SYMmetric structure foR closure algebras

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Abstract. The aim of this paper is to investigate the variety of symmetric closure algebras, that is, closure algebras endowed with a De Morgan operator. Some general properties are derived. Particularly, the lattice of subvarieties of the subvariety of monadic and linear symmetric algebras is described and an equational basis for each subvariety is given.

1. Introduction

This paper deals with the variety of symmetric closure algebras (SC), that is, closure algebras with a symmetric operator (a De Morgan negation), introduced in [9]. An important characteristic of these algebras is that the set of open elements forms a symmetric Heyting algebra [21] (a Heyting algebra with a de Morgan negation [24]). Section 2 provides all necessary background on closure algebras and symmetric Heyting algebras. We describe the subdirectly irreducible algebras of the variety of symmetric closure algebras in Section 3. The rest of the paper is devoted to the study of the subvariety SM of monadic symmetric algebras and the variety SL of linear closure algebras. We determine its lattice of subvarieties and we find equational bases for each subvariety of SL.

Throughout this paper, B and H will denote the equational classes of all Boolean algebras and all Heyting algebras, respectively. If H is a class of similar algebras we will use the following notation: Si(H) (resp. Sim(H), Simp(H)) for the class of subdirectly irreducible (resp. finite subdirectly irreducible, finite simple) algebras in H; H(H) for the class of algebras that are homomorphic images of algebras in H; and S(H) for the class of algebras that are subalgebras of algebras in H. The lattice of congruences of an algebra A ∈ H is denoted by Con(A) or Con(H).

In general, for a variety H and A, B ∈ H, A H B means that A is a H-subalgebra of B. The subalgebra generated by a part X of A ∈ H is denoted by [X] H. If H is a subclass in a variety V, we will denote V(H) the subvariety generated by H, and Λ(V) (or simply Λ) the lattice of subvarieties of V.

2. Preliminaries

In a paper of paramount importance titled “The algebra of topology” J. C. C. McKinsey and A. Tarski [17] started the investigation of a class of algebraic structures which they named closure algebras. A closure algebra is an algebra (A; ∨, ∧, −, ∇, 0, 1) such that (A; ∨, ∧, −, 0, 1) is a Boolean algebra and ∇ is a closure operator, that is, ∇ is a unary
operator on $A$ that satisfies the “Kuratowski axioms”, for all $x, y \in A$: $\bigvee(0) = 0$, $x \leq \bigvee(x)$, $\bigvee(\bigvee(x)) = \bigvee(x)$, $\bigvee(x \lor y) = \bigvee(x) \lor \bigvee(y)$.

The simplest example of closure algebras is the variety $\mathcal{M}$ of monadic Boolean algebras.

Closure algebras have been extensively studied by several authors. Particularly, W. Blok in an exhaustive and very deep work, developed in [6] the general properties of the lattice of subvarieties of the variety of closure algebras.

An important feature in the structure of a closure algebra is the set of open elements. In a continuation of their work on closure algebras, McKinsey and Tarski showed in [18] that the set of open elements of a closure algebra is a Heyting algebra. Conversely, any Heyting algebra can be embedded as the lattice of open elements of a closure algebra.

With the operators $\bigvee$ and $\lnot$ we can define a new unary operator $Q$ (interior operator) by means of $Q(x) = \lnot \lnot(-x)$, for all $x \in A$. This operator satisfies the conditions: $Q(1) = 1$, $x \geq Q(x)$, $Q(Q(x)) = Q(x)$. $Q(x \land y) = Q(x) \land Q(y)$. Closure algebras can be defined by means of these equations and in that case, by defining $\bigvee(x) = -Q(-x)$ we obtain the closure operator.

The equational class of closure algebras will be denoted by $\mathcal{C}$. These algebras were named interior algebras by W. Blok in [6]. If $A \in \mathcal{C}$ then $Q(A)$ is a $(0,1)$-sublattice of $A$, and it is a Heyting algebra if we define $a \rightarrow b = Q(-a \lor b)$, for every $a, b \in A$. If $b \in Q(A)$, $b$ is said to be open.

It is known that if $A \in \mathcal{H}$, then the lattice $\text{Con}(A)$ of congruences of $A$ is isomorphic to the lattice $\mathcal{F}(A)$ of all filters of $A$. If $F \in \mathcal{F}(A)$ then the congruence $\theta$ associated to $F$ is defined by $(a, b) \in \theta \iff a \land u = b \land u$ for some $u \in F$.

If $A \in \mathcal{C}$ and $F$ is a filter in $A$, $F$ is said to be an open filter if $Q(x) \in F$ whenever $x \in F$. It is known ([18]) that $\text{Con}_{\mathcal{C}}(A)$ is isomorphic to the lattice $\mathcal{F}_{Q}(A)$ of all open filters of $A$, and it is not difficult to see that $\mathcal{F}_{Q}(A)$ and $\mathcal{F}(Q(A))$ are isomorphic. So we have:

**Theorem 2.1.** ([5]) Let $A \in \mathcal{C}$. Then $\text{Con}_{\mathcal{C}}(A)$ and $\text{Con}_{\mathcal{H}}(Q(A))$ are isomorphic.

Recall that a Heyting algebra $A$ is subdirectly irreducible if and only if $A = A_{1} \oplus 1$, with $A_{1} \in \mathcal{H}$ and $A_{1} \oplus 1$ is the lattice obtained by adjoining a new 1 to $A_{1}$.

Thus from Theorem 2.1 and this remark it follows immediately the following corollary:

**Corollary 2.2.** Let $A \in \mathcal{C}$. Then $A$ is subdirectly irreducible if and only if $Q(A)$ is subdirectly irreducible as Heyting algebra. Moreover, $A$ is subdirectly irreducible if and only if $Q(A) = A_{1} \oplus 1$, for some $A_{1} \in \mathcal{H}$.

A symmetric Heyting algebra is a system $(A; \lor, \land, \Rightarrow, \sim, 0, 1)$ such that $(A; \lor, \land, \Rightarrow, \sim, 0, 1)$ is a Heyting algebra and $(A; \lor, \land, \sim, 0, 1)$ is a De Morgan algebra ([21], [12], [24], [4]). The variety of symmetric Heyting algebras will be denoted by $\mathcal{S} \mathcal{H}$.

Observe that if $A, B \in \mathcal{S} \mathcal{H}$ and $h : A \rightarrow B$ is a homomorphism then $h^{-1}(1) = \text{Ker}(h)$ is a filter with the property that if $a \Rightarrow b \in \text{Ker}(h)$, then $\sim b \Rightarrow \sim a \in \text{Ker}(h)$. Any filter $F$ of an algebra $A \in \mathcal{S} \mathcal{H}$ satisfying the condition that if $a \Rightarrow b \in F$ then $\sim b \Rightarrow \sim a \in F$, is called a kernel of $A$. A. Monteiro [21] proved that if, for a given kernel $F$ in a symmetric Heyting algebra $A$ we define $x \theta_{F} y$ if and only if $x \Rightarrow y \in F$ and $y \Rightarrow x \in F$, then $\theta_{F}$ is a congruence.
Moreover, every congruence on $A$ is determined by a kernel, and the mapping $F \mapsto \theta_F$ is a lattice isomorphism between the lattice of kernels of $A$ and the lattice $\text{Con}_{\mathcal{SH}}(A)$.

A slightly more amenable to work with, the following theorem gives an equivalent condition for a filter to be a kernel. As usual, $\neg a = a \Rightarrow 0$.

**Theorem 2.3.** ([21]) A filter $F$ of a symmetric Heyting algebra $A$ is a kernel if and only if $\neg \sim a \in F$ whenever $a \in F$.

In the case of a principal filter $F_a = \{x \in A : a \leq x\}$, this condition is equivalent to the condition $a \land \neg a = 0$.

Consider the following terms: $(\neg \sim)^0 x = x$ and $(\neg \sim)^{n+1} x = \neg \sim (\neg \sim)^n x$. Observe that $\neg \sim (a \land b) = \neg \sim a \land \neg \sim b$ and $\neg \sim \neg \sim a \leq a$, for all $a, b \in L \in \mathcal{SH}$. Thus, for $n \in \omega$, $(\neg \sim)^n (a \land \neg \sim a) \geq (\neg \sim)^{n+1} (a \land \neg \sim a)$.

We say that $L \in \mathcal{SH}$ is of finite range if for all $a \in L$ there exists $n \in \omega$ such that $(\neg \sim)^n (a \land \neg \sim a) = (\neg \sim)^{n+1} (a \land \neg \sim a)$.

Let $\text{Cen}(A) = \{x \in A : \neg \sim x = \sim x\}$ be the center of $A$, that is, the sublattice of complemented elements of $A$.

**Theorem 2.4.** ([24]) For $A \in \mathcal{SH}$ of finite range and $|A| \geq 1$, the following are equivalent:
1. $A$ is simple.
2. $A$ is subdirectly irreducible.
3. $A$ is directly indecomposable.
4. $\text{Cen}(A) = \{0, 1\}$ or $\text{Cen}(A) = \{0, a = \sim a, \neg a, 1\}$.

If $A$ satisfies the Kleene condition $(\sim x \land \sim y \leq \sim x \lor \sim y)$ then in Theorem 2.4, condition [4] is just $\text{Cen}(A) = \{0, 1\}$.

To close this section we recall the following important results.

**Theorem 2.5.** ([21]) Any finite symmetric Heyting algebra is a direct product of (finite) simple symmetric Heyting algebras.

**Corollary 2.6.** Every finite subdirectly irreducible symmetric Heyting algebra is simple.

The variety $\mathcal{SH}$ is generated by its finite members ([21]). Thus

**Corollary 2.7.** $\mathcal{SH} = \text{V} (\text{Si}_{\text{fin}}(\mathcal{SH})) = \text{V} (\text{Simp}_{\text{fin}}(\mathcal{SH}))$.

3. Symmetry on closure algebras

In this section we study the variety of symmetric closure algebras introduced in [9]. This variety consists of closure algebras with a De Morgan negation “$\neg$” such that $Q(\sim Q(x)) = \sim Q(x)$, and in particular, the set of open elements form a symmetric Heyting algebra.

First we need some properties of the variety of symmetric Boolean algebras $\mathcal{SB}$ ([4], [21]). A symmetric Boolean algebra is an algebra $(A; \sim)$ such that $A$ is a Boolean algebra and $\sim$ is a De Morgan negation.

Observe that these algebras are symmetric Heyting algebras. In addition, $\sim x \lor \sim x = \sim (\sim x \land x) = \sim 0 = 1$, and $\sim x \land \sim x = \sim (\sim x \lor x) = \sim 1 = 0$. So $\sim x = - x$. Then $\sim$ is a dual isomorphism of Boolean algebras.
A cyclic Boolean algebra of order two is an algebra \( (A; T) \) such that \( A \) is a Boolean algebra and \( T \) is a unary operation which is an automorphism such that \( T^2 = I_A \) (see [22]). If \( (A; \sim) \in \mathcal{MB} \) and we put \( T(x) = \sim x \Rightarrow 0 = \sim x \lor 0 = \sim x = \sim (\sim x) \), then \( T \) is an automorphism such that \( T^2 = I_A \), that is, \( (A; T) \) a cyclic Boolean algebra of order two. Conversely, if \( (A; T) \) is a cyclic Boolean algebra or order two and we put \( \sim x = \neg T(x) \), then \( (A; \sim) \) is a symmetric Boolean algebra. We say that this two varieties are equivalent in the sense of R. Lewin ([14]).

Congruences in the variety \( \mathcal{MB} \) (or in the variety of cyclic Boolean algebras of order two) are given by symmetric filters, that is, by those filters \( F \) satisfying the condition \( T(x) \in F \) (or equivalently \( \sim x \in F \)) whenever \( x \in F \).

There are two subdirectly irreducible algebras in \( \mathcal{MB} \), which in addition, are simple, namely: the algebra \( 2 = \{0, 1\} \) and the algebra \( 2 \times 2 \), where \( \sim (0, 1) = (0, 1) \) and \( \sim (1, 0) = (1, 0) \), (or \( T(0, 1) = (1, 0) \)).

Now we are in a position to define symmetric closure algebras.

**Definition 3.1.** We say that an algebra \( (A; \lor, \land, -, Q, \sim, 0, 1) \) is a symmetric closure algebra if the following conditions are satisfied:

1. \( (A; \lor, \land, -, Q, 0, 1) \in \mathcal{C} \).
2. \( (A; \lor, \land, -, Q, 0, 1) \in \mathcal{MB} \).
3. \( Q(\sim Q(x)) = \sim Q(x) \).

Let \( \mathcal{SC} \) denote the variety of symmetric closure algebras. Observe that if \( A \in \mathcal{SC} \), then \( Q(A) \in \mathcal{SH} \).

**Lemma 3.2.** For \( A, A_1 \in \mathcal{SC} \) and \( h : A \to A_1 \) an \( \mathcal{SC} \)-homomorphism, it holds that

(i) \( h(Q(A)) \subseteq Q(A_1) \), (ii) \( h(Q(A)) : Q(A) \to Q(A_1) \) is an \( \mathcal{SH} \)-homomorphism, and

(iii) if \( h \) is onto, then \( h|_{Q(A)} \) is onto.

Our objective now is to characterize the congruences and the subdirectly irreducible algebras in \( \mathcal{SC} \).

A filter \( F \) is called an open kernel if \( F \) is open, that is, \( Q(x) \in F \) whenever \( x \in F \), and \( \sim b \to \sim a \in F \) whenever \( a \to b \in F \).

