Logic and Implication

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Introduction

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- Protoalgebraic logics and their subclasses are based on a general notion of equivalence.
- Implication has a crucial role in reasoning (entailment, consequence, preservation of truth,...)
- We will now present an AAL theory based on implication.

Two examples of non-classical logics

$[0,1]_{Ł}$: the standard MV-matrix with domain $[0,1]$, filter $\{1\}$ and operations

$$\begin{aligned} x \to y &= \min\{1, 1 - x + y\} \\ x \,\&\, y &= \max\{0, x + y - 1\} \\ x \vee y &= \max\{x, y\} \\ \neg x &= 1 - x \end{aligned}$$

Ł: the logic axiomatized by modus ponens and 4 (5) Łukasiewicz axioms

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Theorem

Let $Ł_\infty$ be the extension of $Ł$ by the rule

$$\{\neg\varphi \to \varphi \mathbin{\&} \overset{n}{\ldots} \mathbin{\&} \varphi \mid n \geq 1\} \rhd \varphi$$

Then $\Gamma \vdash_{Ł_\infty} \varphi$ iff $\Gamma \models_{[0,1]_{Ł}} \varphi$ holds for all Γs.

Outline

1. (In)finitary logics
2. Disjunctions
3. Implications
4. Disjunctions and implications
5. Completeness properties

What is a logic?

Var: an infinite countable set of propositional variables

$\mathcal{L}$: a countable type

$\boldsymbol{Fm}_{\mathcal{L}}$: the absolutely free $\mathcal{L}$-algebra with generators Var

elements of $Fm_{\mathcal{L}}$ are called $\mathcal{L}$-formulas

A logic L is a relation between sets of $\mathcal{L}$-formulas and $\mathcal{L}$-formulas s.t.:

we write '$\Gamma \vdash_{\mathrm{L}} \varphi$' instead of '$\langle \Gamma, \varphi \rangle \in \mathrm{L}$'

- If $\varphi \in \Gamma$, then $\Gamma \vdash_{\mathrm{L}} \varphi$. (Reflexivity)
- If $\Gamma \vdash_{\mathrm{L}} \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash_{\mathrm{L}} \varphi$. (Monotonicity)
- If $\Delta \vdash_{\mathrm{L}} \Gamma$ and $\Gamma \vdash_{\mathrm{L}} \varphi$, then $\Delta \vdash_{\mathrm{L}} \varphi$. (Cut)
- If $\Gamma \vdash_{\mathrm{L}} \varphi$, then $\sigma[\Gamma] \vdash_{\mathrm{L}} \sigma(\varphi)$ for each substitution σ. (Structurality)

A logic L is finitary if $\Gamma \vdash_{\mathrm{L}} \varphi$, then there is a finite $\Gamma' \subseteq \Gamma$ s.t. $\Gamma' \vdash_{\mathrm{L}} \varphi$

Axiomatization

Axiomatic system $\mathcal{AS}$: set of axioms and rules closed under substitutions

Proof of φ from Γ in $\mathcal{AS}$: a well-founded tree labeled by formulas s.t.

- its root is labeled by φ and leaves by axioms or elements of Γ and
- if a node is labeled by ψ and $\Delta \neq \emptyset$ is the set of labels of its preceding nodes, then $\Delta \rhd \psi$ is a rule.

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Ł_∞ is countably axiomatizable but not finitary

Logical matrices and semantical consequence

$\mathcal{L}$-matrix: a pair $\mathbf{A} = \langle \boldsymbol{A}, F\rangle$ where

- $\boldsymbol{A}$ is an $\mathcal{L}$-algebra and
- $F \subseteq A$ called the filter of $\mathbf{A}$

Definition (Semantical consequence)

A formula φ is a semantical consequence of a set Γ of formulas w.r.t. a class $\mathbb{K}$ of $\mathcal{L}$-matrices, $\Gamma \models_{\mathbb{K}} \varphi$ in symbols, if

for each $\langle \boldsymbol{A}, F\rangle \in \mathbb{K}$ and each $\boldsymbol{A}$-evaluation e, we have:
$e(\varphi) \in F$ whenever $e[\Gamma] \subseteq F$.

