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A DUALITY FOR MONADIC $(n+1)$-VALUED MV-ALGEBRAS

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ABSTRACT. Categorical equivalences between the varieties of monadic $(n+1)$-valued MV-algebras and the classes of monadic Boolean algebras endowed with certain family of their filters are given. Using these equivalences, it is proved that every monadic $(n+1)$-valued MV-algebra can be represented by a rich algebra.

1. INTRODUCTION AND PRELIMINARIES

Wajsberg algebras (see [7, 11, 23]) are an equivalent reformulation of Chang MV-algebras based on implication instead of disjunction. MV-algebras were introduced by Chang [4, 5] to prove the completeness of the infinite valued Łukasiewicz propositional calculus. The classes of $(n+1)$-valued MV-algebras were introduced by R. Grigolia in [13], who also gave their equational characterization. For each $n>0$, this variety is generated by the chain of length $n+1$ and the algebras belonging to this variety are the algebraic models of the $(n+1)$-valued Łukasiewicz propositional calculus. Lukasiewicz 3-valued and 4-valued algebras coincide with 3-valued and 4-valued MV-algebras, respectively.

Y. Komori [16] introduced the CN-algebras as algebraic models of Łukasiewicz infinite-valued propositional calculus formulated in terms of the operations implication and negation. A. J. Rodriguez [23] called Wajsberg algebras what was previously known as CN-algebras (see also [11]). $(n+1)$-valued Wajsberg algebras are equivalent to $(n+1)$-valued MV-algebras. The variety of $(n+1)$-bounded W-algebras is generated by chains of length less or equal than $n+1$. In this paper Wajsberg algebras will be used instead of MV-algebras.

For each integer $n>0$, it is shown in [19] that there exists a categorical equivalence between the variety of $(n+1)$-valued MV-algebras and the class of Boolean algebras endowed with a certain family of filters. Another similar categorical equivalence is given by A. Di Nola and A. Lettieri in [9]. In this paper, the mentioned equivalence is extended to the variety of monadic $(n+1)$-valued MV-algebras. Using this equivalence, it is proved that every monadic $(n+1)$-valued MV-algebra can be represented by a rich algebra. When $n=2$, the results given by Luiz Monteiro in [21] about the representation of monadic 3-valued Lukasiewicz algebras by rich algebras are obtained.

The basic results about MV-algebras can be found, for instance, in [7]. For a reformulation in the context of Wajsberg algebras (or CN-algebras) see [23, 11, 16].

A Wajsberg algebra (or W-algebra, for short) is an algebra $A=\langle A,\to,\neg,1\rangle$ of type $(2,1,0)$ satisfying the following identities: $1\to x=x$, $(x\to y)\to((y\to z)\to(x\to z))=1$, $(x\to y)\to y=(y\to x)\to x$ and $(\neg y\to\neg x)\to(x\to y)=1$. The reduct $(A,\vee,\wedge,\neg,0,1)$ is a Kleene algebra where $0=\neg 1$, $x\vee y=(x\to y)\to y$, $x\wedge y=\neg(\neg x\vee\neg y)$ and $x\le y$ if and only if $x\to y=1$. If we set $x\oplus y=\neg y\to x$ and $x\odot y=\neg(x\to\neg y)$ then $\langle A,\oplus,\odot,0\rangle$ is an MV-algebra. The set $B(A)=\{x\in A: x\odot x=x\}$ is a Boolean algebra. Indeed, $B(A)$ is the Boolean algebra of the complemented elements of the lattice reduct of A. The elements of

$B(A)$ are called the boolean elements of A. For all $x \in A$ and each non negative integer m we set:
$0x = 0$ and $(m+1)x = (mx) \oplus x$;
$x^0 = 1$ and $x^{m+1} = (x^m) \odot x$.

For every $x \in A$ and all integer $m \geq 0$, the following properties hold:
(W1) $\neg(x^m) = m(\neg x)$,
(W2) $(p \to q)^m \leq mp \to mq$.

A subset $F \subseteq A$ is an *implicative filter* of A if $1 \in F$ and for all $a, b \in A$, $a, a \to b \in F$ implies $b \in F$. Implicative filters are lattice filters which are closed by the operation $\odot$. The family of all implicative filters of A is an algebraic lattice under set-inclusion, and it is isomorphic to the algebraic lattice of all congruence relations on A. For every implicative filter F of A and each $x \in A$ we represent with $[x]_F$ the set of all elements $y \in A$ such that x and y are $F-congruent$. An implicative filter of A is *prime* if it is a lattice prime filter of A. We denote by $\chi(A)$ the set of all prime implicative filters of A. An implicative filter P of A is prime if and only if A/P is a chain.

In what follows let $n \geq 1$ be an integer.
The unit interval $[0,1]$ endowed with the operations $x \to y := min\ \{1, 1-x+y\}$ and $\neg x := 1-x$ is a Wajsberg algebra. We denote by L_{n+1} the subalgebra of $[0,1]$ whose universe is $\left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\right\}$. It is verified that L_{t+1} is a subalgebra of L_{n+1} if and only if t divides n.

An $(n+1)$-bounded Wajsberg algebra A is a Wajsberg algebra which verifies $x^n = x^{n+1}$, for every $x \in A$.
An $(n+1)$-valued Wajsberg algebra A is an $(n+1)$-bounded Wajsberg algebra which verifies $n(x^j \oplus (\neg x \odot \neg x^{j-1})) = 1$, for every $x \in A$ and $1 < j < n$ does not divide n.

