1)\dots(x+a_k)} (x+a_k - b_k) = \frac{1}{x} - \frac{b_1 \dots b_K}{x(x+a_1)\dots(x+a_K)},$$

which follows from writing the right hand side as $A_0 - A_K$ and noticing that each term on the left is $A_{k-1} - A_k$. Replace x by $(n+x)^2$, a_k by $-k^2$, b_k by $k(\frac{1}{2} - k)$ and $K = n-1$. Multiply everything by $(n+x)(-1)^{n-1}$ and add from $n=1$ to N . Then $b_1 \dots b_{k-1} = (-1)^{k-1} \frac{(2k-2)!}{4^{k-1}}$ and

$$\begin{aligned} \sum_{n=1}^N \frac{(-1)^{n-1}}{(n+x)} - \sum_{n=1}^N \frac{(-1)^{n-1} b_1 \dots b_{n-1}}{(2n-1+x)\dots(1+x)} \\ = \sum_{n=1}^N \sum_{k=1}^{n-1} \frac{(-1)^{n-1}}{(n+k+x)\dots(n-k+x)} b_1 \dots b_{k-1} \left((n+x)^2 - \frac{k}{2} \right). \quad (5) \end{aligned}$$

We define

$$\varepsilon_{n,k}(x) := (-1)^{n+k} \frac{(2k-2)!(\frac{n+x}{2} + \frac{1}{4})}{4^{k-1}(n+k+x)\dots(n-k+1+x)}.$$

Then the last formula of (5) is equal to

$$\sum_{n=1}^N \sum_{k=1}^{n-1} (\varepsilon_{n,k}(x) - \varepsilon_{n-1,k}(x)) = \sum_{k=1}^N \varepsilon_{N,k}(x) - \sum_{k=1}^N \varepsilon_{k,k}(x).$$

Now notice that $\sum_{k=1}^N \varepsilon_{N,k}(x) \rightarrow 0$ if $N \rightarrow \infty$ and $0 < x < 1$. Thus by analytic continuation

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n+x)} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} b_1 \dots b_{n-1}}{(2n-1+x)\dots(1+x)} - \sum_{k=1}^{\infty} \varepsilon_{k,k}(x),$$

which is (3). □

To prove Theorem 2 we need the following lemma.

Lemma 5. *If $1 \leq n$, $-1 < \sigma < 1$, $t > 0$ then*

(i)

$$\int_0^{\infty} \frac{x^{-s}}{(2n+x)(2n-1+x)\dots(1+x)} dx = \frac{\pi}{(2n-1)! \sin(\pi s)} \sum_{j=0}^{2n-1} \binom{2n-1}{j} (-1)^j (j+1)^{-s}.$$

(ii) *For the same values of n and t and with $-1 < \sigma \leq \sigma_0 < 1$ there exists a constant $c_0 = c_0(\sigma_0)$ (depending only on σ_0) such that the absolute value of the integral in (i) is bounded by*

$$\frac{c_0}{|\sin(\pi s)|} \frac{n^{1-\sigma} e^{t\pi/2}}{(2n)!}.$$

Proof. Apply Lemma 1 to the left-hand side of (i) to obtain

$$\frac{\pi}{\sin(\pi s)} \left(\frac{1^{-s}}{(2n-1)!0!} - \frac{2^{-s}}{(2n-2)!1!} + \dots - \frac{(2n)^{-s}}{(2n-1)!} \right),$$

which is the right-hand side of (i). (ii) is proved as follows. By Lemma 1 we have to evaluate $\int_{\gamma} \frac{(-z)^{-s}}{(2n+z)(2n-1+z)\dots(1+z)} dz$ where γ is a positively oriented curve enclosing $-1, -2, \dots, -2n$.

Take γ to be the rectangle with vertices $-\varepsilon + i2n, -2n-1 + i2n, -2n-1 - i2n, -\varepsilon - i2n$, ($0 < \varepsilon < 1$). We parametrize $\gamma = \gamma(\tau)$, $\tau \in [0, cn]$ and $|\gamma'(\tau)| = 1$ with c depending on ε but bounded for any ε . Therefore we have

$$\left| \int_{\gamma} \right| \leq \int_0^{cn} \left| \frac{(-\gamma(\tau))^{-s}}{(2n+\gamma(\tau))\dots(1+\gamma(\tau))} \right| |\gamma'(\tau)| d\tau \leq$$

$$\max_{z \in \gamma} \left| \frac{1}{(2n+z)\dots(1+z)} \right| \int_0^{cn} |(-\gamma(\tau))^{-s}| d\tau,$$

and

$$\int_0^{cn} |(-\gamma(\tau))^{-s}| d\tau \leq \int_0^{cn} |\gamma(\tau)|^{-\sigma} e^{t \text{Arg}(-\gamma(\tau))} d\tau \leq e^{\frac{t\pi}{2}} \int_0^{cn} |\gamma(\tau)|^{-\sigma} d\tau.$$