Filters, theories, models of a logic

Let L be a logic in $\mathcal{L}$ and $\boldsymbol{A}$ an $\mathcal{L}$-algebra

A set $T \subseteq Fm_{\mathcal{L}}$ is a theory if
for each $\varphi \in Fm_{\mathcal{L}}$ we have
$T \vdash_{\mathrm{L}} \varphi$ implies $\varphi \in T$

we write $T \in \mathrm{Th}(\mathrm{L})$

A set $F \subseteq A$ is a filter if
for each $\Gamma \cup \{\varphi\} \in Fm_{\mathcal{L}}$ we have
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Fact 1: $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ is a closure system

Fact 2: $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{Fm}_{\mathcal{L}}) = \mathrm{Th}(\mathrm{L})$

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L-matrix: a matrix $\langle \boldsymbol{A}, F\rangle$, where $F \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ $\qquad \mathbf{MOD}(\mathrm{L})$

1st completeness theorem: $\Gamma \vdash_{\mathrm{L}} \varphi$ iff $\Gamma \models_{\mathbf{MOD}(\mathrm{L})} \varphi$

Leibniz congruence and reduced models

A congruence θ of $\boldsymbol{A}$ is logical in a matrix $\langle \boldsymbol{A}, F\rangle$ if for each $a, b \in A$:

if $a \in F$ and $\langle a, b\rangle \in \theta$, then $b \in F$.

Definition

Let $\mathbf{A} = \langle \boldsymbol{A}, F\rangle$ be an $\mathcal{L}$-matrix. By $\Omega_{\boldsymbol{A}}(F)$ we denote the largest logical congruence on $\boldsymbol{A}$ and we call it Leibniz congruence of $\mathbf{A}$.

Definition

A matrix $\langle \boldsymbol{A}, F\rangle$ is reduced, if $\Omega_{\boldsymbol{A}}(F) = \mathrm{Id}_{\boldsymbol{A}}$.

For a logic L, by $\mathbf{MOD}^*(\mathrm{L})$ we denote the class of reduced L-matrices.

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2nd completeness theorem: $\Gamma \vdash_{\mathrm{L}} \varphi$ iff $\Gamma \models_{\mathbf{MOD}^*(\mathrm{L})} \varphi$

Bases and ∩-prime elements in closure systems

Let $\mathcal{C}$ be a closure system over A.

$X \in \mathcal{C}$ is ∩-prime if for each $Y, Z \subseteq \mathcal{C}$:

$$\text{if } X = Y \cap Z, \text{ then } X = Y \text{ or } X = Z.$$

$X \in \mathcal{C}$ is completely ∩-prime if for each set $\mathcal{Y} \subseteq \mathcal{C}$:

$$\text{if } X = \bigcap_{Y \in \mathcal{Y}} Y, \text{ then } X = Y \text{ for some } Y \in \mathcal{Y}.$$

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$\mathcal{B} \subseteq \mathcal{C}$ is a basis of $\mathcal{C}$ if for every $Y \in \mathcal{C}$ and $a \in A \setminus Y$ there is $Z \in \mathcal{B}$ such that $Y \subseteq Z$ and $a \notin Z$.

Lemma (Lindenbaum Lemma)

If L is finitary, then completely ∩-prime theories form a basis of $\mathrm{Th}(\mathrm{L})$.

RSI and RFSI reduced models

A matrix $\mathbf{A} \in \mathbb{K}$ is (finitely) subdirectly irreducible relative to $\mathbb{K}$, $\mathbf{A} \in \mathbb{K}_{\mathrm{R(F)SI}}$ in symbols, if for every (finite non-empty) subdirect representation $\mathbf{A}$ with a family $\{\mathbf{A}_i \mid i \in I\} \subseteq \mathbb{K}$ there is $i \in I$ such that π_i is an isomorphism.