If $\langle A, \to, \neg, 1\rangle$ is an $(n+1)$-valued Wajsberg algebra then $\langle A, \vee, \wedge, \neg, \sigma_1, \sigma_2, \ldots, \sigma_n, 0, 1\rangle$ is an $(n+1)$-valued Łukasiewicz algebra, where the operators σ_i, for $1 \leq i \leq n$, are defined in terms of the Wajsberg operations (see [15]).
The following results are developed in [19] and establish the equivalences mentioned above.
Let B be a Boolean algebra. We denote by $B^{[n]}$ the set of all increasing monotone functions from $\{1, 2, \ldots, n\}$ into B. $B^{[n]}$ with the operations of the lattice defined pointwise, the chain of constants $0 = c_0 < c_1 < \ldots < c_{n-1} < c_n = 1$ where, for each $0 \leq k \leq n$, $c_k(i)$ is equal to 1 if $i \geq n+1-k$ and equal to 0 otherwise, the negation defined by $(\neg f)(i) = \neg f(n+1-i)$ for each $1 \leq i \leq n$ and the modal operators $\sigma_i(f)(j) = f(i)$ for all $1 \leq i \leq n$ and $1 \leq j \leq n$, is a Post algebra of order $n+1$ [2]; therefore it is an $(n+1)$-valued Wajsberg algebra [24]. In Theorem 1.1 a direct proof of this results is given, showing explicitly the form of operations. In every $(n+1)$-valued Wajsberg algebra, the prime filters occur in finite and disjoint chains, then by the Martínez's Unicity Theorem [20] the implication is determined by the order.

Theorem 1.1. [19] *Let B be a Boolean algebra and $n \geq 1$ be an integer. Then $\langle B^{[n]}, \mapsto, \neg, \mathbb{I}\rangle$ is an $(n+1)$-valued Wajsberg algebra where* $B^{[n]} =$

$\{f:\{1,2,\ldots,n\}\longrightarrow B : f(i)\le f(j)$ *for all* i,j *such that* $i\le j\}$, $\mathbb{I}$ *is the constant function equal to* 1 *and, for* $f,g\in B^{[n]}$ *and* $1\le k\le n$, $(\neg f)(k)=\neg f(n+1-k)$ *and* $(f\mapsto g)(k)=\bigwedge_{i=1}^{n-k+1}(f(i)\to g(i+k-1))$.

Remark 1.1. We denote by $Div(n)$ the set of all positive divisors of n. Let $d\in Div(n)$. For each integer j, $1\le j\le n$, there exists an only integer $q_{d,j}$, $1\le q_{d,j}\le d$, such that $(q_{d,j}-1)\frac{n}{d}<j\le q_{d,j}\frac{n}{d}$. Indeed, $q_{d,j}$ is the first element of the set $X=\{q\in\mathbb{N}:1\le q\le d,\ j\le q\frac{n}{d}\}$. That is to say that the only block corresponding to the divisor d of n that contains j is that determined by $q_{d,j}$. Thus, for any $d\in Div(n)$, we can *think* an n-tuple to be composed by d blocks, each one of them with $\frac{n}{d}$ elements.

In what follows, for each $f\in B^{[n]}$, $d\in Div(n)$ and any integer $1\le q\le d$, we shall write $\xi_{d,q}(f)$ instead of $f(q\frac{n}{d})\to f((q-1)\frac{n}{d}+1)$.

Corollary 1.1. [19] *Let B be a Boolean algebra, let* $n\ge 1$ *be an integer and let h be a function from the lattice of divisors of n into the lattice of filters of B. The set* $\{f\in B^{[n]}:\xi_{d,q}(f)\in h(d)$, *for each* $d\in Div(n)$ *and all* $1\le q\le d\}$ *is denoted by* $M(B,h)$. *Then* $\langle M(B,h),\mapsto,\neg,\mathbb{I}\rangle$ *is an* $(n+1)$*-valued Wajsberg subalgebra of* $B^{[n]}$. *Also, if* $h(d)=B$ *for each* $d\in D=Div(n)-\{n\}$ *then* $M(B,h)$ *is a Post algebra of order* $n+1$.

Theorem 1.2. [19] *Let* $\langle A,\to,\neg,1\rangle$ *be an* $(n+1)$*-valued Wajsberg algebra. For each* $d\in Div(n)$ *let* $h_A(d)=P_d\cap B(A)$, *where* $P_d=\bigcap\{P\in\chi(A):A/P\subseteq L_{d+1}\}$. *Then* $\varphi:A\longrightarrow M(B(A),h_A)$ *is a W-isomorphism, being* $\varphi(x)(i)=\sigma_i(x)$ *for all* $x\in A$ *and every integer* $1\le i\le n$.

Definition 1.1. (a) *A pair* $\langle B,h\rangle\in B^{n+1}$ *if B is a Boolean algebra and h is a function from the lattice of divisors of n into the lattice of filters of B such that* $h(n)=\{1\}$ *and* $h(gcd\{d,r\})=h(d)\vee h(r)$, *for every* $d,r\in Div(n)$ ($gcd\{d,r\}$ *is the greatest common divisor of the set* $\{d,r\}$).

(b) *Objects* $\langle B_1,h_1\rangle$ *and* $\langle B_2,h_2\rangle$ *in* B^{n+1} *are isomorphic if there exists a boolean isomorphism* $\varphi:B_1\longrightarrow B_2$ *which verifies* $\varphi^{-1}(h_2(d))=h_1(d)$ *for all* $d\in Div(n)$.

Remark 1.2. Let $\langle A,\to,\neg,1\rangle$ be an $(n+1)$-valued Wajsberg algebra. Then $\langle B(A),h_A\rangle\in B^{n+1}$, where $h_A(d)=P_d\cap B(A)$ being $P_d=\bigcap\{P\in\chi(A):A/P\subseteq L_{d+1}\}$, for each $d\in Div(n)$.

Theorem 1.3. [19] *Let* $\langle B,h\rangle\in B^{n+1}$ *and let* $A=M(B,h)$. *Then* $\langle B,h\rangle$ *and* $\langle B(A),h_A\rangle$ *are isomorphic objects in* B^{n+1}.

Let $\mathscr{W}^{n+1}$ be the category of $(n+1)$-valued W-algebras and W-homomorphisms. Let $\mathscr{B}^{n+1}$ be the category whose objects are pairs in B^{n+1} and whose morphisms are defined in the following way: if $O_1=\langle B_1,h_1\rangle$ and $O_2=\langle B_2,h_2\rangle$ are objects in this category, θ is a morphism from O_1 into O_2 if it is a boolean homomorphism from B_1 into B_2 which verifies $h_1(d)\subseteq\theta^{-1}(h_2(d))$ for any $d\in Div(n)$.

