Implication Zroupoids: An Abstraction from De Morgan Algebras

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ACKNOWLEDGEMENTS

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PART 1: Historical Motivation and a new characterization of De Morgan algebras

- **Bernstein (1934):** A system of axioms for Boolean algebras was given by Bernstein, using only **implication**.
- Even though that system was not equational, it could easily be converted to an equational one, if we use an additional constant as part of the signature. This led me to ask the following natural question:
- PROBLEM: What about De Morgan algebras? Is it possible to define the variety of De Morgan algebras using only the implication and a constant.
  In this talk, I will address this question, as well as the ramifications of its positive solution.
- But, first let me start with some well known definitions.

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De Morgan, Kleene and Boolean algebras:

Definition

- An algebra $\mathbf{A} = \langle A, \wedge, ^c, 0\rangle$ is a De Morgan algebra if $\mathbf{A}$ satisfies the following conditions, where we define $x \vee y := (x^c \wedge y^c)^c$ and $1 := 0^c$:

  (d1) $\langle A, \vee, \wedge, 0, 1\rangle$ is a distributive lattice with $0, 1$

  (d2) $x^{cc} \approx x$.

  $\mathcal{DM}$ denotes the variety of De Morgan algebras.
- A De Morgan algebra $\mathbf{A}$ is a Kleene algebra if $\mathbf{A}$ satisfies:

  (K) $x \wedge x^c \leq y \vee y^c$.

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De Morgan Algebras: From a New Perspective

- In the article:
  De Morgan algebras: New perspectives and Applications, Scientiae Mathematicae Japonicae (2012)],
- I characterized De Morgan algebras in the language $\{\to, 0\}$, where $\to$ is binary, and 0 is a constant symbol.
- To tell you about this new perspective, I need to introduce a "new" class of algebras:

Definition

An algebra $\mathbf{A} = \langle A, \to, 0 \rangle$ is a $\mathcal{DM}^{\to}$-*algebra* if $\mathbf{A}$ satisfies the following axioms, where $x' := x \to 0$:

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The next theorem gives a positive solution to the problem mentioned earlier.

Theorem

The variety $\mathcal{DM}^{\rightarrow}$ is term-equivalent to the variety $\mathcal{DM}$. More precisely,

(a) For $\mathbf{A} \in \mathcal{DM}$, let $\mathbf{A}^{\rightarrow}$ be the algebra $\langle A, \rightarrow, 0 \rangle$ where $\rightarrow$ is defined by $x \rightarrow y := (x \wedge y^c)^c$. Then $\mathbf{A}^{\rightarrow} \in \mathcal{DM}^{\rightarrow}$.

(b) For $\mathbf{A} \in \mathcal{DM}^{\rightarrow}$, let $\mathbf{A}^*$ be the algebra $\langle A, \wedge, ^c, 0 \rangle$ such that $x \wedge y := (x \rightarrow y')'$, where $x' := x \rightarrow 0$; and $x^c := x'$. Then $\mathbf{A}^* \in \mathcal{DM}$.

(c) If $\mathbf{A} \in \mathcal{DM}$, then $(\mathbf{A}^{\rightarrow})^* = \mathbf{A}$.

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- Let us prove (a):
- Let $\mathbf{A} = \langle A, \wedge, ^c, 0\rangle$ be a De Morgan algebra and let $\mathbf{A}^{\rightarrow}$ be the algebra as in (a) of the above theorem.
  Let x, y, z be arbitrary elements of $\mathbf{A}^{\rightarrow}$.
- First, note that $x' := x \to 0 := (x \wedge 0^c)^c = x^c \vee 0 = x^c$.
  Also, note that $x \vee y = (x^c \wedge y^c)^c = x^c \to y$.
- Now, let us prove (I):
- $(x \to y) \to z = [(x \wedge y^c)^c \wedge z^c]^c = (x \wedge y^c) \vee z$
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  $= (z^c \to x) \wedge (y \to z) = [(z^c \to x) \wedge (y \to z)^{cc}]^{cc}$
  $= [(z^c \to x) \to (y \to z)^c]^c = [(z' \to x) \to (y \to z)']'$,
  which proves (I).
- Next, to prove (j):
  $(x \to y) \to x = [(x \wedge y^c)^c \wedge x^c]^c = [x^c \wedge (x \wedge y^c)^c]^c$
  $= x \vee (x \wedge y^c) = x$, which proves (J).