Theorem

Given any logic L *and* $\mathbf{A} = \langle A, F\rangle \in \mathbf{MOD}^*(\mathrm{L})$, *we have:*

1. $\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$ *iff* F *is completely* $\cap$*-prime in* $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$.
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3rd compl. theorem (for finitary logics): $\Gamma \vdash_{\mathrm{L}} \varphi$ iff $\Gamma \models_{\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}} \varphi$

Classes of infinitary logics

A logic L has the

- CIPEP (completely $\cap$-prime extension property) if completely $\cap$-prime theories form a basis of $\mathrm{Th(L)}$
- IPEP ($\cap$-prime ext. property) if $\cap$-prime theories form a basis of $\mathrm{Th(L)}$

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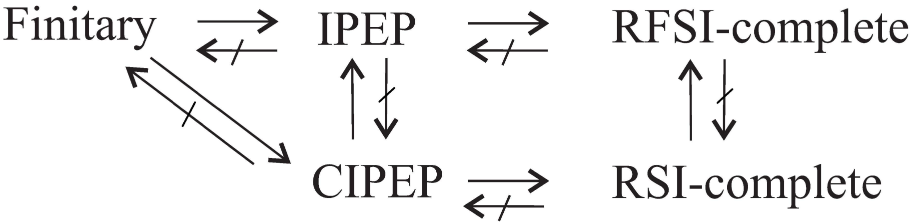

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2. Disjunctions
3. Implications
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Definition and useful conventions

Let $\nabla(p, q, \overrightarrow{r})$ be a set of formulas. We write

$$\begin{aligned}\varphi \nabla \psi &= \{\delta(\varphi, \psi, \vec{\alpha}) \mid \delta(p, q, \vec{s}) \in \nabla \text{ and } \vec{\alpha} \in Fm_{\mathcal{L}}\} \\ \Sigma_1 \nabla \Sigma_2 &= \bigcup \{\varphi \nabla \psi \mid \varphi \in \Sigma_1, \psi \in \Sigma_2\}\end{aligned}$$

Three kinds of disjunctions

A (parameterized) set of formulas ∇ is (p-)protodisjunction if

(PD) $\varphi \vdash \varphi \nabla \psi$ and $\psi \vdash \varphi \nabla \psi$

A (p-)protodisjunction ∇ is a

- weak (p-)disjunction if it satisfies:

  wPCP $\varphi \vdash_{\mathrm{L}} \chi$ and $\psi \vdash_{\mathrm{L}} \chi$ implies $\varphi \nabla \psi \vdash_{\mathrm{L}} \chi$

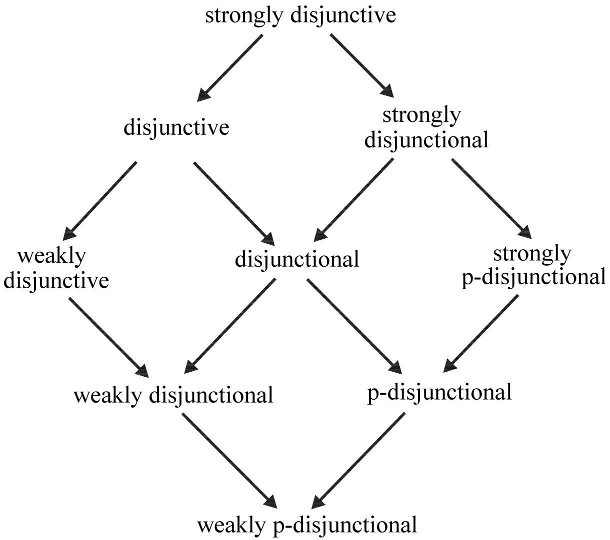
- (p-)disjunction if it satisfies:

  PCP $\Gamma, \varphi \vdash_{\mathrm{L}} \chi$ and $\Gamma, \psi \vdash_{\mathrm{L}} \chi$ implies $\Gamma, \varphi \nabla \psi \vdash_{\mathrm{L}} \chi$