Observe that an open filter \( F \) is an open kernel if and only if \( F \) is symmetric. Indeed, if \( F \) is an open kernel and \( x \in F \), then \( 1 \to x \in F \) and thus \( \sim x \to 0 \in F \). So \( Q(\sim x) \in F \) and then \( \sim \sim x \in F \). Consequently \( F \) is symmetric. Conversely, let \( F \) be an open filter such that \( x \in F \), \( \sim x \in F \), and suppose that \( x \to y \in F \). Then \( Q(\sim x \lor y) \in F \), and thus \( \sim x \lor y \in F \). This implies that \( \sim (\sim x \lor y) = \sim x \lor \sim y \in F \). Hence \( Q(\sim x \lor \sim y) \in F \), and consequently, \( \sim y \to \sim x \in F \). Therefore \( F \) is a kernel.

Let \( N_{\mathcal{SC}}(A) \) denote the family of all open kernels of \( A \). Then we have the following theorem ([9, 10]).

**Theorem 3.3.** For any \( A \in \mathcal{SC} \), \( \text{Con}_{\mathcal{SC}}(A) \cong N_{\mathcal{SC}}(A) \).

It is easy to see that if \( F \) is an open kernel of an algebra \( A \in \mathcal{SC} \), then \( Q(F) \) is a kernel of \( Q(A) \) considered in \( \mathcal{SH} \). Conversely, if \( F \) is an kernel of \( Q(A) \), then the filter generated in \( A \) by \( F, [F]_A \), is an open kernel of \( A \) and \( Q([F]_A) = F \). Then we have the following corollary.
Corollary 3.4. Let $A \in \mathcal{S}C$. Then $\mbox{Con}_{\mathcal{S}C}(A) \equiv N_{\mathcal{S}C}(A) \equiv N_{\mathcal{S}H}(Q(A)) \equiv \mbox{Con}_{\mathcal{S}H}(Q(A))$.

Corollary 3.5. Let $A \in \mathcal{S}C$. Then $A$ is subdirectly irreducible if and only if $Q(A)$ is subdirectly irreducible in $\mathcal{S}H$.

Then by Theorem 2.4 we have the following theorem.

Theorem 3.6. Let $|A| \geq 1$, with $A \in \mathcal{S}C$ and $Q(A)$ of finite range. Then the following conditions are equivalent:
1. $A$ is simple.
2. $A$ is subdirectly irreducible.
3. $A$ is directly indecomposable.
4. $\mbox{Cen}(Q(A)) = \{0, 1\}$ or $\mbox{Cen}(Q(A)) = \{0, a, \neg a, 1\}$.

Corollary 3.7. Let $A \in \mathcal{S}C$, $A$ finite. Then $A$ is subdirectly irreducible if and only if $A$ is simple.

As a consequence of the above results, we have the following.

Theorem 3.8. Let $A \in \mathcal{S}C$, $A$ finite. Then $A$ is a direct product of finite simple algebras.

4. Subvarieties and finite generation of $\mathcal{S}C$

In this section we will prove that the variety $\mathcal{S}C$ is generated by its finite members. We will follow a path similar to that of McKinsey and Tarski in [18] for closure algebras. We will also give some general results about the relationship between the lattice of subvarieties of symmetric closure algebras and the lattice of subvarieties of symmetric Heyting algebras.

Lemma 4.1. Let $(A; \land, \lor, \neg, Q, \sim, 0, 1)$ be a symmetric closure algebra and let $a_1, \ldots, a_r \in A$. Then there exists $A_1 \subseteq A$ and an interior operator $Q_1$ on $A_1$ such that the following conditions hold:
(i) $(A_1; \land, \lor, \neg, Q_1, \sim, 0, 1)$ is a symmetric closure algebra.
(ii) $a_i \in A_1$ for $i = 1, \ldots, r$.
(iii) $A_1$ contains at most $2^{2\#\mathcal{S}C}$ elements.
(iv) If $x \in A_1$ and $Q(x) \in A_1$ then $Q_1(x) = Q(x)$.

Proof. Let $A_1$ be the symmetric Boolean subalgebra generated in $A$ by $a_1, \ldots, a_r$. Then $|A_1| \leq 2^{2\#\mathcal{S}C}$, and (ii) and (iii) hold. Let $K = \{x \in A_1 : Q(x) \in A_1\}$ and $Q(K) = \{Q(x) : x \in K\}$. Observe that if $x \in Q(K)$ then $\sim x \in Q(K)$. Indeed, since $A_1$ is a symmetric Boolean algebra, $\sim x \in A_1$, whenever $x \in Q(K)$. But $A$ is a symmetric closure algebra, so $Q(x) = \sim x \in Q(K)$. Let $L$ be the De Morgan algebra generated by $Q(K)$ in $A_1$. Then $L$ defines an interior operator $Q_1$ on $A_1$. Let us see that if $x \in A_1$ and $Q(x) \in A_1$ then $Q(x) = Q_1(x)$. First we prove that $L = [Q(K)]_{D_0}$. Indeed, if $x \in L$ then there exist subsets $\{H_i\} \subseteq Q(K)$ and $\{S_i\} \subseteq \sim Q(K)$ such that
$$x = \bigwedge_{H_i, y_i \in H_i} y_i \lor \bigvee_{S_i, y_j \in S_i} y_j.$$ 
But we know that $\sim (Q(K)) \subseteq Q(K)$. So we can state that there exist $\{R_i\} \subseteq Q(K)$ such that
$$x = \bigwedge_{R_i, y_i \in R_i} y_i.$$

Actas del IX Congreso Dr. Antonio A. R. Monteiro, 2007
and then \( L = [Q(K)]_{D_{0}}, \) Observe that if \( x \in L, Q_{1}(x) = x = Q(x). \) If \( x \in A_{1} \)

\[
Q_{1}(x) = \bigvee \{ a \in L : a \leq x \}.
\]

Since \( Q(a) = a \leq Q(x), \) it follows that \( Q_{1}(x) \leq Q(x). \) In addition, \( x, Q(x) \in A_{1}, \) and thus \( Q(x) \in Q(K). \) So \( Q(x) \in \{ a \in L : a \leq x \}, \) which implies that \( Q(x) \leq Q_{1}(x). \)

**Definition 4.2.** A finite sequence of terms in the language of \( \mathcal{C}, f_{1}, \ldots, f_{r}, \) (all with the same number of variables) is called a chain for the term \( f \) if:

(i) \( f_{r} = f \)

(ii) \( f_{h}(x_{1}, \ldots, x_{n}) = x_{i}, \) or \( f_{h} = f_{i} \wedge f_{j}, \) (\( h > i, j \)), or \( f_{h} = f_{i} \vee f_{j}, \) or \( f_{h} = \sim f_{i}, \) or \( f_{h} = -f_{i}, \) or \( f_{h} = Q(f_{i}), \) or \( f_{h} = 1. \)

The number \( r \) is the length of the chain. The smallest length of the chain is called the order of \( f. \) In the next theorem we use the fact that any equation in \( \mathcal{C}, \) \( p(x_{1}, \ldots, x_{n}) = q(x_{1}, \ldots, x_{n}), \) can be written in the form \( t(x_{1}, \ldots, x_{n}) = 1. \)

**Theorem 4.3.** If \( f(x_{1}, \ldots, x_{n}) = 1 \) is an identity for every finite algebra in \( \mathcal{C}, \) then it is an identity for every algebra in \( \mathcal{C}. \)

**Proof.** Let \( A \in \mathcal{C} \) infinite such that \( f(x_{1}, \ldots, x_{n}) = 1 \) is not an identity for \( A, \) that is, there exist \( a_{1}, a_{2}, \ldots, a_{n} \in A \) such that \( f(a_{1}, \ldots, a_{n}) \neq 1. \) Let \( f_{1}, \ldots, f_{r} \) be a chain for \( f, \) of length \( r, \) where \( r \) is the order of \( f. \) Let us put \( f_{1}(a_{1}, \ldots, a_{n}) = b_{1}, \) \( f_{2}(a_{1}, \ldots, a_{n}) = b_{2}, \), \( \ldots, f_{r}(a_{1}, \ldots, a_{n}) = f(a_{1}, \ldots, a_{n}) = b_{r} \neq 1. \) By the previous lemma, there exists a finite closure algebra \( A_{1} \subseteq A \) such that \( a_{1}, a_{2}, a_{n}, b_{1}, \ldots, b_{r} \in A_{1}. \) Then \( f_{1} = b_{1} = 1 \in A_{1}, \) \( f_{2}(a_{1}, \ldots, a_{n}) = b_{2} \in A_{1}, \) \( \ldots, f_{r}(a_{1}, \ldots, a_{n}) = b_{r} \in A_{1} \) and \( b_{r} \neq 1. \) So \( f(x_{1}, \ldots, x_{n}) = 1 \) is not an identity for the finite algebra \( A_{1}, \) a contradiction.

**Corollary 4.4.** The variety \( \mathcal{C} \) is generated by its finite members.

**Corollary 4.5.**

\[
\mathcal{C} = V(\text{Simp}_{\text{fin}}(\mathcal{C})) = V(\text{Simp}_{\text{fin}}(\mathcal{C})).
\]

Next, as in the case of closure algebras and Heyting algebras (see Blok [6]), we will find a strong relationship between the lattice of subvarieties of \( \mathcal{H} \) and the lattice of \( \mathcal{C}. \) If \( \mathcal{K} \in \Lambda(\mathcal{C}) \) then we denote

\[
Q(\mathcal{K}) = \{ Q(A) : A \in \mathcal{K} \}
\]

and for \( \mathcal{K} \subseteq \Lambda(\mathcal{C}), \) we put

\[
\mathcal{K} \triangleright \mathcal{C} = \{ A \in \mathcal{C} : Q(A) \in \mathcal{K} \}.
\]

It is long but computational to check that

**Theorem 4.6.**

(i) If \( \mathcal{K} \) is an equational class of symmetric closure algebras, then \( Q(\mathcal{K}) \) is an equational class of symmetric Heyting algebras.

(ii) If \( \mathcal{K} \) is an equational class of symmetric Heyting algebras, then \( \mathcal{K} \triangleright \mathcal{C} \) is an equational class of symmetric closure algebras.

**Theorem 4.7.**

(i) If \( \mathcal{K} \subseteq \mathcal{C}, \) then \( Q(V(\mathcal{K})) = V(Q(\mathcal{K})). \)

(ii) If \( \mathcal{K} \subseteq \Lambda(\mathcal{C}) \) is such that \( \text{Simp}(V(\mathcal{K})) \subseteq \mathcal{K}, \) then \( V(\mathcal{K}) \triangleright \mathcal{C} = V(\mathcal{K} \triangleright \mathcal{C}). \)
(In this case, \( \mathcal{K} \) is not necessarily a variety).

Then we define the functions
\[
\mathcal{O}_1 : \Lambda(\mathcal{I}C) \longrightarrow \Lambda(\mathcal{SH}), \quad \mathcal{O}_1(\mathcal{K}) = Q(\mathcal{K}).
\]
and
\[
\mathcal{O}_2 : \Lambda(\mathcal{SH}) \longrightarrow \Lambda(\mathcal{I}C), \quad \mathcal{O}_2(\mathcal{K}) = \mathcal{K} \cdot \mathcal{Q}.
\]

The functions \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) are well defined and they commute with the operators \( P, H \) and \( S \). By Jónsson's Theorems [13] we have that the distributive lattices \( \Lambda(\mathcal{I}C) \) and \( \Lambda(\mathcal{SH}) \) are complete. Now we will see some important properties of these functions.

**Theorem 4.8.** \( \mathcal{O}_1 \) is a complete surjective lattice homomorphism. \( \mathcal{O}_2 \) is a lattice embedding. Moreover, \( \mathcal{O}_1 \circ \mathcal{O}_2 = \text{Id} \) and consequently \( \Lambda(\mathcal{SH}) \) is a retract of \( \Lambda(\mathcal{I}C) \).

**Proof.** We prove the theorem for \( \mathcal{O}_2 \), the other case being similar. If \( \mathcal{I}C \) is the trivial variety of symmetric closure algebras, then \( \mathcal{O}_1(\mathcal{I}C) = \mathcal{I}H \), with \( \mathcal{I}H \) the trivial variety of symmetric Heyting algebras. In addition, it is clear that \( \mathcal{O}_1(\mathcal{I}C) = \mathcal{O}_1(\mathcal{I}C) = \mathcal{SH} = \mathcal{I}H \). If \( \{ \mathcal{K}_i : i \in I \} \subseteq \Lambda(\mathcal{I}C) \), then
\[
\mathcal{O}_1 \left( \bigvee_{i \in I} \mathcal{K}_i \right) = \mathcal{O}_1 \left( V \left( \bigcup_{i \in I} \mathcal{K}_i \right) \right) = \mathcal{O} \left( V \left( \bigcup_{i \in I} \mathcal{K}_i \right) \right) = V \left( \bigcup_{i \in I} \mathcal{O}(\mathcal{K}_i) \right) = \bigvee_{i \in I} \mathcal{O}_1(\mathcal{K}_i).
\]

On the other hand, \( \mathcal{O}_1 \) preserves arbitrary infima:
\[
\mathcal{O}_1 \left( \bigwedge_{i \in I} \mathcal{K}_i \right) = \mathcal{O}_1 \left( \bigcap_{i \in I} \mathcal{K}_i \right) = \mathcal{O} \left( \bigcap_{i \in I} \mathcal{K}_i \right) = \bigcap_{i \in I} \mathcal{O}(\mathcal{K}_i) = \bigwedge_{i \in I} \mathcal{O}_1(\mathcal{K}_i)
\]
Finally, \( \mathcal{O}_1 \) is onto, as if \( \mathcal{K} \in \Lambda(\mathcal{SH}) \) then \( \mathcal{O}_1(\mathcal{K} \cdot \mathcal{Q}) = Q(\mathcal{K} \cdot \mathcal{Q}) = \mathcal{K} \).

