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On the poset product representation of BL-algebras

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The variety $\mathscr{BL}$ of BL-algebras is the algebraic counterpart of BL, the logic presented by Hájek which includes a fragment common to the most important fuzzy logics (Łukasiewicz, Gödel and product logics). Likewise any variety, $\mathscr{BL}$ is generated by its totally ordered members, namely BL-chains. Since every BL-algebra can be embedded into the direct product of BL-chains and every BL-chain can be decomposed as an ordinal sum of simpler structures, Jipsen and Montagna proposed in [3] a construction called poset product as a sort of generalization of direct product and ordinal sum.

In [2], based on the results of [4], it is shown that every BL-algebra is a subalgebra of a poset product of MV-chains and product chains with respect to a poset which is a forest. Although this embedding provides a representation for finite BL-algebras, some limitations arise from the infinite case. Moreover, in [1] there are easy examples of BL-chains are not representable in the sense of poset product.

The aim of this communication is to examine some features of the poset product construction in the context of $\mathscr{BL}$. We will first consider the restrictions referred above in order to introduce the notion of idempotent free BL-chain. Then, we will suggest some requirements that a BL-algebra should satisfy so that it admit a representation as a poset product of idempotent free BL-chains.

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Forest Products of MTL-chains

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A *semihoop* is an algebra $\mathbf{A} = (A, \cdot, \to, \wedge, \vee, 1)$ of type $(2,2,2,2,0)$ such that $(A, \wedge, \vee)$ is lattice with 1 as greatest element, $(A, \cdot, 1)$ is a commutative monoid and for every $x, y, z \in A$: *(i)* $xy \leq z$ if and only if $x \leq y \to z$, and *(ii)* $(x \to y) \vee (y \to x) = 1$. A semihoop $\mathbf{A}$ is *bounded* if $(A, \wedge, \vee, 1)$ has a least element 0. A *MTL-algebra* is a bounded semihoop. Hence, MTL-algebras are prelinear integral bounded commutative residuated lattices (c.f. [2]). A MTL-algebra $\mathbf{A}$ is a *MTL chain* if its semihoop reduct is totally ordered. A *forest* is a poset X such that for every $a \in X$ the set

$$\downarrow a = \{x \in X \mid x \leq a\}$$

is a totally ordered subset of X.

Inspired by the concept of poset products of BL-algebras (c.f. [1]) we define the *forest product* of MTL-chains in this manner:

Definition 1. Let $\mathbf{F} = (F, \leq)$ a forest and let $\{\mathbf{M}_i\}_{i \in \mathbf{F}}$ a collection of MTL-chains such that, up to isomorphism, all they share the same neutral element 1. If $(\bigcup_{i \in \mathbf{F}})^F$ denotes the set of functions $h : F \to \bigcup_{i \in \mathbf{F}} \mathbf{M}_i$ such that $h(i) \in \mathbf{M}_i$ for all $i \in \mathbf{F}$, the forest product $\bigotimes_{i \in \mathbf{F}} A_i$ is the algebra $\mathbf{M}$ defined as follows:

(1) The elements of $\mathbf{M}$ are the $h \in (\bigcup_{i \in \mathbf{F}})^F$ such that, for all $i \in \mathbf{F}$ if $h(i) \neq 1$ then for all $j > i$, $h(j) = 0$.
(2) The monoid operation and the lattice operations are defined pointwise.
(3) The residual is defined as follows:

$$(h \to g)(i) = \begin{cases} h(i) \to_i g(i), & \text{if for all } j < i,\ h(j) \leq_j g(j) \\ \\ 0 & \text{otherwise} \end{cases}$$

where de subscript $_i$ denotes the realization of operations and of order in $\mathbf{M}_i$.

In every forest $\mathbf{F}$ the collection $\mathscr{D}(\mathbf{F})$ of lower sets of $\mathbf{F}$ defines a topology over F called the *Alexandrov topology* on $\mathbf{F}$. The purpose of this communication is to study the forest product of MTL-chains in order to prove the following statement: *The forest product of MTL-chains is essentially a sheaf of MTL-algebras over an Alexandrov space whose fibers are MTL-chains.*

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A representation for the n-generated free algebra in the subvariety of BL-algebras generated by $[0,1]_{\mathbf{MV}} \oplus [0,1]_{\mathbf{G}}$

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BL-algebras were introduced by Hájek (see [1]) to formalize fuzzy logics in which the conjunction is interpreted by continuous t-norms over the real interval $[0,1]$. These algebras form a variety, usually called $\mathscr{BL}$. Important examples of its proper subvarieties are the variety $\mathscr{MV}$ of MV-algebras, $\mathscr{P}$ of product algebras and $\mathscr{G}$ of Gödel algebras.