It is easy to see that θ is an isomorphism from O_1 onto O_2 if it is a boolean isomorphism from B_1 onto B_2 which verifies $h_1(d)=\theta^{-1}(h_2(d))$ for each $d\in Div(n)$.

Let B be the functor from $\mathscr{W}^{n+1}$ to $\mathscr{B}^{n+1}$ defined in the following way:

(i) For each object $\mathscr{A}=\langle A,\to,\neg,1\rangle$ in $\mathscr{W}^{n+1}$, $B(\mathscr{A})=\langle B(A),h_A\rangle$, where $B(A)$ is the set of boolean elements of A and for all d divisor of n, $h_A(d)=P_d\cap B(A)$, being $P_d=\bigcap\{P\in\chi(A):A/P\subseteq L_{d+1}\}$.

(ii) If $\mathscr{A}_1$ and $\mathscr{A}_2$ are objects in the category $\mathscr{W}^{n+1}$ and $g : \mathscr{A}_1 \longrightarrow \mathscr{A}_2$ is a $\mathscr{W}^{n+1}$-morphism, $B(g) : \langle B(A_1), h_{A_1}\rangle \longrightarrow \langle B(A_2), h_{A_2}\rangle$ is defined by $B(g) = g/_{B(A_1)}$.

Let M be the functor from $\mathscr{B}^{n+1}$ to $\mathscr{W}^{n+1}$ defined in the following way:

(i) For each object $\langle B,h\rangle$ in $\mathscr{B}^{n+1}$, let $M(\langle B,h\rangle) = \langle M(B,h), \mapsto, \neg, \mathbb{I}\rangle$.

(ii) If $\langle B_1,h_1\rangle$ and $\langle B_2,h_2\rangle$ are objects in the category $\mathscr{B}^{n+1}$ and g is a $\mathscr{B}^{n+1}$-morphism from $\langle B_1,h_1\rangle$ into $\langle B_2,h_2\rangle$ let $M(g) : M(B_1,h_1) \longrightarrow M(B_2,h_2)$ where $M(g)(f) = g \circ f$, for any $f \in M(B_1,h_1)$.

From Theorems 1.2 and 1.3 the functors B and M define a natural equivalence between the categories $\mathscr{W}^{n+1}$ and $\mathscr{B}^{n+1}$.

Monadic $(n+1)$-valued W-algebras [25, 26, 12, 10, 1] are defined as follows.

Definition 1.2. *An algebra $\langle A, \rightarrow, \neg, \forall, 1\rangle$ is a monadic Wajsberg algebra if $\langle A, \rightarrow, \neg, 1\rangle$ is a Wajsberg algebra and $\forall : A \longrightarrow A$ is a function which verifies the following identities:*

(U1) $\forall x \rightarrow x = 1$,

(U2) $\forall(\forall x \rightarrow y) = \forall x \rightarrow \forall y$,

(U3)$\forall(\neg x \rightarrow x) = \neg\forall x \rightarrow \forall x$.

Observe that identity U3 can be write $\forall(2x) = 2\forall x$.

Let $\langle A, \rightarrow, \neg, \forall, 1\rangle$ be a monadic Wajsberg algebra. Often we will write A or $\langle A, \forall\rangle$ instead of $\langle A, \rightarrow, \neg, \forall, 1\rangle$. If $X \subseteq A$, $\forall(X) = \{\forall x : x \in X\}$. Algebras $\forall(A)$ and $B(A)$ are monadic Wajsberg subalgebras of A. In particular $\langle B(A), \forall\rangle$ is a monadic Boolean algebra. For all $x, y \in A$ and all integer $m \geq 0$, the following properties hold:

(U4) $\forall\forall x = \forall x$,

(U5) $x \leq y$ implies $\forall x \leq \forall y$,

(U6) $\forall(x \wedge y) = \forall x \wedge \forall y$,

(U7) $\forall(x \rightarrow y) \leq \forall x \rightarrow \forall y$,

(U8) $\forall\neg\forall x = \neg\forall x$,

(U9) $\forall(x \odot \forall y) = \forall x \odot \forall y$,

(U10) $(\forall x)^m \leq \forall(x^m)$.

Definition 1.3. *A monadic Wajsberg algebra $\langle A, \rightarrow, \neg, \forall, 1\rangle$ is a monadic $(n+1)$-valued Wajsberg algebra (MW^{n+1}-algebra, for short) if $\langle A, \rightarrow, \neg, 1\rangle$ is an $(n+1)$-valued Wajsberg algebra.*

The varieties of monadic $(n+1)$-valued Wajsberg algebras will be denoted by $\mathbf{MW^{n+1}}$.

In [18] the classes of $(n+1)$-bounded Wajsberg algebras with a U-operator (or UW_{n+1}-algebras) are defined as $(n+1)$-bounded Wajsberg algebras with an operator which verifies the properties (U1) and (U2). With $\mathbf{UW_{n+1}}$ we denote the varieties of $(n+1)$-bounded Wajsberg algebras with a U-operator.

Lemma 1.1. $\mathbf{MW^{n+1}} \subseteq \mathbf{UW_{n+1}}$, *for all $n \geq 1$.*

Remark 1.3. (i) If $\langle A, \rightarrow, \neg, \forall, 1\rangle$ is a monadic Wajsberg algebra then $\langle A, \oplus, \odot, \neg, \exists, 0, 1\rangle$ is a monadic MV-algebra (see [10, 1, 25, 12]) where for each $x \in A$, $\exists x = \neg\forall\neg x$.

(ii) If $\langle A, \oplus, \odot, \neg, \exists, 0, 1\rangle$ is a monadic MV-algebra then $\langle A, \rightarrow, \neg, \forall, 1\rangle$ is a monadic Wajsberg algebra where for each $x \in A$, $\forall x = \neg\exists\neg x$.