Observe that an identity characterizing a subvariety of symmetric Heyting algebras can be translated into an identity characterizing a subvariety of symmetric closure algebras. Indeed, if
\[
p_V(a_1, \ldots, a_n) = q_V(a_1, \ldots, a_n)
\]
is the characteristic identity for a subvariety \( V \subseteq \mathcal{SH} \) then the subvariety \( V \cdot \mathcal{Q} \) of \( \mathcal{I}C \) is determined by the equation
\[
p_V(Q(a_1), \ldots, Q(a_n)) = q_V(Q(a_1), \ldots, Q(a_n)).
\]
In this way we can study many subvarieties of symmetric closure algebras obtained from subvarieties of symmetric Heyting algebras.

5. **Symmetric monadic algebras**

In this section we study the variety \( \mathcal{I}M \) of symmetric monadic algebras. This variety consists of those symmetric closure algebras in which \( Q \) is a quantifier, that is, \( Q(x \lor Q(y)) = Q(x) \lor Q(y) \), and it will play an important role for the study of the linear case. We reproduce here some known results, the proof of which can be found in [9].

In [4], M. Abad and L. Monteiro introduced the variety of cyclic monadic algebras. A cyclic monadic algebra is a Boolean algebra endowed with a unary operation which is a (monadic) automorphism of period two. We prove in this section that the variety of symmetric monadic algebras and the variety of cyclic monadic algebras are equivalent in

\[\text{Actas del IX Congreso Dr. Antonio A. R. Monteiro, 2007}\]
the sense of [14]. We describe the lattice of subvarieties of \( \mathcal{M} \) and we determine an equational basis for each subvariety.

As the previous section, \( A \in \mathcal{M} \) is a simple (subdirectly irreducible) algebra if and only if \( Q(A) \) is a simple symmetric Boolean algebra, that is, if and only if either \( Q(A) = 2 \) or \( Q(A) = 2 \times 2 \), where \( \sim (0, 1) = (0, 1) \) and \( \sim (1, 0) = (1, 0) \).

Let \( A \in \mathcal{M} \) and let \( T^* = \sim \). We know that \( T^* \) is a Boolean automorphism. Let us prove that \( T^*(Q(x)) = Q(T(x)) \). It is easy to see that \( \mathcal{M} \) is locally finite, and consequently, \( \mathcal{M} \) is generated by its finite members. Thus it is enough to prove that the equation \( T^*(Q(x)) = Q(T(x)) \) holds in all finite subdirectly irreducible (simple) algebras of \( \mathcal{M} \).

Let \( B \) be a simple algebra in \( \mathcal{M} \) such that \( Q(B) = \{0, 1\} \). Let \( x \in B \). If \( x = 1 \), then \( Q(x) = 1 = x \). Thus \( x = 1 = x \). Consequently, \( \sim x = 1 = 0 = Q(\sim 1) = Q(\sim x) \). If \( x \neq 1 \), then \( Q(x) = 0 \) and \( \sim x = 1 \). Thus \( \sim Q(x) = 0 = x = Q(\sim x) \). The case in which \( Q(B) = \{0, 1\} \) is similar. So we have the following theorem.

**Theorem 5.1.** Let \( A \in \mathcal{M} \). Then \( \langle A; T^* \rangle \) is a cyclic monadic algebra of order two.

Now, let \( (A; T) \) be a cyclic monadic algebra of order two. Let \( \sim^* = \sim T \). We know that \( \sim^* \) is a De Morgan negation. In addition, if \( x \in A \), as \( A \in \mathcal{M} \), \( Q(\sim^* Q(x)) = Q(T(Q(x))) = Q(-T(Q(x))) = -Q(T(x)) = -T(Q(x)) = \sim^* Q(x) \). So \( (A; \sim^*) \) is a symmetric monadic algebra.

Observe that \( -\sim^* = -\sim T = T \). We have proved the following result.

**Theorem 5.2.** Let \( (A; T) \) a cyclic monadic algebra of order two, and let \( \sim^* = \sim T \). Then \( \langle A; \sim^* \rangle \in \mathcal{M} \).

If \( \mathcal{M}_2 \) is the variety of cyclic monadic Boolean algebra of order two, then the varieties \( \mathcal{M} \) and \( \mathcal{M}_2 \) are equivalent in the sense of [14]. This equivalence will allow us to use both operations \( \sim \) and \( T \), related by \( T = \sim \).

We say that a simple algebra \( A \) is of type I if \( Q(A) = \{0, 1\} \), whereas \( A \) is said to be of type II when \( Q(A) = \{0, a = \sim a, -a, 1\} \).

Let \( \text{At}(A) \) denote the set of atoms of a finite algebra \( A \), let \( \mathfrak{S}(\text{At}(A)) = \{a \in \text{At}(A): T(a) = a\} \) denote the set of atoms fixed by the action of \( T \), and let \( \mathfrak{S}'(\text{At}(A)) = \{a \in \text{At}(A): T(a) \neq a\} \) denote the set of atoms non-fixed by \( T \). Observe that if \( a \in \mathfrak{S}'(\text{At}(A)) \) and \( T(a) = b \), \( a \neq b \), then \( T(b) = T(T(a)) = a \). So \( T(a) \in \mathfrak{S}'(\text{At}(A)) \). Thus \( \mathfrak{S}'(\text{At}(A)) \) is an odd number. It is clear that \( \text{At}(A) = \mathfrak{S}(\text{At}(A)) \cup \mathfrak{S}'(\text{At}(A)) \).

Let \( C_{2 \times 2} \) denote the finite simple algebras of type I with \( 2 \times 2 \) non-fixed atoms and \( t \) fixed atoms. A simple finite algebra of type II has an even number of atoms as the number of atoms preceding \( a \) equals the number of atoms preceding \( -a \). We denote \( D_c \) a finite simple algebra of type II with \( 2k \) atoms.

Since \( \mathcal{M} \) is a congruence distributive locally finite variety, we can apply the well-known results of Davey [8] for the lattice of subvarieties \( \Lambda(\mathcal{M}) \). In this section we characterize the poset \( J_{\text{fin}}(\Lambda(\mathcal{M})) \) of finite join-irreducible elements of \( \Lambda(\mathcal{M}) \). Recall that a variety is said to be finitely generated if it is generated by finitely many subdirectly irreducible finite algebras. We consider separately the finitely generated and non finitely generated join-irreducible varieties.
Recall that the ordering in $J_{\text{fin}}(\Lambda(\mathcal{M}))$ is given by $V(A) \leq \Lambda V(B) \iff A \in \text{HS}(B)$, where $A$ is a subdirectly irreducible algebra (see Jónsson [13]). Since every subdirectly irreducible algebra in $\mathcal{M}$ is simple, then $V(A) \leq \Lambda V(B) \iff A \in \text{IS}(B)$. Thus we only have to characterize the subalgebras of a simple algebra.

Let $V_{2 \times s,t} = V(C_{2 \times s,t})$. The above properties give us a description of the ordering of the join-irreducible varieties generated by a simple algebra of type I.

**Theorem 5.3.** ([9, 10]) $V_{2 \times k,l} \leq \Lambda V_{2 \times s,t}$ if and only if $k \leq s$ and $k + l \leq s + t$.

Let $\mathcal{F}$ be the variety generated by the algebras of type I. Then we can describe the poset $J_{\text{fin}}(\Lambda(\mathcal{F}))$ of finite join-irreducible elements of $\Lambda(\mathcal{F})$: Let $C$ be a chain of type $\omega$ and $C^{[2]}$ the set of increasing functions from $2$ into $C$ and consider the set $C^{[2]} \setminus \{(0,0)\}$. Let $\varphi : J_{\text{fin}}(\Lambda(\mathcal{F})) \rightarrow C^{[2]} \setminus \{(0,0)\}$ be defined by $\varphi(V_{2 \times s,t}) = (s,s+t)$.

**Theorem 5.4.** The mapping $\varphi$ is an order isomorphism.

Observe that the subvariety $\mathcal{F}$ is determined by the equation $T(Q(x)) = Q(x)$. Indeed, this equation holds in any subdirectly algebra in $\mathcal{F}$ and does not hold in the simple algebras of type II, as $T(Q(a)) = T(a) = -a \neq a = Q(a)$ for an open element $a \notin \{0,1\}$.

Now we investigate the varieties of type II. Observe that a variety contained in $\mathcal{F}$ cannot be greater that (in $\Lambda(\mathcal{M})$) a variety of type II. The reason is that a simple algebras of type I contains no subalgebras of type II.

Let us look at the ordering for join-irreducible varieties generated by algebras of type II. We denote $U_k = V(D_k)$.

The following theorem completes the description of the ordering in the set of finite join-irreducibles of $\Lambda(\mathcal{M})$.

**Theorem 5.5.** ([9, 10]) $U_k \leq U_l$ if and only if $k \leq l$ and $V_{2 \times s,t} \leq \Lambda U_k$ if and only if $2s + t \leq k$.

Now we will determine the infinite join-irreducible subvarieties of $\mathcal{M}$, and we will prove that any variety is a finite join of join-irreducible varieties in $\Lambda(\mathcal{M})$. This result will play an important role in determining equational bases for subvarieties in $\mathcal{M}$.

Consider the following subvarieties: for $h \geq 0$,

$$S_h = V(C_{2 \times j,k} : k \geq 1, j \leq h) = \bigvee_{k \geq 1, j \leq h} V_{2 \times j,k} = V\left(\bigcup_{k \geq 1} C_{2 \times j,k}\right).$$

From the definition of $\mathcal{F}$, it follows that $\mathcal{F} = \bigvee_{k \geq 1, j \geq 0} V_{2 \times j,k} = \bigvee_{k \geq 1} S_k$. Hence

$$S_0 \subset S_1 \subset S_2 \subset \ldots \subset S_n \subset \ldots \subset \mathcal{F}.$$

We have that $\mathcal{F}$, $\{S_i\}_{i \geq 0}$ and $\mathcal{M}$ are the unique infinite join-irreducible varieties.

**Theorem 5.6.** ([9, 10]) The unique infinite join-irreducible varieties in $\Lambda(\mathcal{M})$ are $\mathcal{M}$, $\mathcal{F}$ and $\{S_i\}_{i \geq 0}$.

The poset $J(\Lambda(\mathcal{M}))$ of join-irreducible elements of $\Lambda(\mathcal{M})$ looks like the diagram in the following figure, in which we simply write $V_{st}$ instead of $V_{2 \times s,t}$.

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*Actas del IX Congreso Dr. Antonio A. R. Monteiro, 2007*
Observe that $U_t$ is covered only by $U_{t+1}$. Also, $s + t \neq 0$. So, in $C^{[2]} \setminus \{(0, 0)\}$, $(s, s + t + 1)$ covers $(s, s + t)$. For $t \geq 1$, $(s + 1, s + t)$ also covers $(s, s + t)$, and these are the only pairs that cover $(s, s + t)$. Recall that $\varphi(V_{2 \times s, t}) = (s, s + t)$. Hence $V_{2 \times s, t+1}$, $V_{2 \times s+1, t-1}$ and $U_{2s+t}$ are the unique elements in $J(\Lambda(I\mathcal{M}))$ which cover $V_{2 \times s, t}$.

The following theorem will be used in the next section.

**Theorem 5.7.** ([9, 10]) Any variety in $\Lambda(I\mathcal{M})$ is a finite join of finitely many join-irreducible varieties in $\Lambda(I\mathcal{M})$.

### 6. Equational Bases for $I\mathcal{M}$

In this section an equation that characterizes each subvariety in $I\mathcal{M}$ is given. First we characterize equationally the join-irreducible varieties, and then we determine an equation for the join of finitely many join-irreducible varieties.

We have already shown that the variety $\mathcal{J}$ is characterized, relative to $I\mathcal{M}$, by the equation $T(Q(x)) = Q(x)$.

Observe that $s(x, y, 1, 0) = x \iff y = Q(x \rightarrow y) \land Q(y \rightarrow x)$ is a switching term for the variety $\mathcal{J}$ (see [24]). Then $x \iff y \in \{0, 1\}$ for any $x, y$ in a subdirectly irreducible algebra $A \in \mathcal{J}$. This remark will be strongly used in what follows.
Let $\Gamma_{S_n}(x_1, \ldots, x_{2n})$ denote the term
\[
\left[ \bigwedge_{i=1}^{n+t} (T(x_{2i-1}) \leftrightarrow x_{2i}) \right] \wedge \left( \bigvee_{i \neq j, j=1}^{n+t} ((x_i \wedge x_j) \leftrightarrow 0) \right) \wedge \left( \bigwedge_{i=1}^{2n+1} (x_i \leftrightarrow 0) \right) \rightarrow \left[ \bigvee_{i=1}^{2n+1} x_i \right] \leftrightarrow \left( \bigwedge_{i=1}^{2n+1} x_i \right)
\]

The next theorem shows a set of equations that characterize the varieties $S_n$ within $\mathcal{M}$.

**Theorem 6.1.** (see [9, 10]) The variety $S_n$ is characterized by the following equations: If $n = 0$, $T(x) = x$, and if $n > 0$, $T(Q(x)) = Q(x)$ and $\Gamma_{S_n}(x_1, \ldots, x_{2n}) = 1$

Consider now the following terms:
\[
T_1^1(x_1) = x_1,
\]
\[
T_1^p(\vec{x}) = \left[ \bigvee_{i,j=1, i \neq j}^{p} \nabla(x_i \wedge x_j) \wedge \left( \bigwedge_{k \neq i, k=1}^{p} \nabla(x_k) \right) \right] \vee \left[ \bigvee_{i=1}^{p} (x_i \wedge \left( \bigwedge_{k \neq i, k=1}^{p} \nabla(x_k) \right)) \right],
\]
\[
T_2^1(x_1) = \nabla x_1 \text{ and } T_2^p(\vec{x}) = \bigwedge_{i=1}^{p} \nabla(x_i),
\]

where $T_1^p(\vec{x})$ stands for $T_1^p(x_1, x_2, \ldots, x_p)$.