For each integer $n \geq 0$, we will write $Free_{\mathscr{BL}}(n)$ to refer to the free n-generated BL-algebra, which is generated by the algebra $(n+1)[0,1]_{\mathbf{MV}}$, that is, the ordinal sum of $n+1$ copies of the standard MV-algebra. This fact allows us to characterize the free n-generated BL-algebra $Free_{\mathscr{BL}}(n)$ as the algebra of functions $f : (n+1)[0,1]^n_{\mathbf{MV}} \to (n+1)[0,1]_{\mathbf{MV}}$ generated by the projections. Using this, in [3] and [2] there is a representation of the free-n-generated BL-algebra in terms of elements of free Wajsberg hoops ($\perp$-free subreducts of Wajsberg algebras), organized in a structure based on the ordered partitions of the set of generators and satisfying certain geometrical constraints.

In this work we will concentrate in the subvariety $\mathscr{V} \subseteq \mathscr{BL}$ generated by the ordinal sum of the algebra $[0,1]_{\mathbf{MV}}$ and the Gödel hoop $[0,1]_{\mathbf{G}}$, that is, generated by $\mathbf{A} = [0,1]_{\mathbf{MV}} \oplus [0,1]_{\mathbf{G}}$. Though it is well-known that $[0,1]_{\mathbf{G}}$ is decomposable as an infinite ordinal sum of two-elements Boolean algebra, the idea is to treat it as a whole block. The elements of this block are the dense elements of the generating chain and the elements in $[0,1]_{\mathbf{MV}}$ are usually called regular elements of $\mathbf{A}$. The main advantage of this approach, is that unlike the work done in [3] and [2], when the number n of generators of the free algebra increase the generating chain remains fixed. This provides a clear insight of the role of the two main blocks of the generating chain in the description of the functions in the free algebra: the role of the regular elements and the role of the dense elements.

We give a functional representation for the free algebra $Free_{\mathscr{V}}(n)$. To define the functions in this representation we need to decompose the domain $[0,1]_{\mathbf{MV}} \oplus [0,1]_{\mathbf{G}}$ in a finite number of pieces. In each piece a function $F \in Free_{\mathscr{V}}(n)$ coincides either with McNaughton functions or functions on the free algebra in the variety of Gödel hoops (which we define using a base different from the one given by Gerla in [4]) in the following way:

- For every $\bar{x} \in ([0,1]_{\mathbf{MV}})^n$, $F(\bar{x}) = f(\bar{x})$, where f is a function of $Free_{\mathscr{MV}}(n)$.

For the rest of the domain, the functions depend on this function $f : ([0,1]_{\mathbf{MV}})^n \to [0,1]_{\mathbf{MV}}$:

- On $([0,1]_{\mathbf{G}})^n$: If $f(\bar{1}) = 0$, then $F(\bar{x}) = 0$ for every $\bar{x} \in ([0,1]_{\mathbf{G}})^n$, and if $f(\bar{1}) = 1$, then $F(\bar{x}) = g(\bar{x})$, for a function $g \in Free_{\mathscr{G}}(n)$, for every $\bar{x} \in ([0,1]_{\mathbf{G}})^n$.

Let $B = \{x_{\sigma(1)}, \ldots x_{\sigma(m)}\}$ be a non empty proper subset of the set of variables $\{x_1, \ldots, x_n\}$ and R_B be the subset of $([0,1]_{MV} \oplus [0,1]_G)^n$ where $x_i \in B$ if and only if $x_i \in [0,1]_G$. For every $\bar{x} \in R_B$ we also define $\tilde{x}$ as:

$$\tilde{x}_i = \begin{cases} x_i & \text{if} \quad x_i \notin B \\ \\ 1 & \text{if} \quad x_i \in B \end{cases}$$

- On R_B: If $f(\tilde{x}) < 1$ then $F(\bar{x}) = f(\tilde{x})$, and if $f(\tilde{x}) = 1$, then there is a regular triangulation Δ of $f^{-1}(1) \wedge R_B$ which determines the simplices $S_1, \ldots, S_n$ and l Gödel functions $h_1, \ldots, h_n$ in $n-m$ variables $x_{\sigma(m+1)}, \ldots, x_{\sigma(n)}$ such that
  $F(\bar{x}) = h_i(x_{\sigma(m+1)}, \ldots, x_{\sigma(n)})$ for each point $(x_{\sigma(1)}, \ldots x_{\sigma(m)})$ in the interior of S_i.

This representation allows us to give a simple characterization of the maximal filters in this free algebra.

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States of free product algebras

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Within the framework of t-norm based fuzzy logics as developed by Hájek [2], *states* were first introduced by Mundici [3] as maps averaging the truth-value in Łukasiewicz logic. Such functions on MV-algebras (the algebraic counterpart of Łukasiewicz logic), suitably generalize the classical notion of finitely additive probability measures on Boolean algebras, besides corresponding to convex combinations of valuations in Łukasiewicz propositional logic. One of the most important results of MV-algebraic state theory is Kroupa-Panti theorem [4, §10], showing that every state of an MV-algebra is the Lebesgue integral with respect to a regular Borel probability measure. Many attempts of defining states in different structures have been made. In particular, in [1], the authors provide a definition of state for the Lindenbaum algebra of Gödel logic that results in corresponding to the integration of the truth value functions induced by Gödel formulas, with respect to Borel probability measures on the real unit cube $[0,1]^n$. Such states correspond to convex combinations of finitely many truth-value assignments.