- strong (p-)disjunction if it satisfies:

  sPCP $\Gamma, \Sigma \vdash_{\mathrm{L}} \chi$ and $\Gamma, \Pi \vdash_{\mathrm{L}} \chi$ implies $\Gamma, \Sigma \nabla \Pi \vdash_{\mathrm{L}} \chi$

The disjunctional hierarchy of logics



Characterizations

Theorem

Let L *be a logic with a presentation* $\mathcal{AS}$ *and* ∇ *a p-protodisjunction s.t.*

$$\psi \nabla \varphi \vdash_{\mathrm{L}} \varphi \nabla \psi \qquad\qquad \varphi \nabla \varphi \vdash_{\mathrm{L}} \varphi.$$

Then ∇ *is a strong p-disjunction iff for each* χ *and each* $\Gamma \rhd \varphi \in \mathcal{AS}$*:*

$$\Gamma \nabla \chi \vdash_{\mathrm{L}} \varphi \nabla \chi.$$

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We can show that $\quad \varphi \vee \chi, (\varphi \to \psi) \vee \chi \vdash_{Ł} \psi \vee \chi$

Thus $\vee$ is a (strong) disjunction in Ł and so

$$\frac{\neg\varphi \to \varphi^n \vdash_{Ł} \neg(\varphi \vee \chi) \to (\varphi \vee \chi)^n \qquad \chi \vdash_{Ł} \neg(\varphi \vee \chi) \to (\varphi \vee \chi)^n}{(\neg\varphi \to \varphi^n) \vee \chi \vdash_{Ł} \neg(\varphi \vee \chi) \to (\varphi \vee \chi)^n}$$

Then $\quad \{(\neg\varphi \to \varphi^n) \vee \chi \mid n \geq 0\} \vdash_{Ł_\infty} \varphi \vee \chi$

Thus $\vee$ is a strong disjunction in $Ł_\infty$

Characterizations

∇-prime filter in $\boldsymbol{A}$: if $\varphi \nabla^{\boldsymbol{A}} \psi \subseteq F$, then $\varphi \in F$ or $\psi \in F$

Theorem

Let L *be a logic with a p-protodisjunction* ∇. *TFAE:*

- L *has the* IPEP *and (strong)* ∇ *is a p-disjunction.*
- L *has the* IPEP *and* ∇*-prime and* $\cap$*-prime theories coincide.*
- L *has the* PEP, *i.e.,* ∇*-prime filters form a basis of* $\mathrm{Th}(\mathrm{L})$.

Infinitary Lindenbaum Lemma

Theorem (Infinitary Lindenbaum Lemma)

Let L *be a countably axiomatizable strongly disjunctional logic. Then* L *has the* $\mathrm{(I)PEP}$*, i.e,* $\cap/\nabla$*-prime theories form a basis of* $\mathrm{Th(L)}$*.*

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What is an implication?

Let $\overrightarrow{r}$ be a sequence of atoms and $\Rightarrow(p, q, \overrightarrow{r})$ a set of formulas.

Given formulas φ and ψ, we set

$$\varphi \Rightarrow \psi \quad = \quad \{\delta(\varphi, \psi, \vec{\alpha}) \mid \delta(p, q, \vec{s}) \in \Rightarrow \text{ and } \vec{\alpha} \in Fm_{\mathcal{L}}\}$$

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A set $\Rightarrow(p, q, \overrightarrow{r}) \subseteq Fm_{\mathcal{L}}$ is a weak p-implication in a logic L if:

$$\begin{array}{ll} \text{(R)} & \vdash_{\mathrm{L}} \varphi \Rightarrow \varphi \\ \text{(T)} & \varphi \Rightarrow \psi, \psi \Rightarrow \chi \vdash_{\mathrm{L}} \varphi \Rightarrow \chi \\ \text{(MP)} & \varphi, \varphi \Rightarrow \psi \vdash_{\mathrm{L}} \psi \\ \text{(sCng)} & \varphi \Rightarrow \psi, \psi \Rightarrow \varphi \vdash_{\mathrm{L}} c(\chi_1, \ldots, \varphi, \ldots, \chi_n) \Rightarrow c(\chi_1, \ldots, \psi, \ldots, \chi_n) \\ & \hfill \text{for each } \langle c, n \rangle \in \mathcal{L} \text{ and } i \leq n. \end{array}$$