The proofs of (b), (c), and (d) will be an exercise for you due tomorrow!

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- Let $\mathbf{A} = \langle A, \wedge, ^c, 0 \rangle$ be a De Morgan algebra and let $\mathbf{A}^{\to}$ be the algebra as in (a) of the above theorem.
  Let x, y, z be arbitrary elements of $\mathbf{A}^{\to}$.
- First, note that $x' := x \to 0 := (x \wedge 0^c)^c = x^c \vee 0 = x^c$.
  Also, note that $x \vee y = (x^c \wedge y^c)^c = x^c \to y$.
- Now, let us prove (I):
- $(x \to y) \to z = [(x \wedge y^c)^c \wedge z^c]^c = (x \wedge y^c) \vee z$
  $= z \vee (x \wedge y^c) = (z \vee x) \wedge (z \vee y^c) = (z \vee x) \wedge (y^c \vee z)$
  $= (z^c \to x) \wedge (y \to z) = [(z^c \to x) \wedge (y \to z)^{cc}]^{cc}$
  $= [(z^c \to x) \to (y \to z)^c]^c = [(z' \to x) \to (y \to z)']'$,
  which proves (I).
- Next, to prove (j):
  $(x \to y) \to x = [(x \wedge y^c)^c \wedge x^c]^c = [x^c \wedge (x \wedge y^c)^c]^c$
  $= x \vee (x \wedge y^c) = x$, which proves (J).

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Definition

A $\mathcal{DM}^{\to}$-algebra $\mathbf{A} = \langle A, \to, 0\rangle$ is a $\mathcal{KL}^{\to}$*-algebra* if $\mathbf{A}$ satisfies the following axiom:

(K1) $(y \to y) \to (x \to x) \approx x \to x$.

A $\mathcal{DM}^{\to}$-algebra $\mathbf{A} = \langle A, \to, 0\rangle$ is a $\mathcal{BA}^{\to}$*-algebra* if $\mathbf{A}$ satisfies the following axiom:

(B1) $x \to x \approx 0'$.

Let $\mathcal{KL}^{\to}$ and $\mathcal{BA}^{\to}$ denote respectively the varieties of $\mathcal{KL}^{\to}$-algebras and $\mathcal{BA}^{\to}$-algebras.

Corollary

(1) *The variety $\mathcal{KL}$ is term-equivalent to the variety $\mathcal{KL}^{\to}$,*

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An application

As mentioned earlier, in 1934, Bernstein gave an axiom system using only implication; but it was not equational since one of the axioms was existential.

Theorem (Modified Version of Bernstein's Theorem)

The following (equational) axioms form a 2-base for the variety of Boolean algebras in the language $\{\to, 0\}$, where $x' = x \to 0$:

(J) $(x \to y) \to x \approx x$

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PART 2: Implication Zroupoids-An abstaction from De Morgan algebras

The axiom (I) has played a significant role in the characterizations of De Morgan, Kleene, and Boolean algebras. So (I) deserves to be investigated in its own right.

Definition

- A *groupoid with zero* (zroupoid, for short) is an algebra $\mathbf{A} = \langle A, \to, 0, \rangle$, where $\to$ is a binary operation and 0 is a constant.
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EXAMPLES OF I-ZROUPOIDS:

- $\mathcal{BA}^{\rightarrow}$(of Boolean algebras),
- $\mathcal{KL}^{\rightarrow}$ (of Kleene algebras)
- $\mathcal{DM}^{\rightarrow}$ (of De Morgan algebras)
- There are exactly 3 two-element I-zroupoids as shown below

$\mathbf{2_b}$:

$\rightarrow$	0	1
0	1	1
1	0	1

$\mathbf{2_z}$:

$\rightarrow$	0	1
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A new class of examples of I-zroupoids

Let $\mathcal{SL}$ denote the subvariety of $\mathcal{I}$ defined by

(1) $x' \approx x$,

(2) $x \to y \approx y \to x \quad$ (C).