Theorem 1.4. [10, Corollary 14] *If $\langle A, \forall \rangle$ is a totally ordered monadic Wajsberg algebra, then $\forall$ is the identity.*

The following result is consequence of [18, Theorem 2.2] and Lemma 1.1.

Lemma 1.2. *The variety* $\mathbf{MW^{n+1}}$ *is semisimple.*

Theorem 2.3 in [18] for UW_{n+1}-algebras yields the following result in the class of monadic $(n+1)$-valued Wajsberg algebras.

Theorem 1.5. *Let A be a non trivial MW^{n+1}-algebra. Then A is a simple MW^{n+1}-algebra if, and only if, $\forall(A)$ is a simple $(n+1)$-valued Wajsberg algebra if, and only if, $\forall(A) \cap B(A)$ is simple Boolean algebra.*

The following properties hold for every non trivial Wajsberg algebra A.

(P1) A is a simple $(n+1)$-valued Wajsberg algebra if and only if A is isomorphic to L_{r+1} for some integer $r \geq 1$, r divisor of n.

(P2) A is an $(n+1)$-valued Wajsberg algebra if and only if A can be represented (as subdirect product) in $\prod\limits_{i/n} L_{i+1}^{\chi_{i+1}}$, where $\chi_{i+1} = \{D \in \chi(A) : A/D \simeq L_{i+1}\}$.

Corollary 1.2. *$\langle L_{n+1}^I, \forall \rangle$ is a simple MW^{n+1}-algebra, where I is a nonempty set and for each $f : I \longrightarrow L_{n+1}$, $\forall f$ is the constant function defined by $(\forall f)(x) = inf\{f(x) : x \in I\}$.*

Theorem 1.6. *If A is a simple MW^{n+1}-algebra, then it is isomorphic to a subalgebra of $\langle L_{n+1}^I, \forall \rangle$, for some nonempty set I.*

Proof. The proof is a special case of Theorem 2.4 in [18] using Theorem 1.5, properties (P1) and (P2), Corollary 1.2 and Theorem 1.4. □

Corollary 1.3. *Let $\langle A, \forall \rangle$ be an MW^{n+1}-algebra. Then $\forall(kx) = k\forall x$ for every $x \in A$ and all integer $1 \leq k \leq n$.*

Proof. It is easy to prove that the identities are valid in a simple MW^{n+1}-algebra; so they are valid in all MW^{n+1}-algebra, follows from Lemma 1.2. □

Lemma 1.3. *Let $\langle A, \forall \rangle$ be an MW^{n+1}-algebra. Then for every $x \in A$ the following properties hold:*

(U11) $\forall(x^k) = (\forall x)^k$, *for each integer* $1 \leq k \leq n$,

(U12) $\forall(\sigma_i(x)) = \sigma_i(\forall x))$, *for every* $i \in \{1, 2, \ldots, n\}$.

Proof. (U11) follows from properties W1, W2, U5, U8, U9 and U10. (U12) follows from Corollary 1.3, U11 and [15, Theorem 5.23]. □

It is proved in [12] that monadic $(n+1)$-valued MV-algebras are polynomially equivalent to monadic $(n+1)$-valued Łukasiewicz algebras for $n = 2$ and $n = 3$, respectively.

2. THE DUALITY FOR MONADIC $(n+1)$-VALUED WAJSBERG ALGEBRAS

Theorem 2.1. *Let $\langle B, \forall \rangle$ be a monadic Boolean algebra and $n \geq 1$ be an integer. Then $\langle B^{[n]}, \mapsto, \neg, \mathbb{V}, \mathbb{I} \rangle$ is a monadic $(n+1)$-valued Wajsberg algebra where $B^{[n]} = \{f : \{1, 2, \ldots, n\} \longrightarrow B : f(i) \leq f(j)$ for all i, j such that $i \leq j\}$, $\mathbb{I}$ is the constant function equal to 1 and, for $f, g \in B^{[n]}$ and $1 \leq k \leq n$, $(\neg f)(k) = \neg f(n+1-k)$, $(f \mapsto g)(k) = \bigwedge\limits_{i=1}^{n-k+1} (f(i) \to g(i+k-1))$ and $(\mathbb{V} f)(i) = \forall(f(i))$.*

Proof. From Theorem 1.1 $\langle B^{[n]}, \mapsto, \neg, \mathbb{I}\rangle$ is an $(n+1)$-valued Wajsberg algebra. Moreover, for every $f, g \in B^{[n]}$ and integers i, k, $1 \leq i, k \leq n$, the following properties hold:

(1) $\mathbb{V}f \leq f$

$(\mathbb{V}f)(i) = \forall f(i) \leq f(i)$

(2) $\mathbb{V}(f \mapsto \mathbb{V}g) = \mathbb{V}f \mapsto \mathbb{V}g$

$$(\mathbb{V}(f \mapsto \mathbb{V}g))(k) = \forall((f \mapsto \mathbb{V}g)(k)) = \forall\left(\bigwedge_{i=1}^{n-k+1}(f(i) \to (\mathbb{V}g)(i+k-1))\right)$$
$$= \forall\left(\bigwedge_{i=1}^{n-k+1}(f(i) \to \forall(g(i+k-1))\right) = \bigwedge_{i=1}^{n-k+1}\forall(f(i) \to \forall(g(i+k-1)))$$
$$= \bigwedge_{i=1}^{n-k+1}(\forall(f(i)) \to \forall(g(i+k-1))) = \bigwedge_{i=1}^{n-k+1}((\mathbb{V}f)(i) \to (\mathbb{V}g)(i+k-1)))$$
$$= (\mathbb{V}f \mapsto \mathbb{V}g)(k).$$