Now we claim that $\max_{z \in \gamma} \left| \frac{1}{(2n+z)\dots(1+z)} \right| \leq \frac{1}{(2n)!} + \delta(\varepsilon)$ with $\delta(\varepsilon) \rightarrow 0$ if $\varepsilon \rightarrow 0$. This would prove (ii). To prove the above inequality we observe that by symmetry it is enough to compute a bound on the segments $[-\varepsilon, -\varepsilon + i2n], [-2n-1 + i2n, -\varepsilon + i2n]$. For the

first segment, it is easily seen that for any point z on it we have that $|j+z| = \text{dist}(z, -j)$ increases if z moves upwards on the segment. Thus the maximum is obtained on $z = -\varepsilon$. On the second segment it can be seen that for any point z on it we have $2n \leq |z+j|$. This shows that there the maximum is less than $\frac{1}{(2n)^{2n}}$. This proves the claimed inequality. Letting ε tend to zero we get (ii). \square

Proof of Theorem 2. Write, for short, $f(x, n) := \frac{1}{(2n+x)\dots(1+x)} (\frac{3n}{2} + \frac{x}{2} - \frac{1}{4})$. So (1) is written, interchanging summation and integration, as

$$\zeta(s) = \frac{\sin(\pi s)}{(1-2^{1-s})\pi} \sum_{n=1}^N \int_0^\infty \frac{(2n-2)!x^{-s}f(x, n)}{4^{n-1}} dx + \frac{\sin(\pi s)}{(1-2^{1-s})\pi} \sum_{n=N+1}^\infty \int_0^\infty \frac{(2n-2)!x^{-s}f(x, n)}{4^{n-1}} dx, \quad (6)$$

Let $N = [c_1 t]$ with $\frac{\pi}{2 \log 4} < c_1$. For $n \geq N+1$ the last sum of (6) can be estimated using ii) of Lemma 3:

$$\begin{aligned} & \frac{|\sin(\pi s)|}{|(1-2^{1-s})\pi} \sum_{n=N+1}^\infty \left| \int_0^\infty \frac{(2n-2)!x^{-s}f(x, n)}{4^{n-1}} dx \right| \leq \\ & \frac{c_0}{|(1-2^{1-s})\pi} \sum_{n=N+1}^\infty \frac{(2n-2)!}{4^{n-1}} \left(\frac{3n}{2} \frac{n^{1-\sigma} e^{\frac{t\pi}{2}}}{2n!} + \frac{1}{2} \frac{n^{1-(\sigma-1)} e^{\frac{t\pi}{2}}}{2n!} + \frac{1}{4} \frac{n^{1-\sigma} e^{\frac{t\pi}{2}}}{2n!} \right) \\ & \leq \frac{9c_0 e^{t\pi/2}}{|1-2^{1-s}|8\pi} \sum_{n=N+1}^\infty \frac{n^{1-\sigma}}{(2n-1)4^{n-1}} \leq O\left(\frac{e^{t\pi/2} N^{-\sigma}}{4^N}\right) = O(t^{-\sigma} e^{-(c_1 \log 4 - \frac{\pi}{2})t}) \end{aligned}$$

For the first sum in (6) we use (i) of Lemma 3. \square

Remark 1. From Theorem 1 one has the following curious formula:

Corollary 1. *If $f(s) = \frac{1}{\Gamma(s)} \int_0^\infty \text{ArcTanh}(\frac{1-e^{-x}}{2}) e^{-x} x^{s-1} dx$ then*

$$-\zeta(s) \frac{(1-2^{1-s})}{2} = f(s) - f(s-1) - \frac{1}{2} \left(\frac{1}{1^s} + \frac{1}{3 \cdot 2^s} + \frac{1}{3^2 \cdot 3^s} + \frac{1}{3^3 \cdot 4^s} + \frac{1}{3^4 \cdot 5^s} \dots \right)$$

Hint. Recall that $\int_0^\infty e^{-jx} x^{s-1} dx = \Gamma(s) j^{-s}$ for suitable s and j . Thus Lemma 3 (i) is equal to

$$\frac{\pi}{(2n-1)! \sin(\pi s) \Gamma(s)} \int_0^\infty (1-e^{-x})^{2n-1} e^{-x} x^{s-1} dx.$$