In [23], [1] and [9] it is shown that the equation $T_1^p(\vec{x}) = T_2^p(\vec{x})$ is an equational basis for the variety generated by the simple algebra $B_p \in \mathcal{M}$, for a fixed $p$.

Let $\Gamma_{2n,f}(x_1, \ldots, x_{2n})$ denote the term
\[
\left[ \bigwedge_{i=1}^{n+t} (x_i \leftrightarrow 0) \right] \wedge \left( \bigvee_{i \neq j, j=1}^{n+t} ((x_i \wedge x_j) \leftrightarrow 0) \right) \wedge \left( \bigwedge_{i=1}^{n+t} (T(x_i) \leftrightarrow x_i) \right) \rightarrow \left[ \bigvee_{i=1}^{n+t} x_i \right] \leftrightarrow 1.
\]

The following theorem gives an equational basis for the join-irreducible subvarieties of type I, $V_{2 \times n, f}$.

**Theorem 6.2.** Let $n > 0$. The equations $T_1^{2n+t}(\vec{x}) = T_2^{2n+t}(\vec{x})$ and $\Gamma_{2n,f}(x_1, \ldots, x_{2n}) = 1$ characterize the subvariety $V_{2 \times n, f}$ within $S_n$.

Now we give an equational basis for the subvarieties $U_k = V(D_k)$.

**Lemma 6.3.** ([9, 10]) The subvariety $U_k = V(D_k)$ is characterized by the equation $T_1^k(\vec{x}) = T_2^k(\vec{x})$.

We have determined an equational basis for each join-irreducible subvariety. Now we are going to give an equational basis for every subvariety. From Theorem 5.7, we only have to find an equation for every finite join of join-irreducible subvarieties.

Observe that any variety $V \in J(A(\mathcal{M}))$ is determined by a single equation, that is, there exists an equation of the form $\gamma_V(x_1, \ldots, x_r) = 1$, with $r$ as needed, that determines $V$ within $\mathcal{M}$.
Let \( V = \bigvee_{i=1}^n V_i \) with \( V_i \in J(A(\mathcal{S}, \mathcal{M})) \), for all \( i = 1, \ldots, n \). Let
\[
\gamma_{V}(x_1, \ldots, x_{n_V}) = \bigvee_{i=1}^n (\gamma_{V_i}(x_1^i, \ldots, x_{r_i}^i) \leftrightarrow 1) \land (\sim \gamma_{V_i}(x_1^i, \ldots, x_{r_i}^i) \leftrightarrow 0).
\]
In [9, 10] it is shown that \( \gamma_V(x_1, \ldots, x_{n_V}) = 1 \) is an equational basis for \( V \).

### 7. Linear Symmetric Closure Algebras

This part of the work is devoted to an exhaustive investigation of the variety of those closure algebras whose open elements form a linear symmetric Heyting algebra. We first present some general results obtained in [10] and then we characterize the subvarieties of linear closure algebras. In 7.2 we will carry out a deep study of finitely generated subdirectly irreducible algebras and of the ordering between the varieties generated by them. This will allow us to give in 7.3 a precise description of finitely generated subvarieties of linear symmetric Heyting algebras. In 7.4 we will describe the infinitely generated subvarieties of the locally finite subvarieties. Finally in 7.5 we will give an equational bases for each subvariety of locally finite subvarieties.

#### 7.1. Linear Symmetric Heyting Algebras

A. Monteiro comprehensively investigated the variety of symmetric Heyting algebras and several of its subvarieties in his very important work “Sur les algèbres de Heyting symétriques” [21]. Particularly, he studied the subvariety of linear symmetric Heyting algebras, that is, symmetric Heyting algebras satisfying the identity
\[
(x \Rightarrow y) \lor (y \Rightarrow x) = 1.
\]
Linear symmetric Heyting algebras form an equational class \( \mathcal{SH}_L \). In this section we consider the lattice of subvarieties of this variety. We describe the structure of the poset of its join-irreducible elements and we find equational bases for each subvariety of \( \mathcal{SH}_L \). This description is based on that given in [2] and consequently some of the proofs will be omitted.

Linear symmetric Heyting algebras can be characterized by the condition that the poset of filters containing a prime filter is a chain [21].

The importance of the following examples of linear symmetric Heyting algebras will be clear later.

Let \( C_n, n \geq 2 \), be the Heyting algebra of all fractions \( i/n-1 \), \( i = 0, 1, \ldots, n-1 \) ([21], p. 136), with \( \sim x = 1 - x \), and let \( D_n \) be the Heyting algebra \( C_n \times C_n \), with \( \sim (x, y) = (1 - y, 1 - x) \). \( C_n \) and \( D_n \) are linear symmetric Heyting algebras.

An \( I_n \)-algebra is a symmetric Heyting algebra satisfying the Ivo Thomas identity:
\[
\gamma_n(x_0, x_1, \ldots, x_{n-1}) = \beta_{n-2} \Rightarrow (\beta_{n-3} \Rightarrow (\ldots \Rightarrow (\beta_0 \Rightarrow x_0) \ldots)) = 1,
\]
where \( \beta_i = (x_i \Rightarrow x_{i+1}) \Rightarrow x_0 \) for \( i = 0, 1, \ldots, n-2 \) (see [21], p. 136).

The algebras \( C_n \) and \( D_n \) are examples of symmetric Heyting algebras that satisfy the identity \( \gamma_n = 1 \).

The following result is a characterization of the \( I_n \)-algebras.

**Lemma 7.1.** [21] For a linear Heyting algebra \( A \), the following are equivalent:

1. The identity \( \gamma_n = 1 \) holds.
(2) The poset of proper prime filters containing a prime filter $P$ is a chain of length at most $n - 1$.

A. Monteiro proved ([21], p. 138, Th.1.6) that the variety $L_n$ of $I_n$-algebras is generated in $\mathcal{H}_L$ by $D_n \times D_{n-1}$.

It is clear that a finite linear symmetric algebra is an $I_n$-algebra for some $n$. Then we have the following theorem:

**Theorem 7.2.** If $A$ is a finite algebra in $\mathcal{H}_L$, then $A$ is subdirectly irreducible if and only if there exists $n$ such that either $A$ is isomorphic to $D_n$ or $A$ is isomorphic to $C_n$.

Since the variety of linear Heyting algebras is locally finite, then it is easy to see that $\mathcal{H}_L$ is locally finite. In addition, $\mathcal{H}_L$ has the congruence-distributive property, being that the lattice of congruences in an algebra $A$ is a sublattice of the lattice of congruences of the Heyting algebra $A$, and the latter is congruence-distributive.

We conclude this section by recalling the characterization of subalgebras of the algebras $C_n$ and $D_n$.

Let $n \geq 2$. If $n$ is even, then the subalgebras of $C_n$ are the algebras $C_{2k}$, $k \leq n/2$. If $n$ is odd, then $C_k$ is a subalgebra of $C_n$ for every $k \leq n$.

Let $S_Y = C_n - Y$, where $Y \subseteq C_n - \{0, 1\}$. Let $S_{\mathcal{H}}$ be the set of Heyting subalgebras of $C_n$. Then $S_{\mathcal{H}} = \{S_Y : Y \subseteq C_n - \{0, 1\}\}$. For every $j$, $2 \leq j \leq n$, let $Y_j \in S_{\mathcal{H}}$ be such that $|Y| = n - j$. Then $A = S_Y \times S_{\mathcal{H}_{-Y}}$ is a subalgebra of $D_n$ isomorphic to $D_j$. In addition, $D_i \subseteq D_j$ if and only if $i \leq j$. If $A$ is a subalgebra of $D_n$ and $A$ is not isomorphic to $D_k$, for any $k$, then $A \simeq C_t$, for $t \leq n$. We have that $A = \{(x, \alpha(x)), x \in p_1(A)\}$, where $\alpha$ is an isomorphism from $p_1(A)$ onto $p_2(A)$, $p_1$, $p_2$ the projections in $D_n = C_n \times C_n$.

The order in $\mathcal{S}_i(\mathcal{L})$ (and in $\mathcal{S}_i(\mathcal{L})$) is the following:

$$A \leq B \text{ if and only if } A \in \mathcal{S}(B)$$

being that if $A \in \mathcal{S}(\mathcal{L})$ then $A$ is simple, that is, the unique homomorphic images are the trivial ones.

Let $\mathcal{D}_h$ and $\mathcal{C}_n$ denote the varieties generated by $D_n$ and $C_n$, respectively, that is, $\mathcal{D}_h = V(D_n)$ and $\mathcal{C}_n = V(C_n)$, and for a distributive lattice $R$, $\mathcal{J}(R)$ denotes the ordered set of all join-irreducible elements of the distributive lattice $R$.

Let $\mathcal{K} = V(\bigcup_{n \geq 2} C_n)$. This is the variety called by A. Monteiro the variety of totally linear symmetric Heyting algebras.

Let $\mathcal{P} = V(\bigcup_{n \geq 1} C_{2n})$.

It is clear that $\mathcal{P} \subseteq \mathcal{K}$. Furthermore, $\mathcal{P} \neq \mathcal{K}$. Indeed, for $A \in \mathcal{S}_i(\mathcal{L})$, $A$ is isomorphic to $C_{2n}$, and then, for odd $r$, there isn’t $A \in \mathcal{S}_i(\mathcal{L})$, such that $C_r \in \mathcal{S}(A)$. Thus, $\mathcal{P} \subset \mathcal{K}$

Therefore, $\mathcal{J}(\Lambda(\mathcal{P}))$ is the poset indicated in fig. 1.

**Theorem 7.3.** $\mathcal{P}$ and $\mathcal{H}_L$ are join-irreducible in $\Lambda(\mathcal{L})$.
Observe that from the definitions of \( K \), \( P \) and \( \mathcal{H}_L \), it follows that they are not completely join-irreducible. Furthermore, they are not finitely generated. The following theorem can be found in [8].

**Theorem 7.4.** \( K \), \( P \) and \( \mathcal{H}_L \) are the unique join-irreducible varieties that are not finitely generated. Every variety \( V \in \Lambda(\mathcal{H}_L) \) is a join of finitely many varieties in \( \mathcal{J}(\Lambda(\mathcal{H}_L)) \).

Now we will find equational bases for each subvariety of \( \mathcal{H}_L \).

Consider the terms \( \gamma_K(x) = \neg x \Rightarrow \neg \neg x \) and \( \gamma_P(x) = \neg \neg (\neg x \Rightarrow x) \Rightarrow \neg \neg (x \Rightarrow \neg \neg x) \).

In [2] it is proved that

**Theorem 7.5.** [21]. The equation \( \gamma_K(x) = 1 \) characterizes the variety \( K \) within \( \mathcal{H}_L \), and the equation \( \gamma_P(x) = 1 \) characterizes the variety \( P \) within \( K \).

As we pointed out in Section 1, the variety \( I_n \) is generated by \( D_n \times D_{n-1} \). Nevertheless, we proved that \( D_{n-1} \) is a subalgebra of \( D_n \), thus \( I_n \) is the variety generated by \( D_n \), that is, the variety \( I_n \) is the variety \( D_n \), and consequently, the Ivo Thomas identity \( \gamma_n(x_0, \ldots, x_{n-1}) = 1 \) determines the variety \( D_n \) for \( n \geq 2 \).
Theorem 7.7. The equations \( \gamma_n(x_0, \ldots, x_{n-1}) = 1 \) and \( \gamma(x) = 1 \) determine the variety \( C_n \) for \( n \) odd, and the equations \( \gamma_n(x_0, \ldots, x_{n-1}) = 1 \), \( \gamma(x) = 1 \), and \( \gamma_n(x) = 1 \) determine the variety \( C_n \) for \( n \) even.

7.2. The linearity in \( \mathcal{SC} \). The structure of the lattice of subvarieties of the variety of linear closure algebras has been extensively studied by Blok in [6]. In this variety, the image of the closure operator is a linear Heyting algebra, and hence the subdirectly irreducible algebras are those closure algebras such that the image of the operator is a chain with a dual atom. More precisely we have the following theorem proved by Blok in [6].

**Theorem 7.7. (W. Blok)**

(i) The equation

\[
(Q(x) \rightarrow Q(y)) \lor (Q(y) \rightarrow Q(x)) = Q(-Q(x) \lor Q(y)) \lor Q(-Q(y) \lor Q(x)) = 1,
\]

determines the variety \( \mathcal{C}_L \) in \( \mathcal{C} \).

(ii) \( \mathcal{SI}(\mathcal{C}_L) = \{ A \in \mathcal{C} : Q(A) \text{ is a chain with a dual atom} \} \).

(iii) \( \mathcal{C}_L = V(\mathcal{SI}(\mathcal{C}_L)) \).

(iv) \( \mathcal{C}_L^n = V(\{ A \in \mathcal{C} : Q(A) \cong C_n, \text{and } A \text{ is finite} \}) \).

(v) \( \mathcal{C}_L = V(\bigcup_{n \geq 1} \mathcal{C}_L^n) \).

(vi) \( \mathcal{C}_L^1 \subset \mathcal{C}_L^2 \subset \ldots \subset \mathcal{C}_L \).

Observe that the class \( \mathcal{C}_L^1 \) is the class of monadic Boolean algebras and the class \( \mathcal{C}_L^2 \) is the class \( \mathcal{C}_T \) of three-valued closure algebras investigated by M. Abad and J. P. Díaz Varela in [1] and [10]. Also observe that the finite subdirectly irreducible algebras in \( \mathcal{C}_L \) are those closure algebras in which the set of open elements is isomorphic to a chain \( C_n \) for some \( n \). Let \( B_{k_1,k_2,\ldots,k_s} \) denote the subdirectly irreducible closure algebra with \( n = \sum_{i=1}^s k_i \) atoms and \( Q(B_{k_1,k_2,\ldots,k_s}) = C_s = \{ 0 = a_0 < a_1 < \ldots < a_{s-1} < a_s = 1 \} \), where the intervals \([a_i, a_{i+1}]\) have \( k_{i+1} \) atoms for \( i = 0, \ldots, s-1 \), that is, there are \( k_1 \) atoms of \( B_{k_1,k_2,\ldots,k_s} \) preceding \( a_1 \), and there are \( \sum_{j=1}^{s-1} k_j \) atoms preceding \( a_s \). Thus, the finite subdirectly irreducible algebras in \( \mathcal{C}_L \) are the algebras \( B_{k_1,k_2,\ldots,k_s} \). In addition, these algebras generate the variety \( \mathcal{C}_L \), that is, \( \mathcal{C}_L \) is generated by its finite members.