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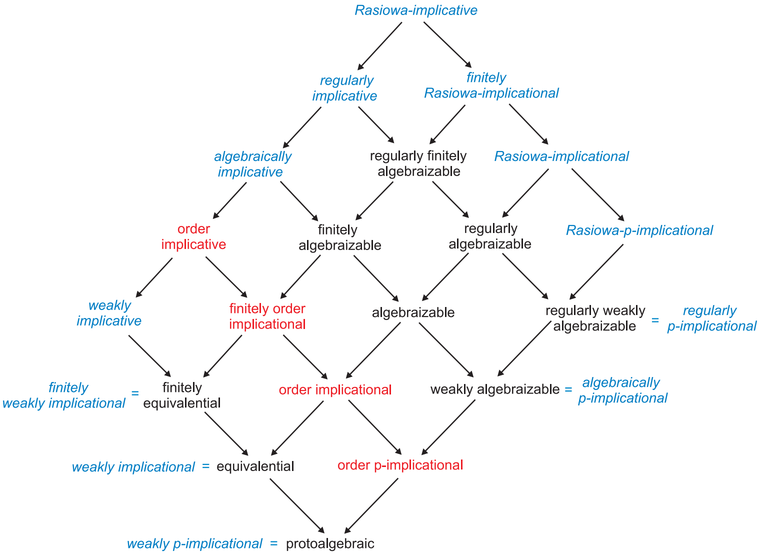
$\rightarrow$ is a weak implication in $\mathrm{Ł}_\infty$

Implicational hierarchy



Order in matrices and reduced matrices

Let $\Rightarrow$ be a weak p-implication in L and $\mathbf{A} = \langle A, F \rangle \in \mathbf{MOD}(\mathrm{L})$

Fact 1: The relation $a \leq_{\mathbf{A}}^{\Rightarrow} b$ is a preorder:

$$a \leq_{\mathbf{A}}^{\Rightarrow} b \qquad \text{iff} \qquad a \Rightarrow^{A} b \subseteq F$$

Fact 2: $\mathbf{A} \in \mathbf{MOD}^{*}(\mathrm{L})$ iff $\leq_{\mathbf{A}}^{\Rightarrow}$ is an order

Definition: F and $\mathbf{A}$ are $\Rightarrow$-linear, $\mathbf{A} \in \mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L})$, if $\leq_{\mathbf{A}}^{\Rightarrow}$ is linear

What is a semilinear implication?

Definition

Let L be a logic and $\Rightarrow$ one of its weak p-implications.
We say that $\Rightarrow$ is semilinear if

$$\mathrm{L} = \models_{\mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L})}.$$

L is a semilinear logic if it has a semilinear implication.

A general characterization of semilinear logics

Theorem

Let L *be a logic and* $\Rightarrow$ *a weak p-implication. TFAE:*

1. $\Rightarrow$ *is semilinear in* L, *i.e.* $\mathrm{L} = \models_{\mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L})}$.
2. L *has the* LEP *w.r.t.* $\Rightarrow$, *i.e.* $\Rightarrow$*-linear theories are a basis of* $\mathrm{Th}(\mathrm{L})$.
3. L *is* RFSI*-complete and any of the following conditions hold:*
   - 3a. L *has the* SLP *w.r.t.* $\Rightarrow$, *i.e., for each* $\Gamma \cup \{\varphi, \psi, \chi\} \subseteq Fm_{\mathcal{L}}$:

$$\frac{\Gamma, \varphi \Rightarrow \psi \vdash_{\mathrm{L}} \chi \qquad \Gamma, \psi \Rightarrow \varphi \vdash_{\mathrm{L}} \chi}{\Gamma \vdash_{\mathrm{L}} \chi}$$