(3) $\mathbb{V}(\neg f \mapsto f) = \neg\mathbb{V}f \mapsto \mathbb{V}f.$

$$(\mathbb{V}(\neg f \mapsto f))(k) = \forall((\neg f \mapsto f)(k))$$
$$= \forall\left(\bigwedge_{i=1}^{n-k+1}(\neg f(n+1-i) \to f(i+k-1))\right)$$
$$= \bigwedge_{i=1}^{n-k+1}\forall(f(n+1-i) \vee f(i+k-1)). \quad (1)$$

On the other hand,

$$(\neg\mathbb{V}f \mapsto \mathbb{V}f)(k) = \bigwedge_{i=1}^{n-k+1}(\neg(\mathbb{V}f)(n+1-i) \to (\mathbb{V}f)(i+k-1)) = \bigwedge_{i=1}^{n-k+1}(\forall f(n+1-i) \vee \forall f(i+k-1)). \quad (2)$$

If $i \leq \lfloor\frac{n-k}{2}\rfloor$ ($\lfloor x \rfloor$ denotes the largest integer less or equal to x, for a real number x) then $i+k-1 \leq n+1-i$ and the equality follows from (1), (2) and U5. Similarly if $i > \lfloor\frac{n-k}{2}\rfloor$ because $n+1-i \leq i+k-1$.

□

Remark 2.1. Let $\langle B, \forall\rangle$ be a monadic Boolean algebra. Algebras $(\forall(B))^{[n]}$ and $\mathbb{V}\left(B^{[n]}\right)$ are isomorphic algebras. Indeed, $(\forall(B))^{[n]} = \{f : \{1, 2, \ldots, n\} \longrightarrow \forall(B) : f(i) \leq f(j)$ for all i, j such that $i \leq j\}$ and $\mathbb{V}\left(B^{[n]}\right) = \{f \in B^{[n]} : \mathbb{V}f = f\} = \{f \in B^{[n]} : \forall(f(i)) = f(i),$ for all $i \in \{1, 2, \ldots n\}\}$. It is clear that $f \in (\forall(B))^{[n]}$ if and only if f is an increasing function from the set $\{1, 2, \ldots, n\}$ into B such that $f(i) \in \forall(B)$ for every $1 \leq i \leq n$; if and only if $f \in B^{[n]}$ and $\forall(f(i)) = f(i)$ for every $1 \leq i \leq n$; if and only if $f \in \mathbb{V}\left(B^{[n]}\right)$.

Corollary 2.1. *Let $\langle B, \forall\rangle$ be a monadic Boolean algebra, $n \geq 1$ be an integer and h^M be a function from the lattice of divisors of n into the lattice of monadic filters of B. Let $M(B, h^M)$ be the set $\{f \in B^{[n]} : f(q\frac{n}{d}) \to f((q-1)\frac{n}{d}+1) \in h^M(d)$, for each $d \in Div(n)$ and all $1 \leq q \leq d\}$. Then $\langle M(B, h^M), \mapsto, \neg, \mathbb{V}, \mathbb{I}\rangle$ is a monadic $(n+1)$-valued Wajsberg subalgebra of $B^{[n]}$.*

Proof. From Corollary 1.1 we only shall prove that $\mathbb{V}$ is closed into $M\left(B, h^M\right)$. Let $f \in M\left(B, h^M\right)$, then $f(q\frac{n}{d}) \to f((q-1)\frac{n}{d}+1) \in h^M(d)$, for every $d \in Div(n)$ and all integer q, $1 \leq q \leq d$. Since $h^M(d)$ is a monadic filter, using U7 we have $(\mathbb{V}f)(q\frac{n}{d}) \to (\mathbb{V}f)((q-1)\frac{n}{d}+1) = \forall(f(q\frac{n}{d})) \to \forall(f((q-1)\frac{n}{d}+1)) \geq \forall(f(q\frac{n}{d}) \to f((q-1)\frac{n}{d}+1))$; then $\mathbb{V}f \in M\left(B, h^M\right)$.
□

Remark 2.2. Let $\langle B,\forall\rangle$ be a monadic Boolean algebra, $n \geq 1$ be an integer and h^M be a function from the lattice of divisors of n into the lattice of monadic filters of B. Then, for each $f \in M\left(B,h^M\right)$, $\mathbb{V}f$ is the last element of the set $(f] \cap M\left(B,h^M\right) \cap (\forall(B))^{[n]}$.

Corollary 2.2. *Let $\langle B,\forall\rangle$ be a monadic Boolean algebra, $n \geq 1$ be an integer and h be a function from the lattice of divisors of n into the lattice of filters of B. Then $\langle M(B,h), \mapsto, \neg, \mathbb{V}, \mathbb{I}\rangle$ is a monadic $(n+1)$-valued Wajsberg algebra where $(\mathbb{V}f)(i) = \forall(f(i))$, for each $f \in M(B,h)$ and $1 \leq i \leq n$.*

Proof. Let $\langle B,\forall\rangle$ be a monadic Boolean algebra. If F be a filter of B, then $\forall^{-1}F$ is a monadic filter of B and $\forall^{-1}F \subseteq F$. Moreover, $\forall^{-1}F$ is maximal among all the monadic filters of B included in F. Let h^M be the function from the lattice of divisors of n into the lattice of monadic filters of B defined by $h^M(d) = \forall^{-1}h(d)$, for each $d \in Div(n)$. From Corollary 2.1 and Remark 2.2 we have that $\langle M(B,h^M), \mapsto, \neg, \mathbb{V}, \mathbb{I}\rangle$ is a monadic $(n+1)$-valued Wajsberg algebra where, for each $f \in M\left(B,h^M\right)$, $\mathbb{V}f$ is the last element of the set $(f] \cap M\left(B,h^M\right) \cap (\forall(B))^{[n]}$. Moreover, $\langle M(B,h^M), \mapsto, \neg, \mathbb{I}\rangle$ is a W-subalgebra of $\langle M(B,h), \mapsto, \neg, \mathbb{I}\rangle$ because for every $f \in M(B,h^M)$ is $f(q\frac{n}{d}) \rightarrow f((q-1)\frac{n}{d}+1) \in h^M(d) \subseteq h(d)$, for each $d \in Div(n)$ and all $1 \leq q \leq d$. Let $f \in M(B,h)$; then $\mathbb{V}f$ is the last element of the set $(f] \cap M\left(B,h^M\right) \cap (\forall(B))^{[n]}$ because, if there exists $g \in M(B,h)$ such that $g \leq f$ and $g \in M\left(B,h^M\right) \cap (\forall(B))^{[n]}$, then $g = \mathbb{V}g \leq \mathbb{V}f$. Therefore $\mathbb{V}$ is the quantifier onto $M(B,h)$ determined by the subalgebra $M\left(B,h^M\right) \cap (\forall(B))^{[n]}$. □