Using this in formula (1), interchanging summation and integration and using that

$$\sum_{n=1}^\infty \frac{\frac{3n}{2} - \frac{1}{4}}{4^{n-1} (2n-1)} \alpha^{2n-1} = \text{ArcTanh}\left(\frac{\alpha}{2}\right) - \frac{3\alpha}{\alpha^2 - 4},$$

and

$$\sum_{n=1}^\infty \frac{\alpha^{2n-1}}{4^{n-1} (2n-1)} = 2 \text{ArcTanh}\left(\frac{\alpha}{2}\right),$$

we get

$$\zeta(s) = \frac{f(s) - f(s-1)}{(1-2^{1-s})} - \frac{3}{(1-2^{1-s})\Gamma(s)} \int_0^\infty \frac{(1-e^{-x})e^{-x}x^{s-1}}{(1-e^{-x})^2 - 4} dx.$$

This formula proves the corollary after some simplifications. \square

4. ON THE COEFFICIENT $a_{j,N}$

Recall the definition of $a_{j,N}$, the coefficient of $\frac{(-1)^j}{(1+j)^s}$ in Theorem 2:

$$a_{j,N} = \sum_{n=[j/2]+1}^N \frac{1}{(2n-1)4^{n-1}} \binom{2n-1}{j} (3n-j-3/2).$$

The author noticed numerically that this coefficient behaved like a jump function. More precisely

$$a_{j,N} \approx \begin{cases} 2 & \text{if } 0 \leq j \leq N-2 \\ 1 & \text{if } j = N-1 \\ 0 & \text{if } N \leq j \leq 2N-1. \end{cases}$$

Here we give some evidence of this fact. An unknown referee has kindly provided part of the proof below.

As $a_{0,N} = 2(1-4^{-N})$ we assume that $1 \leq j$. Also if $n \leq [j/2]$ then $\binom{2n-1}{j} = 0$ since $j > 2n-1$. Thus we write for $1 \leq j$

$$a_{j,N} = 6 \sum_{n=1}^N \frac{1}{4^n} \binom{2n-1}{j} - 4 \sum_{n=1}^N \frac{1}{4^n} \binom{2n-2}{j-1},$$

where we have used that $\binom{2n-1}{j} = \frac{2n-1}{j} \binom{2n-2}{j-1}$ as long as $1 \leq j$.

Now we will show that

$$2 = a_{j,N} + \sum_{n=N+1}^{\infty} \frac{1}{(2n-1)4^{n-1}} \binom{2n-1}{j} (3n-j-3/2) = a_{j,N} + \text{Tail}(N, j). \quad (7)$$

The tail will be seen to be small in a sense explained below. But the middle formula of (7) can be written as (curves are oriented in the usual way)

$$\begin{aligned} & 6 \sum_{n=1}^{\infty} \frac{1}{4^n} \binom{2n-1}{j} - 4 \sum_{n=1}^{\infty} \frac{1}{4^n} \binom{2n-2}{j-1} \\ &= \frac{1}{2\pi i} \left(6 \sum_{n=1}^{\infty} \int_{|z|=1/2} \frac{(1+z)^{2n-1}}{4^n z^{j+1}} dz - 4 \sum_{n=1}^{\infty} \int_{|z|=1/2} \frac{(1+z)^{2n-2}}{4^n z^j} dz \right) \\ &= \frac{1}{2\pi i} \int_{|z|=1/2} \frac{1}{4z^j \left(1 - \frac{(1+z)^2}{4}\right)} \left(\frac{6(1+z)}{z} - 4 \right) dz. \end{aligned}$$

Now one deforms the curve $|z| = 1/2$ to $|z| = r$, computing the residues at $z = 1, -3$ ($-2, 0$ respectively). Notice that the integral over the curve $|z| = r$ tends to zero if $r \rightarrow \infty$. This proves (7).

The evidence that our function $a_{j,N}$ behaves like a jump function is given by:

- Tail(N, j) = $O(1)$. For any fixed $0 < \delta < 1$, Tail(N, j) $\rightarrow 0$ uniformly in j if $1 \leq j < N\delta$ and $N \rightarrow \infty$.
- For any fixed $0 < \delta < 1$, Tail(N, j) $\rightarrow 2$ uniformly in j if $N \cdot (1 + \delta) < j \leq 2N$ and $N \rightarrow \infty$.