   - 3b. L *has the transferred* SLP *w.r.t.* $\Rightarrow$,
   - 3c. $\Rightarrow$*-linear filters coincide with* $\cap$*-prime filters in each* $\mathcal{L}$*-algebra,*
   - 3d. $\mathbf{MOD}^{*}(\mathrm{L})_{\mathrm{RFSI}} = \mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L})$.
4. L *has the* IPEP *and* $\Rightarrow$*-linear theories coincide with* $\cap$*-prime ones.*
5. L *is* RSI*-complete and* $\mathbf{MOD}^{*}(\mathrm{L})_{\mathrm{RSI}} \subseteq \mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L})$.

$Ł_\infty$ is semilinear

We can easily prove that: $\vdash_{Ł} (\varphi \to \psi) \vee (\psi \to \varphi)$

We have shown that $\vee$ is a disjunction in $Ł_\infty$

Thus from $\Gamma, \varphi \to \psi \vdash_{Ł_\infty} \chi$ and $\Gamma, \psi \to \varphi \vdash_{Ł_\infty} \chi$ we get $\Gamma \vdash_{Ł_\infty} \chi$

We also know that $Ł_\infty$ has IPEP and so it is RFSI-complete

Thus $Ł_\infty$ is semilinear and $\mathbf{MOD}^*(Ł_\infty)_{\mathrm{RFSI}} = \mathbf{MOD}^\ell_\to(Ł_\infty)$

Outline

1 (In)finitary logics

2 Disjunctions

3 Implications

4 Disjunctions and implications

5 Completeness properties

Interplay with disjunction

Proposition

Let L *be a logic with a weak p-implication* $\Rightarrow$ *and p-protodisjunction* ∇.

- *If* $\Rightarrow$ *enjoys the* SLP, *then it holds:*

$$(\mathrm{P}^{\Rightarrow}_{\nabla}) \quad \vdash_{\mathrm{L}} (\varphi \Rightarrow \psi) \nabla (\psi \Rightarrow \varphi).$$

- *If* ∇ *is a p-disjunction, then it holds:*

$$(\mathrm{MP}^{\Rightarrow}_{\nabla}) \quad \varphi \Rightarrow \psi, \varphi \nabla \psi \vdash_{\mathrm{L}} \psi \quad \text{and} \quad \varphi \Rightarrow \psi, \psi \nabla \varphi \vdash_{\mathrm{L}} \psi.$$

Interplay with disjunction

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- *If* L *satisfies* $(\mathrm{P}^{\Rightarrow}_{\nabla})$*, then each* ∇*-prime filter in* $\boldsymbol{A}$ *is* $\Rightarrow$*-linear.*

Interplay with disjunction

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- *If* L *satisfies* $(\mathrm{P}^{\Rightarrow}_{\nabla})$, *then each* ∇*-prime filter in* $\boldsymbol{A}$ *is* $\Rightarrow$*-linear.*
- *If* L *satisfies* $(\mathrm{MP}^{\Rightarrow}_{\nabla})$, *then each* $\Rightarrow$*-linear filter in* $\boldsymbol{A}$ *is* ∇*-prime.*

Disjunction-based characterization of semilinear logics

Theorem

Let L *be a logic with a weak p-implication* $\Rightarrow$ *and a p-protodisjunction* ∇*. TFAE:*

1. L *satisfies* $(\mathrm{MP}^{\Rightarrow}_{\nabla})$ *and* $\Rightarrow$ *is semilinear in* L
2. L *satisfies* $(\mathrm{P}^{\Rightarrow}_{\nabla})$ *and has the* PEP *w.r.t.* ∇
3. L *satisfies* $(\mathrm{P}^{\Rightarrow}_{\nabla})$*, has the* IPEP *and* ∇ *is p-disjunction in* L*.*

Disjunction-based characterization of semilinear logics

Theorem

Let L *be a countably axiomatizable logic with a weak p-implication* $\Rightarrow$ *and a p-protodisjunction* ∇*. TFAE:*

1. L *satisfies* $(\mathrm{MP}_{\nabla}^{\Rightarrow})$ *and* $\Rightarrow$ *is semilinear in* L
2. L *satisfies* $(\mathrm{P}_{\nabla}^{\Rightarrow})$ *and has the* PEP *w.r.t.* ∇
3. L *satisfies* $(\mathrm{P}_{\nabla}^{\Rightarrow})$ *and* ∇ *is strong p-disjunction in* L.