Let us consider the following subvarieties:

\[ V_{k_1,k_2,\ldots,k_s} = V(B_{k_1,k_2,\ldots,k_s}). \]

In order to give a characterization of the ordering for the subvarieties \( V_{k_1,k_2,\ldots,k_s} \), we will consider the set \( \mathbb{N}_f \) of all finite sequences of positive integers. Given \( X = x_1, \ldots, x_s \in \mathbb{N}_f \) we will abbreviate \( V_X \) for \( V_{x_1,\ldots,x_s} \) and \( B_X \) for \( B_{x_1,\ldots,x_s} \). For \( X, Y \in \mathbb{N}_f \), we have that \( V_X \leq V_Y \) if and only if \( B_X \in \text{HS}(B_Y) \). Hence we can define an ordering in \( \mathbb{N}_f \) by means of

\[ X \leq_N Y \iff B_X \in \text{HS}(B_Y). \]

The mapping \( \varphi : \text{SI}_1(\mathcal{C}_L) \rightarrow \mathbb{N}_f \) defined by \( \varphi(B_X) = X \) is clearly an order isomorphism.
The following theorem gives us a useful characterization of the order relation defined on $\mathbb{N}_f$. Let $X, Y \in \mathbb{N}_f$ with $X = x_1, \ldots, x_k$ and $Y = y_1, \ldots, y_l$.

**Theorem 7.8.** (W. Blok) $X \leq_n Y$ if and only if there exist $1 = i_1 < i_2 < \ldots < i_k \leq l$ such that $x_j \leq y_{i_j}$ for $j = 1, 2, \ldots, k$.

In [9] an equation is given for each subvariety of the variety $\mathcal{C}_L^n = V(\{B_X, \text{where } X \text{ is a } k-\text{tuple, } k \leq n\})$

Consider now the variety $\mathcal{C}_L$ of linear closure symmetric algebras. This variety consists of those symmetric closure algebras that satisfy the linearity condition on the set of its open elements, that is,

$$(Q(x) \rightarrow Q(y)) \lor (Q(x) \rightarrow Q(y)) = 1.$$ 

This variety is the symmetric counterpart of linear closure algebras. Observe that if $A \in \mathcal{C}_L$ then $Q(A)$ is a linear symmetric Heyting algebra, so the results of the previous section will be useful.

Since the variety of linear closure algebras is not locally finite then $\mathcal{C}_L$ is not locally finite. Of course, it is congruence-distributive.

The proof of the following theorem is similar to that of its analogous for linear closure algebras (see [6]).

**Theorem 7.9.** Any subvariety $V$ of $\mathcal{C}_L$ is generated by its finite members.

7.3. **Finitely generated subvarieties of $\mathcal{C}_L^n$.** Now we will study the subvarieties of $\mathcal{C}_L^n$, where $\mathcal{C}_L^n$ is the subvariety of $\mathcal{C}_L$ characterized by

$$(CLn) \quad \bigvee_{i=0}^n (Q(x_i) \rightarrow Q(x_{i+1})) = \bigvee_{i=0}^n Q(-Q(x_i) \lor Q(x_{i+1})) = 1.$$ 

Observe that if $A \in \text{Si}_{\text{fin}}(\mathcal{C}_L)$, then $Q(A) \in \text{Si}_{\text{fin}}(\mathcal{C}_L)$ and then $Q(A)$ is isomorphic to a subalgebra of $D_n$, for some $n$. From this we can deduce that $\mathcal{C}_L^n$ is locally finite. Indeed, let $A \in \text{Si}(\mathcal{C}_L^n)$. $A$ finitely generated. Let $g_1, g_2, \ldots, g_m$ be generators of $A$. Then

$$A = [g_1, g_2, \ldots, g_m]_{\mathcal{C}_L} \subseteq [g_1, g_2, \ldots, g_n, Q(A)]_{\mathcal{C}_L}.$$ 

But $|Q(A)| \leq n^2$, so $|A| \leq 2^n$. Hence $A$ is finite. Consequently we can conclude that $\mathcal{C}_L^n$ is locally finite, and thus, generated by its finite members.

In the following we will study the ordering between the finitely generated subvarieties. To this end we need to characterize the simple algebras generating finitely generated join-irreducible subvarieties.

If $Q(A) \cong C_t$, $t \leq n$, then $A \cong \mathfrak{C}_X B_X$ with $X = (x_1, x_2, \ldots, x_t)$. In addition, if $t$ is even, there exists $a \in Q(A)$ such that $a \approx a$. Thus $x_1 = x_2 = x_{t-1}$, and in general, if $i \leq t/2$, $x_i = x_{i-i+1}$. If $t$ is odd then for $i \leq t/2$, we have $x_i = x_{t-i+1}$, as $a \approx (a_{i-1})$, and in general, $\sim (a_{i-1})$ where $Q(A) = \{0 = a_0 < a_1 < \ldots < a_{t-1} = a_t\}$. Observe also that $T = \sim$, when restricted to the atoms of $A$, $T : \text{At}(A) \rightarrow \text{At}(A)$, is a bijection, by means of which the set $A_{x_i} = \text{At}(a_i) \setminus \text{At}(a_{i-1})$ of atoms that precede $a_i$ and do not precede $a_{i-1}$ corresponds to the set $A_{y_i} = \text{At}(-a_{t-i+1}) \setminus \text{At}(-a_{t-1})$. If $t$ is even then the atoms of $A$ are non-invariant by $T$, as $T(A_{x_i}) \cap A_{x_i} = \emptyset$. 

*Actas del IX Congreso Dr. Antonio A. R. Monteiro, 2007*
After these considerations we will analyze the structure of the finite subdirectly irreducible closure algebras of $\mathcal{AC}^n$. Let $A \in \text{Sfin}(\mathcal{AC}^n)$. We consider three cases.

I. $Q(A) \cong D_n$. In this case there exists $a \in B(Q(A)) \setminus \{0,1\}$ such that $A \cong \mathcal{I}^* (a) \times [a]$. Observe in addition that $[a] \cong B_X$ with $X \in \mathcal{N}_f$, $X = (x_1, x_2, \ldots, x_{n-1}, x_n)$, and that $[a] \cong B_{X'}$ with $X' \in \mathcal{N}_f$ and $X' = (x_n, x_{n-1}, \ldots, x_2, x_1)$. Taking $\sim_d (x, y) = (d^{-1}(y), d(x))$, where $d : B_X \to B_{X'}$ is the application such that $x \leq y$ if and only if $d(y) \leq d(x)$, we state that $A \cong \mathcal{I}^* (B_X \times B_{X'}, \sim_d)$. Indeed, it is clear that $d$ is a negation over $B_X \times B_{X'}$. Let us see that every negation that can be defined on $B_X \times B_{X'}$ that maps $a \longrightarrow a$ equals $d$, up to isomorphism.

Let $(B_X \times B_{X'}, \sim_1)$ and $(B_X \times B_{X'}, \sim_2)$, two structures of symmetric closure algebras with the same universe $B_X \times B_{X'}$. Let $T_1 = \sim_1 = - \sim_1$ and $T_2 = \sim_2 = - \sim_2$. Since $T_1(a) = -a$ with $i = 1, 2$ and $a \in B(Q(A)) \setminus \{0,1\}$, it follows that all atoms are non-invariant, that is, both structures are isomorphic as symmetric Boolean algebras, by means of an isomorphism $\alpha$ such that $\alpha T_1 = T_2$. So $\alpha = T_2 T_1^{-1} = T_2 T_1$ as $T_1^2 = I_d$. In addition

$$\alpha'_{Q(A)}(a_i, a_j) = T_1 T_2(a_i, a_j) = \sim_1 \sim_2 (a_i, a_j),$$

that is, $\alpha'_{Q(A)} = I_d'_{Q(A)}$. Hence $\alpha$ is an $\mathcal{I}'$–isomorphism. Thus the structure is unique up to isomorphism. We then denote

$$D^n_X \cong (B_X \times B_{X'}, \sim_d),$$

where $(n + 1)^2 = |Q(A)|$.

II. $Q(A) \cong C_n$, $n$ even. Then there exists $a \in C_n$ such that $\sim a = a$. So every element is non-invariant, and consequently, as in the previous case, there exists a unique structure of symmetric closure algebras, up to isomorphism. Observe that $A \cong \mathcal{I}^* B_X$ and from the action of $\sim$, $x_i = x_{n-i+1}$, for all $i = 1, \ldots, \frac{n}{2}$.

III. $Q(A) \cong C_n$, $n$ odd. As in the previous case, $A \cong \mathcal{I}^* B_X$ and $x_i = x_{n-i+1}$, for all $i = 1, \ldots, \frac{n+1}{2}$. Let $a_0 = 0, a_1, \ldots, a_n$ the open elements of $A$. Observe that if $b \in At(a_{n+1})$, then $T(b) = - b \leq - a_{n+1}$. Since $At(a_{n+1}) \cap At(-a_{n+1}) = \emptyset$, it follows that $T(b) \neq b$. So if $b \in At(a_{n+1})$, $b$ is non-invariant. In the same way it can be proved that if $b \in At(-a_{n+1})$, $b$ is non-invariant. Hence, if $b$ is an invariant atom, $b \in At(a_{n+1}) \setminus At(a_{n+1})$. Thus there can be at most $\frac{n+1}{2}$ invariant atoms in $A$. In addition, if we fix a negation with its invariant atoms and its non-invariant atoms and with the structure of closure algebra of $B_X$, there exists a unique structure of symmetric closure algebra for $A$. So there exist so many structures of closure algebras as negations that can be defined on $B_X$, such that

- $\sim (a_i) = a_{n-i}$,
- $\sim : At(a_{n+1}) \to At(-a_{n+1})$ is a bijection.

Then we have that if $x_{n+1} = 2s + t$ with $2s$ non-invariant atoms and $t$ invariant atoms, then the notation $C^{2s+t}_X$ describes a unique structure of algebra $A$, up to isomorphism.

We will denote $\mathcal{W}^n_X = V(D^n_X)$, $\mathcal{Y}^n_X = V(C_X)$, $\mathcal{Y}^n_{X, 2s+t} = V(C^{2s+t}_X)$. Also $\frac{x}{2} = (x_1, \ldots, x_{\frac{n}{2}})$ if $n$ is odd and $\frac{x}{2} = (x_1, \ldots, x_{\frac{n-1}{2}})$ if $n$ is even.
The following lemmas will be important to obtain the ordering on the algebras $C_X^{2s+t}$.

**Lemma 7.10.** Let $x \in C_X$. Then $x \wedge Q(x) \wedge Q(\sim x) \in [0, a_2^{-1}]$ if $n$ is even and $x \wedge Q(x) \wedge Q(\sim x) \in [0, a_2^{-1}]$ if $n$ is odd.

**Proof.** Let $n$ be even. If $Q(x) = a_2^{-1}$, there is nothing to prove. If $Q(x) > a_2^{-1}$, then $Q(x) \geq a_2^{-1}$, so $Q(\sim x) \leq a_2^{-1}$. Hence $x \wedge Q(x) \wedge Q(\sim x) \leq a_2^{-1}$. The case $n$ odd is similar. □

By the previous lemma, it is immediate that if $x \in C_X$ then $\sim (Q(x) \wedge Q(\sim x)) = \sim Q(x) \vee \sim Q(\sim x) \in [a_2^{-1}, 1]$ if $n$ is even and $\sim (Q(x) \wedge Q(\sim x)) = \sim Q(x) \vee \sim Q(\sim x) \in [a_2^{-1}, 1]$ if $n$ is odd.

**Lemma 7.11.** Let $x \in C_X^n$, with $n$ even. Then

1. $x = Q(x), x \wedge x = x$ and $x \vee (Q(y) \wedge Q(\sim y)) = x$ for all $y \in C_X^n$, if and only if $x = a_2^{-1}$.
2. $x = Q(x), x \vee x = x$ and $x \wedge (\sim Q(y) \vee \sim Q(\sim y)) = x$ for all $y \in C_X^n$, if and only if $x = a_2^{-1}$.

**Proof.** If $x = Q(x)$, then $x$ is open, that is, $x = a_i$. Then $\sim x = a_{n-i}$. Since $x \leq x$ then $i \leq n-i$ and $i \leq \frac{n}{2}$. From the hypothesis, $x \vee (Q(a_2^{-1}) \wedge Q(\sim a_2^{-1})) = x \vee a_2^{-1} = x$. Thus, $x \geq a_2^{-1}$. So $x = a_2^{-1}$. Conversely if $x = a_2^{-1}$, it is easy to check that these properties hold. The rest of the proof is analogous. □

**Lemma 7.12.** Let $x \in C_X^n$, with $n$ odd. If $Q(x) = Q(\sim x)$ and $x \vee (Q(y) \wedge Q(\sim y)) = x$ for all $y \in C_X^n$, then $x \in (a_{n-1}, a_{n+1})$.

**Proof.** $\Leftarrow$ If $x \in (a_{n-1}, a_{n+1})$, it is easy to see that $\sim x < (a_{n-1}, a_{n+1})$. So $Q(x) = Q(\sim x) = a_{n-1}$. In addition, $x > a_{n-1}$, and by the previous lemma $Q(y) \wedge Q(\sim y) \leq a_{n-1}$. Hence $x \vee (Q(y) \wedge Q(\sim y)) = x$.