Theorem

$\mathcal{SL} = \mathbf{V}(\mathbf{2_s})$= The variety of $\vee$-semilattices with least element 0.

From Freese and Nation we get the following corollary.

Corollary

*The class of congruence lattices **Con A**, where **A** $\in \mathcal{I}$, does not satisfy any non-trivial lattice identities. In particular, the variety $\mathcal{I}$ is neither congruence-distributive, nor congruence-modular.*

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The class of congruence lattices **Con A**, *where* $\mathbf{A} \in \mathcal{I}$, *does not satisfy any non-trivial lattice identities. In particular, the variety* $\mathcal{I}$ *is neither congruence-distributive, nor congruence-modular.*

A new class of examples of I-zroupoids

Let $\mathcal{SL}$ denote the subvariety of $\mathcal{I}$ defined by

(1) $x' \approx x$,

(2) $x \to y \approx y \to x$ (C).

Theorem

$\mathcal{SL} = \mathbf{V}(\mathbf{2_s})$= *The variety of* $\vee$*-semilattices with least element* 0.

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A Fundamental Property of I-zroupoids

Theorem

Let $\mathbf{A} \in \mathcal{I}$ *and* $x, y \in \mathbf{A}$. *Then* $x''' \to y = x' \to y$.

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A Glivenko-like theorem for $\mathcal{I}$

Let $\mathbf{A} \in \mathcal{I}$. Let $A'' = \{a'' \mid a \in A\}$ and let $\mathbf{A}'' := \langle A'', \to, 0\rangle$.

Theorem

Let $\mathbf{A} = \langle A, \to, 0\rangle \in \mathcal{I}$, then

(a) $\mathbf{A} \models (x'' \to y'') \to x'' \approx x''$ *iff* $\mathbf{A}''$ *is a De Morgan algebra,*

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Some Important Subvarieties of the Variety of **Implication Zroupoids**

Definition

Let $\mathcal{I}_{3,1}$, $\mathcal{I}_{2,0}$ and $\mathcal{I}_{1,0}$ denote, respectively, the subvarieties of $\mathcal{I}$ satisfying $x''' \approx x'$, $x'' \approx x$, and $x' \approx x$.

More characterizations of the variety $\mathcal{I}_{2,0}$:

Theorem

Let $\mathbf{A}$ *be a* I*-zroupoid. Then T.F.A.E.:*

(a) $\mathbf{A} \in \mathcal{I}_{2,0}$

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PART 3: Derived algebras of $\mathcal{I}$ and Connections to other classes of algebras

Let $\mathbf{A} = \langle A; \to, 0\rangle \in \mathcal{I}$. We define the operations $\wedge$ and $\vee$ on $\mathbf{A}$ by:

- $x \wedge y := (x \to y')'$,
- $x \vee y := (x' \wedge y')'$.

With each implication zroupoid $\mathbf{A}$, we associate the following algebras, referred to as "derived algebras":

- $\mathbf{A^m} := \langle A, \wedge, 0\rangle$,
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Theorem

Let $\mathbf{A} \in \mathcal{I}$. *Then* $\mathbf{A}^{\mathbf{m}}$ *is a semigroup.*

In fact, more is true:

Corollary

Let $\mathbf{A} \in \mathcal{I}$. *Then* $\mathbf{A}^{\mathbf{mj}}$ *is a bisemigroup.*

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More properties of the derived algebras $\mathbf{A}^{\mathbf{mj}}$ in the variety I

Theorem

Let $\mathbf{A} \in \mathcal{I}$. *Then*

(a) $\mathbf{A}^{\mathbf{mj}}$ *satisfies:*

(1) $(x \vee y)' \approx x' \wedge y'$,

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The derived algebra $\mathbf{A}^{\mathbf{mj}}$ as a lattice