The aim of this contribution is to introduce and study states for the Lindenbaum algebra of product logic, the remaining fundamental t-norm fuzzy logic for which such a notion is still lacking. Denoting with $\mathscr{F}_{\mathbb{P}}(n)$ the free n-generated product algebra, that is isomorphic to the Lindenbaum algebra of product logic formulas built from n propositional variables, a *state* will be a map from $\mathscr{F}_{\mathbb{P}}(n)$ to $[0,1]$ satisfying the following conditions: (i) $s(1)=1$ and $s(0)=0$; (ii) $s(f\wedge g)+s(f\vee g)=s(f)+s(g)$; (iii) If $f\leq g$, then $s(f)\leq s(g)$; (iv) If $f\neq 0$, then $s(f)=0$ implies $s(\neg\neg f)=0$.

Our main result is an integral representation for states of free product algebras. More precisely, a $[0,1]$-valued map s on $\mathscr{F}_{\mathbb{P}}(n)$ is a state in our sense if and only if there is a unique regular Borel probability measure μ such that, for every $f\in\mathscr{F}_{\mathbb{P}}(n)$,

$$s(f)=\int_{[0,1]^n} f\,\mathrm{d}\mu.$$

Moreover, we prove that every state belongs to the convex closure of product logic valuations. Indeed, in particular, extremal states will result in corresponding to the homomorphisms of $\mathscr{F}_{\mathbb{P}}(n)$ into $[0,1]$, that is to say, to the valuations of the logic.

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NPc-algebras and Gödel hoops

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This is a joint work with S. Aguzzoli, M. Busaniche and B. Gerla. See [2].

The algebraic models of paraconsistent Nelson logic were introduced by Odinstov under the name of N4-lattices.

NPc-lattices are a variety of residuated lattices that turn out to be termwise equivalent to eN4-lattices, the expansions of N4-lattices by a constant e. Thus paraconsistent Nelson logic can be studied within the framework of substructural logics (see [3]).

The first aim of this work is to prove a categorical equivalence between the category of NPc-lattices and its morphisms and a category whose objects are pairs of Brouwerian algebras and certain filters that we call regular. For this we follow the ideas of Odintsov in [4].

We then define Gödel NPc-lattices as those NPc-lattices with a prelinear negative cone, which turn to be equivalent to pairs of Gödel hoops and regular filters. For the case of finite Gödel NPc-lattices, by the duality of finite Gödel hoops and finite trees (see [1]) we also obtain a dual category, which consists in pairs of finite trees and certain subtrees.

Therefore we have a duality between a category of algebras and a category of combinatorics, which we use to characterize the free algebras in the variety of Gödel NPc-lattices.

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On an operation with regular elements

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In a previous work (see [1]) we have studied, in the context of residuated lattices, the operation B given by the greatest Boolean below a given element. In particular, our results hold for the class $\mathbb{ML}$ of meet-complemented lattices. Given an $\mathbf{M} \in \mathbb{ML}$ and $a \in M$, a is said to be *Boolean* iff there is an element $b \in M$ such that $a \wedge b = 0$ and $a \vee b = 1$, where, as usual, we use $\wedge$, $\vee$, 0, and 1 for the infimum, the supremum, the bottom, and the top of $\mathbf{M}$, respectively. The given definition is easily seen to be equivalent to saying that a is Boolean iff $a \vee \neg a = 1$. Accordingly, given an $\mathbf{M} \in \mathbb{ML}$, we have postulated the existence of an operation B such that, for all $a \in M$,

$Ba = max\{b \in M : b \leq a \text{ and } b \vee \neg b = 1\}$,

using the notation $\mathbb{ML}^B$ for the corresponding class. It is equivalent to postulate that there exists an operation B such that the following hold:

(BE1) $Bx \preccurlyeq x$,
(BE2) $Bx \vee \neg Bx \approx 1$, and
(BI) if $y \leq x$ & $y \vee \neg y \approx 1 \Rightarrow y \preccurlyeq Bx$,

where, resembling the terminology used in a Natural Deduction setting, the notations I and E come from Introduction and Elimination, respectively.

Now we deal with the operation that results when substituting the notion of regular for the notion of Boolean. Accordingly, given an $\mathbf{M} \in \mathbb{MSL}$, that stands for the class of meet-complemented meet-semilattices, we may postulate the existence of an operation R such that, for all $a \in M$,

$Ra = max\{b \in M : b \leq a \text{ and } \neg\neg b = b\}$,

using the notation $\mathbb{MSL}^R$ for the corresponding class.

Note that the minimum regular above is not a new operation, as it equals $\neg\neg$. On the other hand, operation R seems to deserve inspection.

We prove that R is the right adjoint of double meet-negation. We also prove that the resulting class is an equational class satisfying the Stone equality, i.e. $\neg x \vee \neg\neg x \approx 1$. Finally, it is also the case that the class of distributive meet-negated lattices with the greatest regular below is the same as the class of Stone distributive meet-negated lattices with the greatest Boolean below.

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