$\Rightarrow$ If $x \vee (Q(y) \wedge Q(\sim y)) = x$ for all $y \in C_X^n$, then $x \vee (Q(a_{n-1}) \wedge Q(\sim a_{n-1})) = x$, and consequently $x \vee (a_{n-1} \wedge a_{n+1}) = x$ for all $y \in C_X^n$, that is, $x \geq a_{n+1}$. Since $Q(x) = Q(\sim x)$, $x$ is not open. So $x > a_{n-1}$. If $Q(x) \neq a_{n-1}$ then $x > a_{n-1}$ is not open and $\sim x < a_{n+1} = a_{n-1}$. Therefore $Q(\sim x) < a_{n+1} < a_{n-1} \leq Q(x)$. So $Q(x) \neq Q(\sim x)$, a contradiction. Hence $Q(x) = Q(\sim x) = a_{n+1}$. Thus $x > a_{n-1}$ and thus $a_{n+1} < x < a_{n+1}$. From this, $a_{n+1} < x < a_{n+1}$, which concludes the proof. □

After this description of subdirectly irreducible (simple) algebras, we will provide an order for the join-irreducible finitely generated varieties. First observe that $\mathcal{Y}_X^n \not\subseteq \mathcal{Y}_X^m$ for any $X$ and $n$, being that the algebras $C_X$ satisfy the Kleene condition $\gamma_X(Q(x)) = 1$ for its open elements and $D_X^n$ does not. Similarly, $\mathcal{Y}_X^n \not\subseteq \mathcal{Y}_X^m$ for $n-1$ odd and $m-1$ even, since $\mathcal{Y}_X^n$ satisfies the condition $\gamma_X(Q(x)) = 1$ for its open elements and $\mathcal{Y}_X^m$ does not.

The ordering for $J_{fin}(\mathcal{A}(\mathcal{Y}_x \subseteq \mathcal{Y}_x))$ is given by the following theorems.

**Theorem 7.13.** Let $n, m$ be odd. Then $\mathcal{Y}_X^{n, 2s+t} \subseteq \mathcal{Y}_X^{m, 2s+t}$ if and only if $\frac{X}{2} \leq N$, $\frac{X}{2} \leq x$, $x_{n+1} \leq y_{n+1}$, $k \leq s$ and $k+l \leq s+t$. 

*Actas del IX Congreso Dr. Antonio A. R. Monteiro, 2007*
Proof. ⇒ If $\mathcal{Y}_X^{n,2k+l} \preceq \mathcal{Y}_Y^{m,2k+l}$ there exists an $\mathcal{H}$-embedding $i : C_X^{2k+l} \rightarrow C_Y^{2k+l}$. Observe that $i$ is a $\mathcal{H}$-embedding, and in addition $i|Q(C_X^{2k+l})$ is an $\mathcal{H}$-embedding. If $a_0 = 0, a_1, \ldots, a_n$ are the open elements of $C_X^{2k+l}$ and $b_0 = 0, b_1, \ldots, b_m$ are the open elements of $C_Y^{2k+l}$ then

$$i(Q(x) \wedge Q(\sim x)) = Q(i(x)) \wedge Q(\sim (i(x))) \leq b_{n-1}.$$ 

That is, $i[0, a_{n-1}] : [0, a_{n-1}] \rightarrow [0, b_{n-1}]$ is a $\mathcal{H}$-embedding. So $\frac{x^{n}}{2} \leq \frac{y^{n}}{2}$. Also, as $i$ is an $\mathcal{H}$-morphism, the images of the elements satisfying the equations of Lemma 7.12 satisfy the same equations. Then $i((a_{n-1}, a_{n-1})) \leq (b_{n-1}, b_{n-1})$. If $(a_{n-1}, a_{n-1}) = \emptyset$ the case is trivial. Suppose $(a_{n-1}, a_{n-1}) \neq \emptyset$. If $z \in (a_{n-1}, a_{n-1})$ then $Q(z) = a_{n-1}$ and $i(a_{n-1}) = i(Q(z)) = Q(i(z)) = b_{n-1}$. So $i(a_{n-1}) = i(\sim a_{n-1}) = \sim b_{n-1} = b_{n-1}$. Then $i[0, a_{n-1}] : [a_{n-1}, a_{n-1}] \rightarrow [b_{n-1}, b_{n-1}]$ is an $\mathcal{H}$-embedding and, by the previous section, $x_{n-1} \leq y_{n-1}$, $k \leq s$ and $k + l \leq s + t$.

$\Leftarrow$ Let $x \leq y$ and let $P_1$ the partition associated to a $\mathcal{H}$-subalgebra isomorphic to $[0, a_{n-1}]$ in the $\mathcal{H}$-algebra $[0, b_{n-1}]$ (with the $\mathcal{H}$-structure inherited from $C_Y^{2k+l}$). Observe that $T(P_1) = \sim (P_1)$ generates a partition in $T(C_Y^{2k+l}) \setminus T(0, b_{n-1}) = T(0, -b_{n-1})$. If $x_{n-1} \leq y_{n-1}$, $k \leq s$ and $k + l \leq s + t$, then there exists a partition $P_2$ of the atoms of $T((0, b_{n-1}) \setminus T(0, b_{n-1}))$ that generates a $\mathcal{H}$-subalgebra isomorphic to $[a_{n-1}, a_{n-1}]$ in $[b_{n-1}, b_{n-1}]$, (recall that there exists a bijection between the atoms of $[b_{n-1}, b_{n-1}]$, and $T((0, b_{n-1}) \setminus T(0, b_{n-1}))$). Then, it is easy to see that $P = P_1 \cup T(P_1) \cup P_2$ is a partition that generates a symmetric closure subalgebra isomorphic to $C_X^{2k+l}$. □

Theorem 7.14. Let $n, m$ be even. Then $\mathcal{Y}_X^{n} \leq \mathcal{Y}_Y^{m}$ if and only if $\frac{x^{n}}{2} \leq \frac{y^{n}}{2}$.

Consider the term $\gamma_Y^c = [(\gamma_Y \leftrightarrow 1) \lor (T(\gamma_Y) \leftrightarrow 1)] \lor [(\gamma_Y^c \leftrightarrow 1) \lor (T(\gamma_Y^c) \leftrightarrow 1)]$

Lemma 7.15. $D_Y^n$ satisfies the equation $1$, where $\gamma_Y = 1$ is the equation that characterizes $\mathcal{C}_Y$, the variety of linear closure algebras with $Y = (y_1, \ldots, y_r)$. (See [9, 10]).

Proof. Recall that $D_Y^n \equiv \mathcal{C}_Y \times \mathcal{C}_Y^r$. Let $(x_i, y_i)_{i=1}^r \in C_Y \times C_Y^r$, with $r$ as needed. Then

$\gamma_Y((x_1, y_1), \ldots, (x_r, y_r)) = \gamma_Y((x_1, \ldots, x_r), (y_1, \ldots, y_r)) = (1, a)$.

Then $\gamma_Y \leftrightarrow 1 = (1, a) \leftrightarrow 1, 1 = (1, c)$ with $c$ open. In addition

$T(\gamma_Y) \leftrightarrow 1 = T((1, a)) \leftrightarrow 1, 1 = \sim (1, a) \leftrightarrow 1, 1 = \sim (0, -a) \leftrightarrow 1, 1 = (d^{-1}(-a), d(0)) \leftrightarrow 1, 1 = (d', 1) \leftrightarrow 1, 1 = (c', 1)$.

So

$\gamma_Y \leftrightarrow 1 \lor T(\gamma_Y) \leftrightarrow 1 = (1, c) \lor (c', 1) = (1, 1)$.

In a similar way it can be proved that

$\gamma_Y^c \leftrightarrow 1 \lor T(\gamma_Y^c) \leftrightarrow 1 = 1$,

which concludes the proof. □

Theorem 7.16. $\mathcal{Y}_X^{n} \leq \mathcal{Y}_Y^{m}$ if and only if $X \leq \mathcal{N}_r Y$ and $X^* \leq \mathcal{N}_r Y^*$ or $X^* \leq \mathcal{N}_r Y$ and $X \leq \mathcal{N}_r Y^*$.
Proof. \( \Rightarrow \) Let \( i : D_X^y \to D_Y^y \), \( a, -a \in B(D_X^y) \setminus \{0, 1\} \), and \( b, -b \in B(D_X^y) \setminus \{0, 1\} \). Then \( i(a) = b \) and \( i(-a) = -b \) or \( i(a) = -b \) and \( i(-a) = b \). In the first case, \( i[0, a] : [0, a] \to [0, b] \) and \( i[0, -a] : [0, -a] \to [0, -b] \) are \( G \)-embeddings and so \( X \leq Y \) and \( X^* \leq Y^* \). In the other case, \( i[0, a] : [0, a] \to [0, -b] \) and \( i[0, -a] : [0, a] \to [0, b] \) are \( G \)-embeddings and so \( X \leq Y^* \) and \( X^* \leq Y^* \).

\( \Leftarrow \) If \( X \leq Y \) then \( B_X \) is a subalgebra of \( B_Y \). Then there exists a \( G \)-embedding \( i : B_X \to B_Y \). In addition, \( B_X \cong G_d i(B_X) \in S_G(B_Y) \). Thus \( D_X^y \cong G_d i(B_X) \times d i(B_X) \in S_G(D_Y^y) \).

Theorem 7.17. Let \( n \) be odd. Then \( \gamma_X^m \leq \gamma_Y^m \) if and only if \( X \leq N_y Y \) and \( X \leq N_y Y^* \).

Proof. \( \Rightarrow \) By Lemma 7.15, \( C_X^y \) satisfies the equation \( \gamma_X^y = 1 \). Then, as \( Q(C_X^y) \cong C_{ny} \), \( \gamma_Y = 1 \) or \( T(\gamma_Y) = 1 \). But \( TT(\gamma_Y) = T(1) = 1 \). Hence \( \gamma_Y = 1 \) in \( C_X^y \), and consequently \( X \leq N_y Y \).

The other part of the equation proves that \( X \leq Y^* \).

\( \Leftarrow \) Suppose that \( X \leq N_y Y \) and \( X \leq N_y Y^* \). Recall that \( X = X^* \). Then \( X^* \leq N_y Y^* \). Let \( A \cong G \)

\( C_m \times C_m \sim \) with \( (x, y) = (\sim y, \sim x) \). Then \( A \cong G \)

\( D_X^y \), and by the previous theorem, \( A \in S_G \). The \( \gamma^* \) subalgebra of \( A, D = \{ (x, x) \in A \} \) is isomorphic to \( C_m^y \), and hence \( C_m^y \in S_G \).

Observe that the proof of the previous theorem does not make use of the fact that \( n \) is even, so the theorem is also true for \( n \) odd. So we have:

Theorem 7.18. Let \( n \) be odd. Then \( \gamma_X^m \leq \gamma_Y^m \) if and only if \( X \leq N_y Y \) and \( X \leq N_y Y^* \).

Lemma 7.19. Let \( n \) be even. Then \( \gamma_X^{1,2} \leq \gamma_Y^m \) if and only if \( 2s + t \leq y_1 \).

Theorem 7.20. Let \( n \) be odd and \( m \) be even and let \( X^* + 1 = (x_1, x_2, \ldots, x_{n-1}, x_{n+1}) \). Then \( \gamma_X^m \leq \gamma_Y^m \) if and only if \( X^* + 1 \leq N_y Y^* \).

Proof. \( \Rightarrow \) If \( \gamma_X^m \leq \gamma_Y^m \) then we have an embedding \( i : C_X^y \to C_Y^y \). Since in 7.13, if \( a_0 = 0, a_1, \ldots, a_n \) are the open elements of \( C_X^y \) and \( b_0 = 0, b_1, \ldots, b_m \) are the open elements of \( C_Y^y \), then

\[
i(Q(x) \land Q(\sim x)) = Q(i(x)) \land Q(\sim (i(x))) \leq b_y,\]

that is, \( i[0, a_{n+1}] : [0, a_{n+1}] \to [0, b_{n+1}] \) is a \( G \)-embedding. Then \( X^* + 1 \leq N_y Y^* \). In addition, \( i(a_{n+1}) = b_h \). Observe that \( h \neq m \), as \( b_{n+1} = b_{n+1} \), and this does not occur with \( a_{n+1} \) and \( h \neq m \) since otherwise \( b_{n+1} \leq b_h \) and \( i(a_{n+1}) < i(a_{n+1}) \), a contradiction. So \( h < m \). Then \( i[0, a_{n+1}] : [0, a_{n+1}] \to [b_h, b_{n+1}] \) is a \( G \)-embedding, and by the previous lemma, \( x_{n+1} \leq y_{n+1}, h \leq m - 1 \). Thus \( X^* + 1 \leq N_y Y^* \).

The converse is similar to Theorem 7.13.

Then we have given the ordering for each subvariety of \( J_{fin}(A, \mathcal{G}) \).

7.4. Infinite Subvarieties of \( \mathcal{G}^m \). We recall the ordering for infinite subvarieties of linear closure algebras. These results can be seen in [9].

Consider the following \( s \)-tuples

\[
\mathcal{X} = (x_1, x_2, \ldots, x_s),
\]

Actas del IX Congreso Dr. Antonio A. R. Monteiro, 2007
where there exists a (nonempty) subset of indexes \( i_1 < i_2 < \ldots < i_t \leq s \) such that \( x_{i_j} = \ast \) and another (possibly empty) \( h_1 < h_2 < \ldots < h_t \leq s \) such that \( x_{h_i} = k_{h_i} \), with \( k_{h_i} \in \mathbb{N} \), \( t^* + t = s \).

By way of example, for \( s = 5 \), \( X = (3, \ast, \ast, 2, \ast) \) is a 5—tuple.