Necessary and sufficient conditions on algebras $\mathbf{A} \in \mathcal{I}$ under which the *derived* algebra $\mathbf{A}^{\mathbf{mj}} = \langle A, \wedge, \vee, 0 \rangle$ is a lattice:

Theorem

The following are equivalent in $\mathbf{A} \in \mathcal{I}$:

(1) $\mathbf{A}^{\mathbf{mj}}$ *is a lattice,*

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Symmetric Implication Zroupoids: Another important subvariety of $\mathcal{I}$

An algebra $\mathbf{A}$ in $\mathcal{I}$ is symmetric if $\mathbf{A}$ satisfies:

(a) $x'' \approx x$ (that is, $\mathbf{A} \in \mathcal{I}_{2,0}$), and

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$\mathcal{S}$ denotes the variety of symmetric Implication zroupoids.

Theorem

Let $\mathbf{A} \in \mathcal{S}$. Then $\mathbf{A}^{mj}$ satisfies:

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(e) $x \vee (y \wedge z) \approx (x \vee y) \wedge (x \vee z)$,
(f) $x \wedge (x \vee y) \approx x \vee (x \wedge y)$.

Symmetric Implication Zroupoids: Another important subvariety of $\mathcal{I}$

An algebra **A** in $\mathcal{I}$ is symmetric if **A** satisfies:
(a) $x'' \approx x$ (that is, $\mathbf{A} \in \mathcal{I}_{2,0}$), and
(b) $x \wedge y \approx y \wedge x$ ($\wedge$-commutative)
$\mathcal{S}$ denotes the variety of symmetric Implication zroupoids.

Theorem

Let $\mathbf{A} \in \mathcal{S}$. Then $\mathbf{A}^{\mathbf{mj}}$ satisfies:

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A bisemilattice is an algebra $\langle A, \wedge, \vee \rangle$ such that $\langle A, \wedge \rangle$ and $\langle A, \vee \rangle$ are both semilattices.

A distributive bisemilattice is a bisemilattice in which the distributive laws hold.

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Let $\mathbf{A}$ *be in the variety* $\mathcal{S}$*. Then* $\mathbf{A}^{\mathbf{mj}}$ *is a distributive bisemilattice.*

$\mathcal{S}$ meets Birkhoff Systems.

A Birkhoff system is a bisemilattice satisfying the Birkhoff's identity:

$$(BR) \quad x \wedge (x \vee y) \approx x \vee (x \wedge y).$$

Theorem

If $\mathbf{A} \in \mathcal{S}$, then $\mathbf{A}^{\mathbf{mj}}$ is both a distributive bisemilattice and a Birkhoff system.

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Theorem

The only simple algebras in $\mathcal{I}$, up to isomorphism, are $\mathbf{2_z}$, $\mathbf{2_s}$, $\mathbf{2_b}$, $\mathbf{3}_k$ and $\mathbf{4}_d$, where the $\to$ operations of $\mathbf{3}_k$ and $\mathbf{4}_d$ are given below

$\to$:	0	1	2
0	1	1	1
1	0	1	2
2	2	1	2

$\to$:	0	1	2	3
0	1	1	1	1
1	0	1	2	3
2	2	1	2	1
3	3	1	1	3

Corollary

The only (nontrivial) simple algebras in $\mathcal{I}_{2,0}$, up to isomorphism, are $\mathbf{2_s}$, $\mathbf{2_b}$, $\mathbf{3}_k$ and $\mathbf{4}_d$.

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PART 4: Simple Algebras in $\mathcal{I}$

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Corollary

The only (nontrivial) simple algebras in $\mathcal{I}_{2,0}$, up to isomorphism, are $\mathbf{2_s}$, $\mathbf{2_b}$, $\mathbf{3}_k$ and $\mathbf{4}_d$.

Recall that a variety is semisimple if and only if every subdirectly irreducible algebra in it is simple.