Theorem 2.2. *Let $\langle A, \rightarrow, \neg, \forall, 1\rangle$ be a monadic $(n+1)$-valued Wajsberg algebra. Let h_A be the function from the lattice of divisors of n into the lattice of filters of $B(A)$ where, for each $d \in Div(n)$, $h_A(d) = P_d \cap B(A)$, being $P_d = \bigcap\{P \in \chi(A) : A/P \subseteq L_{d+1}\}$. Then $\langle M(B(A),h_A), \mapsto, \neg, \mathbb{V}, \mathbb{I}\rangle$ and $\langle A, \rightarrow, \neg, \forall, 1\rangle$ are isomorphic monadic $(n+1)$-valued Wajsberg algebras.*

Proof. From Theorem 1.2 the function $\varphi : A \longrightarrow M(B(A),h_A)$ is a W-isomorphism, being $\varphi(x)(i) = \sigma_i(x)$ for all $x \in A$ and every integer $1 \leq i \leq n$; moreover, $\mathbb{V}\varphi(x) = \varphi(\forall x)$ because from U12 we have $(\mathbb{V}\varphi(x))(i) = \forall(\varphi(x)(i)) = \forall(\sigma_i(x)) = \sigma_i(\forall x) = (\varphi(\forall x))(i)$. □

Definition 2.1. (i) *A 3-tuple $\langle B,\forall,h\rangle \in MB^{n+1}$ if $\langle B,\forall\rangle$ is a monadic Boolean algebra and h is a function from the lattice of divisors of n into the lattice of filters of B such that $h(n) = \{1\}$ and $h(gcd\{d,r\}) = h(d) \vee h(r)$, for every $d,r \in Div(n)$ ($gcd\{d,r\}$ is the greatest common divisor of the set $\{d,r\}$).*

(ii) *3-tuples $\langle B_1,\forall_1,h_1\rangle$ and $\langle B_2,\forall_2,h_2\rangle$ in MB^{n+1} are isomorphic if there exists a monadic boolean isomorphism $\varphi : B_1 \longrightarrow B_2$ which verifies $\varphi^{-1}(h_2(d)) = h_1(d)$ for all $d \in Div(n)$.*

Remark 2.3. Let $\langle A, \rightarrow, \neg, \forall, 1\rangle$ be a monadic $(n+1)$-valued Wajsberg algebra. Then $\langle B(A), \forall, h_A\rangle \in MB^{n+1}$, where, for each $d \in Div(n)$, $h_A(d) = P_d \cap B(A)$ being $P_d = \bigcap\{P \in \chi(A) : A/P \subseteq L_{d+1}\}$.

Theorem 2.3. *Let $\langle B,\forall,h\rangle \in MB^{n+1}$ and let $A = M(B,h)$. Then $\langle B,\forall,h\rangle$ and $\langle B(A), \mathbb{V}, h_A\rangle$ are isomorphic objects in MB^{n+1}.*

Proof. Let $\langle B,\forall,h\rangle \in MB^{n+1}$ and $A = M(B,h)$. By Corollary 2.2 we know that $\langle A, \mapsto, \neg, \mathbb{V}, \mathbb{I}\rangle$ is a monadic $(n+1)$-valued Wajsberg algebra where $(\mathbb{V}f)(i) = \forall(f(i))$, for all $f \in A$ and every integer $1 \leq i \leq n$.

It is easy to see that $B(A)$ is the subalgebra that consist of all constant functions. If h_A is the function from the lattice of divisors of n into the lattice of filters of $B(A)$ defined by $h_A(d) = P_d \cap B(A)$, being $P_d = \bigcap\{P \in \chi(A) : A/P \subseteq L_{d+1}\}$, then $\langle B(A), \mathbb{V}, h_A\rangle \in MB^{n+1}$ (because Remark 2.3).

Let $\mu : B \longrightarrow B(A)$ such that $\mu(a)$ is the constant function from $\{1,2,\dots,n\}$ into B that takes the value a, for each $a \in B$. In [19, Theorem 3] it is prove that μ is a boolean isomorphism from B onto $B(A)$ which verifies $\mu^{-1}(P_d \cap B(A)) = h(d)$, for each $d \in Div(n)$. Moreover, for each $x \in B$ and all $i \in \{1,2,\dots n\}$, it is $(\mu(\forall x))(i) = \forall x = \forall(\mu(x)(i)) = (\mathbb{V}\mu(x))(i)$. □

Let $\mathscr{MW}^{n+1}$ be the category of monadic $(n+1)$-valued W-algebras and monadic W-homomorphisms. Let $\mathscr{MB}^{n+1}$ be the category whose objects are the 3-tuples in MB^{n+1} and whose morphisms are defined in the following way: if $O_1 = \langle B_1, \forall_1, h_1\rangle$ and $O_2 = \langle B_2, \forall_2, h_2\rangle$ are objects in this category, θ is a morphism from O_1 into O_2 if it is a monadic boolean homomorphism from B_1 into B_2 which verifies $h_1(d) \subseteq \theta^{-1}(h_2(d))$ for any $d \in Div(n)$.

It is easy to see that θ is an isomorphism from O_1 onto O_2 if it is a monadic boolean isomorphism from B_1 onto B_2 which verifies $h_1(d) = \theta^{-1}(h_2(d))$ for each $d \in Div(n)$.