Let \( \chi_s \) denote the set of all such \( s \)—tuples, and consider the following subvarieties:

\[
U_s^X = V \left( \{ B_{r_1, \ldots, r_t} : r_j \geq 1, \; j = 1, \ldots, t^*, \; r_{h_j} = k_{h_j}, \; l = 1, \ldots, t \} \right).
\]

Observe that \( U_2^{(\ast, \ast)} = \mathcal{E}_L^2 = \mathcal{C}_T \), and in general, \( \mathcal{E}_L^n = U_n^X \), where \( X = (\ast, \ast, \ldots, \ast) \) (odd case).

The following propositions prove that these varieties are join irreducible and they are the unique non finitely generated join-irreducible subvarieties in \( \mathcal{E}_L^n \).

**Proposition 7.21.** ([9]) The varieties of the form \( U_s^X \), with \( s \leq n \), are join-irreducible in \( \Lambda(\mathcal{E}_L^n) \).

Let \( \chi_s^d \) denote those \( t \)—tuples \( X \) of the set \( \chi_t \), satisfying the condition \( x_i = x_{t-i+1} \). Observe that this only holds for the even case or the odd case with \( x_{\frac{t+1}{2}} = \ast \). It remains the case \( x_{\frac{t+1}{2}} = \ast \) and \( x_{\frac{t+1}{2}} = 2s + t \) (odd case).

Recall that the ordering in \( \chi_s \), (see [9]) is \( Y_s \leq X_t \) if and only if

- \( s \leq t \)
- There exist \( i = i_1 < i_2 < \ldots < i_k \leq l \) such that \( y_j \leq x_{i_j} \) for \( j = 1, 2, \ldots, k \), where \( y_j \) is a positive integer and \( x_{i_j} \leq x_{i_j} \),
- \( y_j \) is a positive integer and \( x_{i_j} = \ast \),
- \( y_j = \ast \) and \( x_{i_j} = \ast \),

(observe that if \( x_{i_j} \) is a positive integer and \( y_i = \ast \), then \( y_j \leq x_{i_j} \)).

For the symmetric case we define an order in \( \chi_s \), in the following way.

- For \( s, t \) even, \( \frac{Y_s^+}{2} = y_1, \ldots, y_{\frac{s}{2}} \), and \( \frac{X_t^+}{2} = x_1, \ldots, x_{\frac{t}{2}} \). Then we put \( \frac{Y_s}{2} \leq \frac{X_t}{2} \) if and only if

\[
\frac{Y_s^+}{2} \leq \frac{X_t^+}{2}.
\]

- For the case \( t, s \) odd, we have new \( n \)—tuples, those that represent the non-invariant elements of \( Y_{\frac{s+1}{2}} \) (see the previous section). Let us put \( \frac{Y_s^+}{2} = y_1, \ldots, y_{\frac{s+1}{2}} \), and \( \frac{X_t^+}{2} = x_1, \ldots, x_{\frac{t+1}{2}} \). \( Y_{\frac{s+1}{2}} \) can take any of these three values: \( \ast \), \( 2s + t \) and \( \ast h \) wih is the one that corresponds to the monadic variety \( S_h \). Then the ordering is \( \frac{Y_s}{2} \leq \frac{X_t}{2} \) if and only if

\[
\frac{Y_s^+}{2} \leq \frac{X_t^+}{2},
\]

and

- \( y_{\frac{s+1}{2}} = \ast \leq x_{\frac{s+1}{2}} = \ast \),
- \( y_{\frac{s+1}{2}} = \ast_h \leq x_{\frac{s+1}{2}} = \ast_k \) or \( x_{\frac{s+1}{2}} = \ast \), if \( h \leq k \),
- \( y_{\frac{s+1}{2}} = 2h + l \leq x_{\frac{s+1}{2}} = \ast_k \) or \( x_{\frac{s+1}{2}} = \ast \), if \( h \leq k \),
- \( y_{\frac{s+1}{2}} = 2h + l \leq x_{\frac{s+1}{2}} = 2k + r \), if \( h \leq k, h + l \leq k + r \).

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*Actas del IX Congreso Dr. Antonio A. R. Monteiro, 2007*
• The case $s$ odd, $t$ even, $\frac{1}{t} = y_1, \ldots, y_{\frac{t+1}{2}}, y_{\frac{t+1}{2}}$ and $\frac{1}{2} + 1 = x_1, \ldots, x_{\frac{t+1}{2}}.$ Observe that $* \leq *$. Then the ordering is $\frac{1}{t} \leq x_i$, if and only if $\frac{1}{t} + 1 \leq \frac{X_i}{2}$.

To simplify the notation when introducing the irreducible infinite varieties, we order the $n-$tuples $N_i$ of $\mathcal{V}$ by means of $X_i \leq Y_i$, if and only if there exist $x_i < i_2 < \ldots < i_k \leq \frac{t}{2}$ such that $x_j \leq y_j$, for $j = 1, 2, \ldots, \frac{t}{2}.$ where $x_j \leq y_j$, if $x_j$ is a positive integer and $y_j$ is equal to $*$ or a greater positive integer, and for the case $s, t$ odd we put $x_{\frac{t+1}{2}} \leq y_{\frac{t+1}{2}}$ with the ordering:

- $x_{\frac{t+1}{2}}$ a positive integer and $y_{\frac{t+1}{2}} = *$.
- $x_{\frac{t+1}{2}} = 2h + l \leq y_{\frac{t+1}{2}} = *$, if $h \leq k$.
- $x_{\frac{t+1}{2}} = 2h + l \leq y_{\frac{t+1}{2}} = 2k + r$, if $h \leq k, h + l \leq k + r.$

We denote the ordering in $\mathcal{V}$ by $\leq *$. In a similar way as for linear closure algebras we construct the following subvarieties: If $t$ is even

$$P_t^X = V \left( \bigcup_{Y \leq X} C_Y \right), \quad P_t^Y = V \left( \bigcup_{s \leq t, t=\text{odd}} C_Y \right)$$

and is $t$ is odd

$$K_t^X = V \left( \bigcup_{Y \leq X} C_Y \right), \quad K_t = V \left( \bigcup_{s \leq t, t=\text{odd}} C_Y \right).$$

For the case of algebras in which their open elements isomorphic to $D_n$ for some $n$, we define

$$I_t^X = V \left( \bigcup_{Y \leq X} D_Y \right), \quad I_t = V \left( \bigcup_{s \leq t} D_Y \right).$$

Observe that $I_t = \mathcal{V}_t$.

>From the results of the previous section and the beginning of this section, and making use of the ordering for finite join-irreducible varieties, the following theorem can be proved.

**Theorem 7.22.** The varieties $P_t^X, P_t, K_t^X, K_t, I_t^X, I_t$ are the unique infinite join-irreducible varieties in $\Lambda(\mathcal{V}_t)$.

**Theorem 7.23.** Every variety is a finite join of join-irreducible varieties in $\Lambda(\mathcal{V}_t)$.

### 7.5. Equational Bases in $\mathcal{V}_t$.

Our purpose is to determine equations characterizing each join-irreducible variety, and then, equations characterizing a finite join of join-irreducible varieties. By Theorem 7.23 we will have equations for every subvariety in $\Lambda(\mathcal{V}_t)$.

Observe that if $x \neq y, x, y \in A$ and $x, y \in Q(A) \cong C_n$, then, if $x = a_i, y = a_j$ and $x \leq y$, we have that

$$x \leftrightarrow y = Q(-x \lor y) \land Q(-x \lor y) = (a_i \rightarrow a_j) \land (a_j \rightarrow a_i) = 1 \land a_i = a_i \land a_j = a_j.$$

This simple observation will be useful for the proof of the following theorems.

*Actas del IX Congreso Dr. Antonio A. R. Monteiro, 2007*
Consider the following terms:

\[ A_1(z_0, z_1, \ldots, z_{n+1}) = \bigwedge_{i=1}^{n+1} Q(z_i) \rightarrow Q(z_i), \]

\[ A_2(z_0, z_1, \ldots, z_{n+1}) = \bigwedge_{i=1}^{n+1} z_i \lor (Q(z_0) \land Q(z_0)) \rightarrow z_i, \]

\[ A_3(z_0, z_1, \ldots, z_{n+1}) = \bigwedge_{i \neq j, i=1}^{n+1} Q(z_i \land z_j) \rightarrow (z_i \land z_j), \]

\[ A_4(z_0, z_1, \ldots, z_{n+1}) = Q \left( \bigvee_{i=1}^{n+1} z_i \right) \rightarrow \left( \bigvee_{i=1}^{n+1} z_i \right), \]

and

\[ \gamma_{\mathcal{K}_{n+1}}(z_0, z_1, \ldots, z_{n+1}) = (A_1 \land A_2 \land A_3) \rightarrow A_4 \]

**Theorem 7.24.** \( C_n^{m,2k+1} \) satisfies the equation. \( \gamma_{\mathcal{K}_{n+1}}(z_0, z_1, \ldots, z_{n+1}) = 1. \)

**Proof.** We have the following cases:

1. Suppose that \( A_1 \neq 1 \). Then there exists \( z_i \) such that \( Q(z_i) \neq Q(z_i) \). Then \( Q(z_i) \rightarrow Q(z_i) \leq Q(z_i) \land Q(z_i) \leq A_{n+1}. \) So \( \bigwedge_{i=1}^{n+1} A_i \leq (Q(z_i) \land Q(z_i)) \leq Q(z_i) \leq Q(\bigvee_{i=1}^{n+1} z_i) \leq A_4 \), and consequently \( \gamma_{\mathcal{K}_{n+1}} = (\bigwedge_{i=1}^{n+1} A_i) \rightarrow A_4 = 1. \)

2. Suppose that \( A_2 \neq 1 \). Then there exist \( z_i \) and \( z_0 \) such that \( z_i \lor (Q(z_0) \land Q(z_0)) \neq z_i. \) Then \( A_2 \leq Q(z_i) \), and arguing as in the previous case, we have that \( \gamma_{\mathcal{K}_{n+1}} = 1. \)

3. Suppose that \( A_3 \neq 1 \). Then there exist \( z_i \) and \( z_j \) such that \( Q(z_i \land z_j) \neq (z_i \land z_j). \) Thus \( A_3 \leq Q(z_i \land z_j) \) and so \( A_1 \land A_2 \land A_3 \leq A_4 \), and \( \gamma_{\mathcal{K}_{n+1}} = 1. \)

4. Suppose that \( A_1 = A_2 = 1 \). Then, by Lemma 7.11 we have that \( z_i \in (a_{n+1}, a_{n+1}) \) and if \( A_3 = 1 \) then the elements \( z_i \) are distinct atoms (necessarily all the atoms) of \( [a_{n+1}, a_{n+1}]. \) Then

\[ Q(\bigvee_{i=1}^{n+1} z_i) = (\bigvee_{i=1}^{n+1} z_i), \]

and hence \( \gamma_{\mathcal{K}_{n+1}} = 1. \)

**Theorem 7.25.** Let \( m \) be odd. Then \( C_n^{m,2k+1} \) satisfies \( \gamma_{\mathcal{K}_{n+1}} = 1 \) if and only if \( y_{m+1} \leq x_{n+1} \).

**Proof.** \( \Rightarrow \) Suppose that \( y_{m+1} > x_{n+1} \). Let \( b_1, \ldots, b_m \) the open elements of \( C_n^{m,2k+1} \) and \( z_1, \ldots, z_{n+1} \), the atoms of \( [b_{m+1}, b_{m+1}]. \) Let \( \{z_1, \ldots, z_{n+1}\} \) be a proper subset of atoms of \( [b_{m+1}, b_{m+1}] \) and \( z_0 = b_{m+1}. \) Then it is easy to check that \( A_1 = A_2 = A_3 = 1. \) But \( Q(\bigvee_{i=1}^{n+1} z_i) = b_{m+1} \neq (\bigvee_{i=1}^{n+1} z_i), \) and thus \( A_4 \neq 1. \) So \( \gamma_{\mathcal{K}_{n+1}} \neq 1. \) A contradiction.

\( \Leftarrow \) We have already seen in the proof of the previous theorem that if \( A_1, A_2, A_3 \) are distinct

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*Actas del IX Congreso Dr. Antonio A. R. Monteiro, 2007*
from 1, then the equation holds. Observe that if \( y_{\frac{m-1}{2}} < x_{\frac{m-1}{2}} \) then \( A_1 \) or \( A_2 \) or \( A_3 \) are distinct from 1, and consequently the equation holds. If \( y_{\frac{m-1}{2}} = x_{\frac{m-1}{2}} \) and there exist \( z_1, \ldots, z_{\frac{m-1}{2}} \), such that \( A_1, A_2, A_3 = 1 \), then \( z_1, \ldots, z_{\frac{m-1}{2}} \) are the atoms of \([b_{\frac{m-1}{2}}, b_{\frac{m+1}{2}}] \). Then \( A_4 = 1 \) and the equation holds.

As we have already proved, the elements of the form \( x \land A(x_0) \), with \( A(x_0) = Q(x_0) \land Q(\sim x_0) \in C_{X}^n \) belong to the interval \([0, \frac{a_{\frac{n-1}{2}}}{2}] \) in the odd case, and to \([0, \frac{a_{\frac{n+1}{2}}}{2}] \) in the even case. We can algebraize the interval \([0, A(x_0)] \) with \( E \)-terms and we can put \( A = \langle [0, A(x_0)], \land, \lor, \neg, Q, 1_A, 0_A \rangle \), where the \( A \)-operations are defined: \( \land = \land, \lor = \lor, \neg = \neg \), \( Q = Q, 1_A = A(x_0), 0_A = 0 \). With these operations it is easy to see that \([0, A(x_0)] \) is a closure algebra. Observe that \([0, A(x_0)] = [0, a_i] \) with \( i \leq \frac{n}{2} \) or \( i \leq \frac{n-1}{2} \), and \([0, A(a_{\frac{n-1}{2}})] = [0, a_{\frac{n-1}{2}}] \), for the odd case and \([0, A(a_{\frac{n+1}{2}})] = [0, a_{\frac{n+1}{2}}] \), for the even case. So \([0, A(x_0)] \in S(B_{\frac{2}{2}}) \) for every \( x_0 \). Thus \([0, A(x_0)] \) satisfies the equation

\[
\gamma^{A(x_0)}_{x} (x_0, x_1, \ldots, x_r) = 1,
\]

where \( \gamma^{A(x_0)}_{x} \) is the equation \( \gamma_{x}^{+} \) that characterizes the subvariety of closure algebras \( C_{t}^n \) (ver [Tesis][UMA]), where the \( E \)-operations are replaced by the \( A \)-operations. Observe that if \( C_{Y}^{m, 2k+1} \) satisfies this equation, then \([0, b_{\frac{m-1}{2}}] \), satisfies \( \gamma_{x}^{+} \), hence \( \frac{n}{2} \leq \frac{n}{2} \). So we have proved the following theorem.