Corollary

A subvariety $\mathcal{V}$ of $\mathcal{I}$ is semisimple if and only if $\mathcal{V} \subseteq \mathbb{V}(\mathbf{2_z}, \mathbf{2_s}, \mathbf{4}_d)$.

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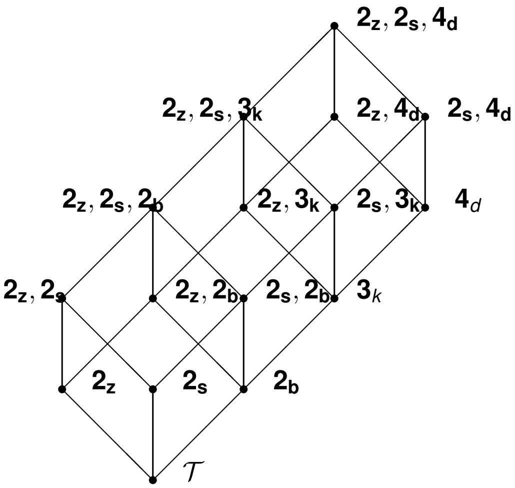
The lattice of semisimple subvarieties of $\mathcal{I}$



- Both the variety $\mathcal{SL}$ of $\vee$ semilattices with a least element 0 and the variety $\mathcal{DM}$ of De Morgan algebras have a partial order induced by the operation $\wedge$.
- So, it is but natural to consider the following relation $\sqsubseteq$ in Implication Zroupoids.

Definition

Let $\mathbf{A} \in \mathcal{I}$. We define the relation $\sqsubseteq$ on A as follows:

$$x \sqsubseteq y \text{ if and only if } x \wedge y = x \quad (\textit{equivalently}, (x \to y')' = x).$$

- PROBLEM: Is there a subvariety $\mathcal{V}$ of $\mathcal{I}$, containing both $\mathcal{SL}$ and $\mathcal{DM}$, such that, for every algebra $\mathbf{A}$ in $\mathcal{V}$, the relation $\sqsubseteq$ on $\mathbf{A}$ is actually a partial order.

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Part of the importance of the variety $\mathcal{I}_{2,0}$, which contains the varieties $\mathcal{SL}$ and $\mathcal{DM}$, is highlighted by the following theorem.

Theorem

The variety $\mathcal{I}_{2,0}$ is a maximal subvariety of $\mathcal{I}$ with respect to the property that the relation $\sqsubseteq$, is a partial order.

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Finite $\mathcal{I}_{2,0}$-chains

- Since the relation $\sqsubseteq$ is a partial order on algebras in $\mathcal{I}_{2,0}$, it is natural to ask the
  Question: How many $\mathcal{I}_{2,0}$-chains are there on a given set of size n?

Definition

$\mathbf{A} \in \mathcal{I}_{2,0}$ is an $\mathcal{I}_{2,0}$-chain if the relation $\sqsubseteq$ is totally ordered on A.

- Let me give some examples of $\mathcal{I}_{2,0}$-chains.
- Note that, in these examples, the number 0 is the constant element.

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Examples of finite $\mathcal{I}_{2,0}$-chains

The only 2-element $\mathcal{I}_{2,0}$-chains, up to isomorphism, are

$\to$:	0	1
0	1	1
1	0	1

with $0 \sqsubset 1$.

(This is the Boolean Algebra $\mathbf{2_b}$), and

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2	0	1	2

with $0 \sqsubset 1 \sqsubset 2$,

$\to$:	-1	0	1
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Characterization of finite $\mathcal{I}_{2,0}$-chains

Theorem

There are n non-isomorphic $\mathcal{I}_{2,0}$-chains of size n, for $n \in \mathbb{N}$.