Proof. As above the tail can be written using residues as

$$\begin{aligned} \text{Tail}(N, j) &= 6 \cdot \frac{1}{2\pi i} \int_{|z|=1/2} \frac{(1+z)^{2N+1}}{4^N z^{j+1} (4 - (1+z)^2)} dz - 4 \cdot \frac{1}{2\pi i} \int_{|z|=1/2} \frac{(1+z)^{2N}}{4^N z^j (4 - (1+z)^2)} dz \\ &= 6 \cdot S_1(N, j) - 4 \cdot S_2(N, j). \end{aligned}$$

We will compare the above integrals with the more suitable

$$\begin{aligned} S'_1(N, j) &= \frac{1}{2\pi i} \int_{|z|=1/2} \frac{1}{4(1-z)} \cdot \frac{(1+z)^{2N+1}}{4^N z^{j+1}} dz, \\ S'_2(N, j) &= \frac{1}{2\pi i} \int_{|z|=1/2} \frac{1}{4(1-z)} \cdot \frac{(1+z)^{2N}}{4^N z^j} dz. \end{aligned}$$

Indeed we will see that uniformly in $0 \leq j \leq 2N$, $i = 1, 2$ one has

$$S_i(N, j) - S'_i(N, j) = o(1) \text{ as } N \rightarrow \infty. \quad (8)$$

Also, we will prove for fixed $0 < \delta < 1$, $i = 1, 2$

$$S'_i(N, j) \rightarrow 0 \text{ uniformly in } j \text{ if } 1 \leq j \leq N\delta; N \rightarrow \infty, \quad (9)$$

$$S'_1(N, j) + S'_1(N, 2N - j) = \frac{1}{2}; \quad S'_2(N, j) + S'_2(N, 2N - j + 1) = \frac{1}{4} \quad (10)$$

Observe that a), b) follows from (8), (9), (10) and the fact that $0 \leq S'_i(N, j)$.

To prove (8) say, for $i = 1$, notice that

$$S_1(N, j) - S'_1(N, j) = \frac{1}{2\pi i} \int_{\gamma_1 + \gamma_2} g(z) \frac{(1+z)^{2N+1}}{4^N z^{j+1}} dz$$

where $g(z)$ is a regular function on $|z| = 1$; γ_1 is the curve given by $\{|z| = 1, \varepsilon(N) \leq \text{Arg}(z) \leq 2\pi - \varepsilon(N)\}$ and $0 < \varepsilon(N)$ is chosen so that $|1+z| \leq 2(1 - \frac{\text{Log} N}{2N})$ on γ_1 . Now it is not difficult to see that $\varepsilon(N)$ tends to zero if $N \rightarrow \infty$. Also we denote γ_2 the curve $\{|z| = 1, -\varepsilon(N) \leq \text{Arg}(z) \leq \varepsilon(N)\}$; so that the length of γ_2 tends to zero as $N \rightarrow \infty$.

Now on γ_1 the above integral is by the maximum modulus principle

$$\int_{\gamma_1} = O\left(\frac{(1 - \frac{\text{Log} N}{2N})^{2N}}{1}\right) = o(1)$$

Now the integral over γ_2 tends to zero because the length of γ_2 tends to zero. This proves $S_1(N, j) - S'_1(N, j) = o(1)$ as $N \rightarrow \infty$. The proof for $i = 2$ is similar.

Now in the definition of $S'_1(N, j)$ deforming the curve $|z| = 1/2$ to a curve $|z| = 2$ and computing the residue at $z = 1$ one has

$$S'_1(N, j) = \frac{1}{2} + \frac{1}{2\pi i} \int_{|z|=2} \frac{1}{4(1-z)} \cdot \frac{(1+z)^{2N+1}}{4^N z^{j+1}} dz$$

This last integral is $-S'_1(N, 2N - j)$ making the change of variable $z = 1/w$. This proves (10) (case $i = 2$ is similar).

Finally to prove (9) notice that

$$|S'_1(N, j)| = \left| \frac{1}{2\pi i} \int_{|z|=r < 1} \frac{1}{4(1-z)} \cdot \frac{(1+z)^{2N+1}}{4^N z^{j+1}} \right| \ll_r \left(\frac{(1+r)^2}{4}\right)^N \frac{1}{r^j} \leq \left(\frac{(1+r)^2}{4 \cdot r^\delta}\right)^N$$

But $\left(\frac{(1+r)^2}{4 \cdot r^\delta}\right) < 1$ for $r = \frac{\delta}{2-\delta}$. Again case $i = 2$ is similar.

□

Notice that $S'_1(N, N) = \frac{1}{4}$. This follows from (10) with $j = N$.

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