**Theorem 7.26.** \( C_{Y}^{m, 2k+1} \) satisfies the equation \( \gamma^{A(x_0)}_{x} = 1 \). Moreover if \( C_{Y}^{m, 2k+1} \) satisfies this equation then \( \gamma^{A(x_0)}_{x} \leq \gamma_{x}^{+} \).

As we have already seen in Lemma 7.12 the elements \( x \in C_{X}^{2k+1, n} \) that satisfy \( Q(x) = Q(\sim x) \) and \( x \lor (Q(y) \land Q(\sim y)) = x \) for all \( y \in C_{X}^{2k+1, n} \), are those that belong to \( (a_{\frac{n-1}{2}}, a_{\frac{n+1}{2}}) \). In addition, \( Q(x) = a_{\frac{n-1}{2}} \) and \( \sim Q(x) = a_{\frac{n+1}{2}} \). In this way we can define, for every \( x \) satisfying this equation, in the interval \( M = [Q(x), \sim Q(x)] \) a structure of symmetric closure algebra if we write \((M, \land, \lor, \neg, Q, 1_M, 0_M) \) where the operations are defined by \( \land _M = \land, \lor _M = \lor, \neg _M = \neg, 1_M = Q(x), 0_M = Q(x) \), and \( -M(y) = (\sim y \land \sim Q(x)) \) \( Q(x) = (T(y) \land \sim Q(x)) \lor Q(x) \). Observe that \( c \) is an invariant atom of \( C_{X}^{2k+1, n} \) if and only if \( c \lor a_{\frac{n-1}{2}} \) is an \( T_M \)-invariant atom of \( M \). Indeed, if \( c \) is \( T \)-invariant then

\[
T_M(c \lor a_{\frac{n-1}{2}}) = (T(c \lor a_{\frac{n-1}{2}}) \lor Q(x)) \lor Q(x) = ((c \lor a_{\frac{n-1}{2}}) \lor a_{\frac{n+1}{2}}) \lor a_{\frac{n-1}{2}} = (c \lor a_{\frac{n-1}{2}}) \lor a_{\frac{n+1}{2}} = c \lor a_{\frac{n-1}{2}}.
\]

Conversely if \( c \lor a_{\frac{n-1}{2}} \) is \( T_M \)-invariant then

\[
c \lor a_{\frac{n-1}{2}} = T_M(c \lor a_{\frac{n-1}{2}}) = (T(c) \lor a_{\frac{n+1}{2}}) \lor a_{\frac{n-1}{2}} = T(c) \lor a_{\frac{n+1}{2}}
\]

and \( T(c) \land \sim a_{\frac{n-1}{2}} = c \). As \( T(c) \) is an atom, then \( T(c) = c \). Hence \((M, T_M)\) is a symmetric monadic algebra isomorphic to \( C_{2, t} \) where \( 2s \) is the number of non–invariant atoms and \( t \) is the number of invariant atoms of \( M \).

By these remarks and the results obtained for monadic algebras we have the following theorem.

*Actas del IX Congreso Dr. Antonio A. R. Monteiro, 2007*
Theorem 7.27. $C_{X}^{2^{i}+t}$ satisfies the equation

$$\beta \beta_{2t}(z_{0}, z_{1}, \ldots, z_{2^n+1}) =$$

$$\left[ \bigwedge_{i=1}^{2^{s}} (Q(\sim z_{i}) \leftrightarrow Q(z_{i})) \right] \land \left[ \bigwedge_{i=1}^{2^{s}} (z_{i} \lor (Q(z_{0}) \land Q(\sim z_{0}))) \leftrightarrow z_{i} \right] \land$$

$$\left[ \bigwedge_{j=1}^{s} (Q(z_{i} \land z_{j}) \leftrightarrow (z_{i} \land z_{j})) \right] \land \left[ \bigvee_{j=1}^{s} (T_{M(z_{i})}(z_{2i-1}) \leftrightarrow z_{2i}) \right] \rightarrow$$

$$\left[ \bigvee_{i=1}^{s} T_{M(z_{i})} \left( \bigvee_{i=1}^{2^{s}+1} z_{i} \right) \rightarrow \left( \bigvee_{i=1}^{2^{s}+1} z_{i} \right) \right] = 1.$$

Proof: 1. Suppose that $A_{1} \neq 1$. Then, as in the proof of Theorem 7.24, there exists $z_{i}$ such that $\bigwedge_{i=1}^{4} A_{i} \leq Q(z_{i})$. But $T_{M(z_{i})}(\bigvee_{i=1}^{2^{s}+1} z_{i}) = (\bigvee_{i=1}^{2^{s}+1} z_{i}) \lor (Q(z_{i}) \lor (Q(z_{i}) \geq Q(z_{i})))$, that is, $\bigwedge_{i=1}^{4} A_{i} \leq Q(z_{i}) \leq A_{s}$. So $\beta \beta_{2t} = 1$.

2. If $A_{2} \neq 1$ or $A_{3} \neq 1$ the proof is similar.

3. Suppose that $A_{1} = A_{2} = A_{3} = 1$. Then $z_{i} \in (a_{s+1}, a_{s+2})$, and thus $T_{M(z_{i})} = T_{M}$. Suppose that there exists $z_{2i-1}$ such that $T_{M(z_{i})}(z_{2i-1}) \neq z_{2i}$. Then $T_{M(z_{i})}(z_{2i-1}) \leftrightarrow z_{2i}$ is an open element less than $Q(z_{2i})$, and arguing as in the first item, $\bigwedge_{i=1}^{4} A_{i} \leq Q(z_{2i}) \leq A_{s}$. Hence $\beta \beta_{2t} = 1$.

4. In the case that $\bigwedge_{i=1}^{4} A_{i} = 1$, the elements $z_{i}$ for $i = 1, \ldots, 2s$ are all distinct and non-invariant. Thus, as for monadic algebras, we have that $\bigvee_{i=1}^{s} T_{M(z_{i})}(\bigvee_{i=1}^{2^{s}+1} z_{i}) = (\bigvee_{i=1}^{2^{s}+1} z_{i})$, and consequently $A_{5} = 1$. Then $\beta \beta_{2t} = 1$.

Theorem 7.28. $C_{X}^{2^{s}+t}$ satisfies the equation

$$\beta \beta_{2t}(z_{0}, z_{1}, \ldots, z_{s+t}) =$$

$$\left( \bigwedge_{i=1}^{s+t} (Q(\sim z_{i}) \leftrightarrow Q(z_{i})) \right) \land \left( \bigwedge_{i=1}^{s+t} (z_{i} \lor (Q(z_{0}) \land Q(\sim z_{0}))) \leftrightarrow z_{i} \right) \land$$

$$\left( \bigwedge_{i \neq j, j=1}^{s+t} (Q(z_{i} \land z_{j}) \leftrightarrow (z_{i} \land z_{j})) \right) \land \left( \bigvee_{j=1}^{s+t} (T_{M(z_{i})}(z_{i}) \leftrightarrow z_{i}) \right) \rightarrow$$

$$\left[ \bigvee_{i=1}^{s+t} z_{i} \rightarrow \left( \bigvee_{i=1}^{s+t} Q(z_{i}) \right) \right] = 1.$$

As a consequence of the previous theorems we have the following.
Theorem 7.29. For $n$ odd, the following equations form an equational basis for $V_{X,2k+1}$:
\[ \gamma_k(Q(x_0)) \land \gamma_r(Q(x_1)) = 1, \beta_2 \beta_3 = 1, \beta_3 \beta_4 = 1, \gamma_{K_{2k+1}} = 1, \gamma^{A(x_0)}_{\frac{3}{2}} = 1. \]

Theorem 7.30. For $n$ even, the following equations form an equational basis for $V_{X}^1$:
\[ \gamma_k(Q(x_0)) = 1, \gamma^{A(x_0)}_{\frac{3}{2}} = 1. \]

Proof. We know that $C^0_X$ satisfies the two first equations. Let $x_0 \in C^0_X$ and suppose that $x_0 = x_0 \lor (Q(y) \land Q(y))$. Then $x_0 \geq a_2$ and $Q(x_0) \leq x_0 \leq a_2 \leq Q(x_0)$. Then if $Q(x_0) = Q(x_0)$, we have that $x_0 = x_0 = Q(x_0) = Q(x_0) = a_2$. So the third equation holds. If $A$ is subdirectly irreducible and satisfies these equations then $Q(A)$ is a chain, $A \cong C^0_X$ and $X^+ \leq X^+$. If $m$ is even, then if the third equation holds, it holds for $x_0 = b_{m-1}$. So $\gamma^{Q(x_0)}_{\frac{3}{2}} = 1$, which implies that $X^+ + 1 \leq X^+_2$, and from the previous section, $C^0_X \in S(C^0_X)$.

Observe that all these varieties are within $C^0_X$ and consequently they satisfy equational characterizing this variety.

**Theorem 7.31.** The variety $V_{X}^1$ is characterized by the following equation
\[ \gamma_p = \left( (\gamma \dashv \gamma \dashv 1) \lor (T(\gamma) \dashv 1) \right) \land \left( (\gamma_r \dashv 1) \lor (T(\gamma_r) \dashv 1) \right) = 1. \]

Now we give an equational basis for each infinitely generated join irreducible subvariety. As for the finitely generated varieties, we use the equations of the infinitely generated varieties in $C^0_X$ (ver [9]).

Observe that in the odd case, we can consider three different types of varieties $P^X_h$, according to the value of $x_{\frac{3}{2}}$:

1. $P^X_h$, where $x_{\frac{3}{2}} = *$, that is, if $C_X \in P^X_h$, then $[a_{2k+1}, a_{2k+1}] \in S_h$, where $S_h$ is the variety introduced in [9].

2. $P^{X,2k}_{2h+t}$, where $x_{\frac{3}{2}} = 2h + t$, that is, if $C_X \in P^{X,2k}_{2h+t}$, then $[a_{2k+1}, a_{2k+1}] \in S(B_{2k+t})$.

3. $P^{X,*}$, where $x_{\frac{3}{2}} = *$, that is, if $C_X \in P^{X,h}$, then $[a_{2k+1}, a_{2k+1}] \in S$. $S$ is the variety introduced in [9].

In the following theorem, we give an equational characterization of these varieties.

**Theorem 7.32.** (1) An equational base for the variety $P_h$ is:
\[ \gamma_k(Q(x_0)) \land \gamma_r(Q(x_1)) = 1, \gamma_k(Q(x_1), \ldots, Q(x_n)) = 1. \]

(2) An equational base for the variety $P^{X,*}$ is:
\[ \gamma_k(Q(x_0)) \land \gamma_r(Q(x_1)) = 1, \gamma_k(Q(x_1), \ldots, Q(x_n)) = 1, \gamma^{A(x_0)}_{\frac{3}{2}} = 1. \]

*Actas del IX Congreso Dr. Antonio A. R. Monteiro, 2007*
(3) An equational base for the variety $P_n^X$ is:
$$\gamma_k(Q(x_0)) \land \gamma_l(Q(x_1)) = 1, \gamma_l(Q(x_1), \ldots, Q(x_s)) = 1, \frac{\gamma^{A(x_0)}}{X} = 1, \beta_2^n = 1.$$

(4) An equational base for the variety $P_n^{2h+2}$ is: $\gamma_k(Q(x_0)) \land \gamma_l(Q(x_1), \ldots, Q(x_s)) = 1, \frac{\gamma^{A(x_0)}}{X} = 1, \beta_2^n = 1, \beta_{h+2} = 1, \gamma^{A(x_0)}_{X} = 1.$

(5) An equational base for the variety $K_n$ is:
$$\gamma_k(Q(x_0)) = 1, \gamma_l(Q(x_1), \ldots, Q(x_s)) = 1.$$

(6) An equational base for the variety $P_n^X$ is:
$$\gamma_k(Q(x_0))(Q(x_1)) = 1, \gamma_l(Q(x_1), \ldots, Q(x_s)) = 1, \frac{\gamma^{A(x_0)}}{X} = 1, \gamma^{A(x_0)}_{X} = 1.$$

(7) An equational base for the variety $I_n$ is:
$$\gamma_l(Q(x_1), \ldots, Q(x_s)) = 1.$$

As in the case of symmetric Heyting algebras the following theorem can be proved for join reducible varieties.

**Theorem 7.33.** Let $V \in \Lambda(\mathcal{E}^L_L), V = \bigvee_{i=1}^n V_i$ with $V_i \in J(\Lambda(\mathcal{E}^L_L))$. Then
$$\gamma_l(x_1, \ldots, x_{n_v}) = \bigvee_{i=1}^n \left( \gamma_l(x_1^i, \ldots, x_{n_v}^i) \leftrightarrow 1 \right) \land \left( \sim \gamma_l(x_1, \ldots, x_{n_v}) \leftrightarrow 0 \right) = 1$$
is an equational basis for $V$ en $\mathcal{E}^L_L$.

**References**


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