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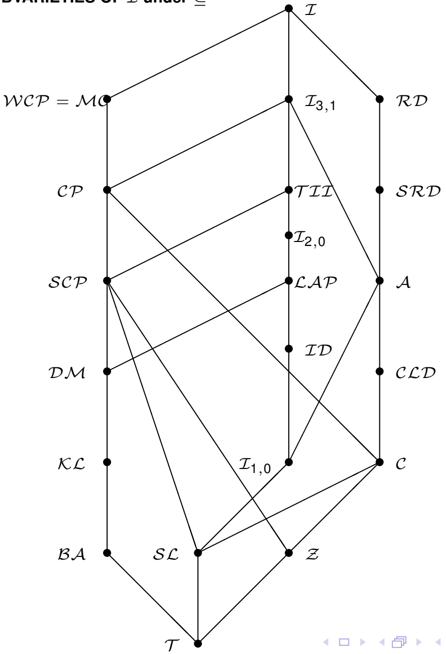
PART 6: Varieties of Implication Zroupoids

POSET OF (SOME KNOWN) SUBVARIETIES OF $\mathcal{I}$ under $\subseteq$



The Identities defining the above subvarieties of $\mathcal{I}$

$(I_{2,0})$ $x'' \approx x$:
$(I_{3,1})$ $x''' \approx x'$;
$(I_{1,0})$ $x' \approx x$;
(ID) $x \to x \approx x$;
(Z) $x \to y \approx 0$;
(MC) $x \wedge y \approx y \wedge x$;
(C) $x \to y \approx y \to x$;
(CP) $x \to y' \approx y \to x'$;
(SCP) $x \to y \approx y' \to x'$;
(WCP) $x' \to y \approx y' \to x$;

(A) $(x \to y) \to z \approx x \to (y \to z)$;
(RD) $(x \to y) \to z \approx (x \to z) \to (y \to z)$;
(SRD) $(x \to y) \to z \approx (z \to x) \to (y \to z)$;
(LD) $x \to (y \to z) \approx (x \to y) \to (x \to z)$; not shown in the picture
(CLD) $x \to (y \to z) \approx (x \to z) \to (y \to x)$;

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$(Z)\quad x \to y \approx 0$;
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$(SCP)\ x \to y \approx y' \to x'$;

$(I_{1,0})\ x' \approx x$;

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PART 7: Associativity-like Identities

The search for new subvarieties of $\mathcal{I}$ led us to look at "associative-like" identities.

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An identity of associative type is an identity $p \approx q$, in the groupoid language $\langle \to \rangle$, where p and q have exactly 3 variables, say x,y,z, and in which the variables are grouped according to one of the following two ways of grouping:

a) $o \to (o \to o)$

b) $(o \to o) \to o$.

The six permuatations of 3 variables give rise to 12 terms:

(1a) $x \to (y \to z)$, (1b) $(x \to y) \to z$

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It turns out that the following 14 identities of associative type are the only ones that are mutually independent on any groupoid:

(A1) $x \to (y \to z) \approx (x \to y) \to z$ (Associative law)

(A2) $x \to (y \to z) \approx x \to (z \to y)$

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Main Theorem for the varieties of associative-type

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(a) *The following are the 8 subvarieties of associative type of $\mathcal{I}$ that are distinct from each other.*

$$\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_6, \mathcal{A}_9, \mathcal{A}_{11} \text{ and } \mathcal{A}_{14}$$

(b) *They satisfy the following relationships*

1. $\mathcal{SL} \subset \mathcal{A}_3 \subset \mathcal{A}_4$
2. $\mathcal{BA} \subset \mathcal{A}_4 \subset \mathcal{I}$
3. $\mathcal{A}_3 \subset \mathcal{A}_1 \subset \mathcal{I}$
4. $\mathcal{A}_3 \subset \mathcal{A}_2 \subset \mathcal{A}_{11}$, $\mathcal{A}_3 \subset \mathcal{A}_6 \subset \mathcal{A}_{11}$ *and* $\mathcal{A}_3 \subset \mathcal{A}_9 \subset \mathcal{A}_{11}$
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PART 7(2) Weak Associative Identities of length m $\leq$ 4 in Symmetric Idempotent Zroupoids

The first systematic analysis of the relationships among the weak associative identities appears to have been done by Feyves in 1969 in the context of loops for the special case of Bol-Moufang type. He listed 60 identities of Bol-Moufang type (of size 4 in 3 variables, with one variable repeated).