Let B be defined from $\mathscr{MW}^{n+1}$ to $\mathscr{MB}^{n+1}$ as follows:

(i) For each object $\mathscr{A} = \langle A, \to, \neg, \forall, 1\rangle$ in the category $\mathscr{MW}^{n+1}$, $B(\mathscr{A}) = \langle B(A), \forall, h_A\rangle$, where $B(A)$ is the set of boolean elements of A and for all d divisor of n, $h_A(d) = P_d \cap B(A)$, being $P_d = \bigcap\{P \in \chi(A) : A/P \subseteq L_{d+1}\}$.

(ii) If $\mathscr{A}_1$ and $\mathscr{A}_2$ are objects in the category $\mathscr{MW}^{n+1}$ and $g : \mathscr{A}_1 \longrightarrow \mathscr{A}_2$ is an $\mathscr{MW}^{n+1}$-morphism, $B(g) : \langle B(A_1), \forall_1, h_{A_1}\rangle \longrightarrow \langle B(A_2), \forall_2 h_{A_2}\rangle$ is defined by $B(g) = g/_{B(A_1)}$.

It is immediate that $B(g)$ is a monadic boolean homomorphism. Moreover, $B(g)$ is an $\mathscr{MB}^{n+1}$-morphism. Indeed, let $a \in h_{A_1}(d)$. If $a \notin B(g)^{-1}(h_{A_2}(d))$ then $g(a) \notin h_{A_2}(d)$, hence there exists a prime implicative filter P of A_2 such that $A_2/P \subseteq L_{d+1}$ and $g(a) \notin P$. Thus $a \notin g^{-1}(P) \cap B(A_1)$. The function $v : A_1/g^{-1}(P) \longrightarrow A_2/P$ defined by $v([x]_{g^{-1}(P)}) = [g(x)]_P$ is an embedding from $A_1/g^{-1}(P)$ into $A_2/P \subseteq L_{d+1}$, i.e., $A_1/g^{-1}(P) \subseteq L_{d+1}$ then $a \notin h_{A_1}(d)$ which is a contradiction. It is easy to verify that B is a functor.

Let M be defined from $\mathscr{MB}^{n+1}$ to $\mathscr{MW}^{n+1}$ as follows:

(i) For each object $\langle B, \forall, h\rangle$ in $\mathscr{MB}^{n+1}$, let $M(\langle B, \forall, h\rangle) = \langle M(B,h), \mapsto, \neg, \mathbb{V}, \mathbb{I}\rangle$, where $\mathbb{V}$ is defined pointwise.

(ii) If $\langle B_1, \forall_1, h_1\rangle$ and $\langle B_2, \forall_2, h_2\rangle$ are objects in $\mathscr{MB}^{n+1}$ and g is an $\mathscr{MB}^{n+1}$-morphism from $\langle B_1, \forall_1, h_1\rangle$ into $\langle B_2, \forall_2, h_2\rangle$ let $M(g) : M(B_1, h_1) \longrightarrow M(B_2, h_2)$ where $M(g)(f) = g \circ f$, for any $f \in M(B_1, h_1)$.

It is clear that $M(g)$ is well defined because, if $f \in M(B_1, h_1)$ then for each $d \in Div(n)$ and all integer q, $1 \leq q \leq d$ we have $\xi_{d,q}(f) \in h_1(d)$; hence $\xi_{d,q}(g \circ f) = g(\xi_{d,q}(f)) \in g(h_1(d)) \subseteq gg^{-1}(h_2(d) \subseteq h_2(d)$. Therefore $g \circ f \in M(B_2, h_2)$. Besides $M(g)$ is a monadic W-homomorphism. It is easy to see that M is a functor.

From Theorems 2.2 and 2.3 follows that the functors B and M define a natural equivalence between the categories $\mathscr{MW}^{n+1}$ and $\mathscr{MB}^{n+1}$.

3. REPRESENTATION BY RICH ALGEBRAS

Using the natural equivalence established in section 2 and the Representation Theorem by rich algebras for monadic Boolean algebras [14], we will prove that every monadic $(n+1)$-valued Wajsberg algebra can be represented by a rich algebra. Specifically, we will prove that every monadic $(n+1)$-valued W-algebra is isomorphic to a subalgebra B of a functional algebra A^I such that, for every $b \in B$ there exists $x_0 \in I$ such that $b(x_0) = \bigwedge_{x\in I} b(x)$.

Let $\langle A, \forall\rangle$ a monadic $(n+1)$-valued Wajsberg algebra.

Claim 3.1 $\langle B(A), \forall, h_A\rangle \in MB^{n+1}$ (see Remark 2.3). Particularly, $\langle B(A), \forall\rangle$ is a monadic Boolean algebra, therefore it can be represented by a rich algebra as follows [14]. A *constant* of $B(A)$ is a boolean homomorphism $c : B(A) \to \forall(B(A))$ such that $c(x) = x$ for every $x \in \forall(B(A))$; the set of all constants of $B(A)$ is denoted by I. The functional algebra $\langle (\forall(B(A)))^I, V\rangle$ is a monadic boolean algebra where $(Vf)(c) = \bigwedge_{c\in I} f(c)$, for each $f \in (\forall(B(A)))^I$. Then $\eta : B(A) \to (\forall(B(A)))^I$ defined by $\eta(b)(c) = c(b)$ for each $b \in B(A)$ is a monadic boolean monomorphism such that $\eta(b)(c) = \bigwedge_{x\in I} (\eta(b))(x)$.

Claim 3.2 The image of a filter in $B(A)$ under $\forall$ is a filter in $\forall(B(A))$. Let h^1 be the function from the lattice of divisors of n into the lattice of filters of $\forall(B(A))$ defined by $h^1(d) = \forall(h_A(d))$. It is easy to show that $\langle \forall(B(A)), \forall, h^1\rangle \in MB^{n+1}$; then, by Corollary 2.2, $\langle M(\forall(B(A)), h^1), \mathbb{V}\rangle$ is a monadic $(n+1)$-valued Wajsberg algebra.