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Definition

Let $n, m, p, q \in \mathbb{N}$ and let X denote a word of length n in which the m (distinct) variables occur alphabetically.

$(\mathrm{nm}X\mathrm{pq})$ denotes the weak associative identity $t \approx s$ of length n with m variables, where t and s are terms, obtained from X, with bracketing numbers p and q respectively.

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Weak Associative Identities of size ≤ 4, relative to $\mathcal{S}$

Let us look at the weak associative laws of size ≤ 4 relative to $\mathcal{S}$.

Clearly, $x \approx x$ is the only identity of lenth 1 which is trivial.

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Weak associative identities of length 3 with 1 variable

The only word of length 3 with 1 variable is:
A: $\langle x, x, x \rangle$.
Ways in which the word A can be bracketed (where 'o' is just a place holder) are:

1: $o \to (o \to o)$,
2: $(o \to o) \to o$.

The only weak associative identity in this category is:

1: (31A12) $x \to (x \to x) \approx (x \to x) \to x$.

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Possible words of length 3 with 2 variables are:

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3: (31C12) =(RALT) $x \to (y \to y) \approx (x \to y) \to y$ (*the right-alternative law*)

Weak associative laws of length 3 with 2 variables:

Possible words of length 3 with 2 variables are:

A: $\langle x, x, y \rangle$,
B: $\langle x, y, x \rangle$,
C: $\langle x, y, y \rangle$.

Ways in which a word of size 3 can be bracketed:

1: $o \to (o \to o)$,
2: $(o \to o) \to o$.

The weak associative identities in this category are:

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Weak associative laws of length 3 with 3 variables:

The only word of length 3 with 3 variables is:

A: $\langle x, y, z \rangle$.

Ways in which a word of length 3 can be bracketed:

1: $o \to (o \to o)$,
2: $(o \to o) \to o$, where 'o' is a place holder.

The only weak associative identitiy in this category is:

(33A12) $x \to (y \to z) \approx (x \to y) \to z$ (associative law).

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Weak associative laws with length 4 and with 1 variable.

The only word of length 4 with 1 variable is:

A: $\langle x, x, x, x \rangle$.

Ways in which a word of length 4 can be bracketed are:

1: $o \to (o \to (o \to o))$,
2: $o \to ((o \to o) \to o)$,
3: $(o \to o) \to (o \to o)$,
4: $(o \to (o \to o)) \to o$,
5: $((o \to o) \to o) \to o$, where 'o' is a place holder.
There are 10 identities in this category.

Weak associative laws with length 4 and with 1 variable.

The only word of length 4 with 1 variable is:

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Ways in which a word of length 4 can be bracketed are:

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There are 10 identities in this category.

Weak associative identities with length 4 and with 2 variables.

Possible words of length 4 with 2 variables are:

A: $\langle x, x, x, y \rangle$, B: $\langle x, x, y, x \rangle$,
C: $\langle x, x, y, y \rangle$, D: $\langle x, y, x, x \rangle$,
E: $\langle x, y, x, y \rangle$, F: $\langle x, y, y, x \rangle$,
G: $\langle x, y, y, y \rangle$.

Ways in which a word of size 4 can be bracketed are:

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There are 70 identities in this category.

Weak associative identities with length 4 and with 2 variables.

Possible words of length 4 with 2 variables are:

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Weak associative laws with length 4 and with 3 variables (=Bol-Moufang identities) of length 4

Possible words of length 4 with 3 variables are:

A: $\langle x, x, y, z \rangle$, B: $\langle x, y, x, z \rangle$,
C: $\langle x, y, y, z \rangle$, D: $\langle x, y, z, x \rangle$,
E: $\langle x, y, z, y \rangle$, F: $\langle x, y, z, z \rangle$.

Ways in which a word of size 4 can be bracketed:

1: $a \to (a \to (a \to a))$, 2: $a \to ((a \to a) \to a)$,
3: $(a \to a) \to (a \to a)$, 4: $(a \to (a \to a)) \to a$,
5: $((a \to a) \to a) \to a$.

There are 60 identities in this category.

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Weak associative laws with length 4 and with 4 variables.

The only word of length 4 with 4 variables is:

A: $\langle t, x, y, z \rangle$.

Ways in which a word of length 4 can be bracketed:

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MAIN THEOREM ABOUT DISTINCT, WEAK ASSOCIATIVE SUBVARIETIES of $\mathcal{S}$

Theorem

(a) *The following are the 6 (distinct) varieties defined, relative to $\mathcal{S}$, arising from the 155 weak associative laws of length $m \leq 4$ that are distinct from each other:*

$$\mathcal{SL},\ 43\mathcal{A}12,\ 43\mathcal{A}23,\ 42\mathcal{A}12,\ 43\mathcal{F}25 \text{ and } \mathcal{S}.$$

(b) *They satisfy the following relationships:*

1. $\mathcal{SL} \subset 43\mathcal{A}23 \subset 43\mathcal{F}25 \subset \mathcal{S}$,
2. $\mathcal{SL} \subset 43\mathcal{A}12 \subset 42\mathcal{A}12 \subset \mathcal{S}$,
3. $\mathcal{BA} \subset 43\mathcal{A}12 \subset 43\mathcal{F}25$,
4. $43\mathcal{A}12 \not\subseteq 43\mathcal{A}23$ *and* $43\mathcal{A}23 \not\subseteq 43\mathcal{A}12$,
5. $42\mathcal{A}12 \not\subseteq 43F25$ *and* $43\mathcal{F}25 \not\subseteq 42\mathcal{A}12$.

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The Hasse diagram of the poset of (distinct) weak associative subvarieties of $\mathcal{S}$ of length $\leq$ 4, together with the variety $\mathcal{BA}$:

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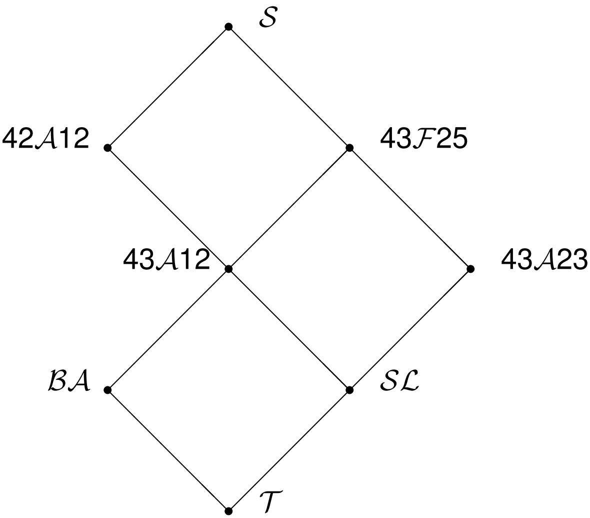

Corresponding Identities:

43A12: $x \to [x \to (y \to z)] \approx x \to [(x \to y) \to z]$

45A23: $x \to [(x \to y) \to z] \approx (x \to x) \to (y \to z]$

43F25: $x \to [(y \to z) \to z] \approx [(x \to y) \to z] \to z$

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Some directions for future research.

PROBLEM 1: Explore the connection of implication zroupoids to semigroups as given by $\mathbf{A^m}$ or $\mathbf{A^j}$, further.
PROBLEM 2: Describe subdirectly irreducible implication zroupoids in $\mathcal{S}$, $\mathcal{I}_{2.0}$, and $\mathcal{I}$.
PROBLEM 3: Is their a characterization of Stone algebras in the language $\{\to, 0\}$?
PROBLEM 5: Investigate the lattice of subvarieties of $\mathcal{I}$ further.
PROBLEM 6: Investigate expansions of $\mathcal{I}$ by adding additional (interesting) operations.

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THANK YOU VERY MUCH FOR LISTENING.