Claim 3.3 If F is a filter in $\forall(B(A))$, then F^I is a filter in $(\forall(B(A)))^I$. Let h^2 be the function from the lattice of divisors of n into the lattice of filters of $(\forall(B(A)))^I$ defined by $h^2(d) = (\forall(h_A(d)))^I$. It is easy to show that $\langle (\forall(B(A)))^I, V, h^2\rangle \in MB^{n+1}$. Therefore, $\langle M((\forall(B(A)))^I, h^2), \mathbb{V}\rangle$ is a monadic $(n+1)$-valued Wajsberg algebra, follows from Corollary 2.2.

Claim 3.4 $\langle M((\forall(B(A)))^I, h^2), \mathbb{V}\rangle$ and $\langle (M(\forall(B(A)), h^1))^I, V\rangle$ are isomorphic algebras.

Let $\Psi : M((\forall(B(A)))^I, h^2) \to (M(\forall(B(A)), h^1))^I$ be the function defined by $((\Psi(g))(c))(i) = g(i)(c)$, for each $g \in M((\forall(B(A)))^I, h^2)$, $c \in I$ and $i \in \{1, 2, \ldots, n\}$.

The function Ψ is well defined and it is a monadic W-isomorphism. Indeed, let $g \in M((\forall(B(A)))^I, h^2)$, $d \in Div(n)$ and $1 \le q \le d$ be an integer. For short let $i_0 = (q-1)\frac{n}{d} + 1$ and $i_1 = q\frac{n}{d}$; then $\xi_{d,q}(g) = g(i_1) \to g(i_0) \in h^2(d) = (\forall(h_A(d)))^I$. Therefore for each $c \in I$ we have $\xi_{d,q}((\Psi(g))(c)) = ((\Psi(g))(c))(i_1) \to ((\Psi(g))(c))(i_0) = g(i_1)(c) \to g(i_0)(c) = (g(i_1) \to g(i_0))(c) \in h^1(d) = \forall(h_A(d))$.

On the other hand, let $f, g \in M((\forall(B(A)))^I, h^2)$, $c \in I$ and $i \in \{1, 2, \ldots, n\}$; then:

(i) $\Psi(f \mapsto g) = \Psi(f) \to \Psi(g)$, indeed:

$(\Psi(f \mapsto g)(c))(i) = (f \mapsto g)(i)(c) = \bigwedge_{k=1}^{n-i+1} (f(k)(c) \to g(k+i-1)(c))$
$= \bigwedge_{c\in I} (\Psi(g)(c))(i) = \bigwedge_{k=1}^{n-i+1} ((\Psi(f)(c))(k) \to (\Psi(g)(c))(k+i-1))$
$= (\Psi(f)(c) \mapsto \Psi(g)(c))(i)$.

(ii) $\Psi(\neg f) = \neg\Psi(f)$, and

(iii) $\Psi(\mathbb{V}g) = V\Psi(g)$, indeed:
$(\Psi(\mathbb{V}g)(c))(i) = ((\mathbb{V}g)(i))(c) = (V(g(i)))(c) = \bigwedge_{c\in I} g(i)(c) = \bigwedge_{c\in I} (\Psi(g)(c))(i)$
$= \left(\bigwedge_{c\in I} (\Psi(g)(c))\right)(i) = ((V\Psi(g))(c))(i).$

(iv) Ψ is bijective.

Claim 3.5 From Theorem 2.2 $\langle A,\forall\rangle$ and $\langle M(B(A),h_A),\mathbb{V}\rangle$ are isomorphic monadic $(n+1)$-valued Wajsberg algebras; the isomorphism is $\varphi : A \longrightarrow M(B(A),h_A)$ defined by $\varphi(x)(i) = \sigma_i(x)$ for all $x \in A$ and every integer $1 \leq i \leq n$.

Claim 3.6 The monomorphism η is a morphism between the objects $\langle B(A),\forall,h_A\rangle$ and $\langle(\forall(B(A)))^I,V,h^2\rangle$ in $\mathscr{MB}^{n+1}$. Thus, $M(\eta)$ is a monadic W-monomorphism from $\langle M(B(A),h_A),\mathbb{V}\rangle$ into $\langle M((\forall(B(A)))^I,h^2),\mathbb{V}\rangle$.

From Claim 3.1 we only have to show $h_A(d) \subseteq \eta^{-1}\left((\forall h_A(d))^I\right)$, for every $d \in Div(n)$. If $x \in h_A(d)$ then $\forall x \in \forall(h_A(d))$, on the other hand, $\forall x = c(\forall x) \leq c(x)$, for each $c \in I$. Therefore $c(x) = \eta(x)(c) \in \forall(h_A(d))$ for every $c \in I$, i.e., $\eta(x) \in (h_A(d))^I$, so $x \in \eta^{-1}\left((\forall h_A(d))^I\right)$.

Claim 3.7 From Claims 3.1 to 3.6 we have the situation that is shown in the following diagram. The function $\gamma = \Psi \circ M(\eta) \circ \varphi$ from A into $(M(\forall(B(A)),h^1))^I$ is a monadic W-monomorphism such that for every $a \in A$ there exists $x_0 \in I$ such that $(\gamma(a))(x_0) = \bigwedge_{c\in I} (\gamma(a))(c)$.

$$\begin{array}{ccc}
\langle A,\forall\rangle & & \\
B\downarrow & & \\
\langle B(A),\forall,h_A\rangle & \overset{\eta}{\longrightarrow} & \langle(\forall(B(A)))^I,V,h^2\rangle \\
M\downarrow & & \downarrow M \\
\langle M(B(A),h_A),\mathbb{V}\rangle & \underset{M(\eta)}{\longrightarrow} & \langle M((\forall(B(A)))^I,h^2),\mathbb{V}\rangle \\
 & & \downarrow \Psi \\
 & & \langle(M(\forall(B(A)),h^1))^I,V\rangle
\end{array}$$

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