Disjunction-based characterization of semilinear logics

Theorem

Let L *be a countably axiomatizable logic with a weak p-implication* $\Rightarrow$ *and a p-protodisjunction* ∇*. TFAE:*

1. L *satisfies* $(\mathrm{MP}^{\Rightarrow}_{\nabla})$ *and* $\Rightarrow$ *is semilinear in* L
2. L *satisfies* $(\mathrm{P}^{\Rightarrow}_{\nabla})$ *and has the* PEP *w.r.t.* ∇
3. L *satisfies* $(\mathrm{P}^{\Rightarrow}_{\nabla})$ *and* ∇ *is strong p-disjunction in* L
4. L *has a countable presentation* $\mathcal{AS}$ *such that:*

$$\vdash_{\mathrm{L}} (\varphi \Rightarrow \psi) \nabla (\psi \Rightarrow \varphi) \qquad \psi \nabla \varphi \vdash_{\mathrm{L}} \varphi \nabla \psi \qquad \varphi \nabla \varphi \vdash_{\mathrm{L}} \varphi.$$

and each $\Gamma \rhd \varphi \in \mathcal{AS}$*:*

$$\Gamma \nabla \chi \vdash_{\mathrm{L}} \varphi \nabla \chi.$$

Outline

1. (In)finitary logics
2. Disjunctions
3. Implications
4. Disjunctions and implications
5. Completeness properties

Three kinds of $\mathbb{K}$-completeness and the first theorem

Definition

Let L be a logic and $\mathbb{K} \subseteq \mathbf{MOD}^*(\mathrm{L})$. We say that L has the property of:

- Strong $\mathbb{K}$-completeness, $\mathrm{S}\mathbb{K}\mathrm{C}$ for short, when for every set of formulas $\Gamma \cup \{\varphi\}$: $\Gamma \vdash_{\mathrm{L}} \varphi$ iff $\Gamma \models_{\mathbb{K}} \varphi$ i.e., $\vdash_{\mathrm{L}} = \models_{\mathbb{K}}$
- Finite strong $\mathbb{K}$-completeness, $\mathrm{FS}\mathbb{K}\mathrm{C}$ for short, when for every finite set of formulas $\Gamma \cup \{\varphi\}$: $\Gamma \vdash_{\mathrm{L}} \varphi$ iff $\Gamma \models_{\mathbb{K}} \varphi$ i.e., $\mathcal{F}\mathrm{C}(\vdash_{\mathrm{L}}) = \mathcal{F}\mathrm{C}(\models_{\mathbb{K}})$
- $\mathbb{K}$-completeness, $\mathbb{K}\mathrm{C}$ for short, when for every formula φ: $\vdash_{\mathrm{L}} \varphi$ iff $\models_{\mathbb{K}} \varphi$

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- $\mathbb{K}$-completeness, $\mathbb{K}\mathrm{C}$ for short, when for every formula φ: $\vdash_{\mathrm{L}} \varphi$ iff $\models_{\mathbb{K}} \varphi$

Example 1: Ł separates $\mathrm{S}\mathbb{K}\mathrm{C}$ and $\mathrm{FS}\mathbb{K}\mathrm{C}$

Example 2: any non structurally complete logic separates $\mathrm{FS}\mathbb{K}\mathrm{C}$ and $\mathbb{K}\mathrm{C}$

Class operators

Slogan: Matrices can be regarded as first-order structures

A homomorphism $f\colon \langle \boldsymbol{A}, F\rangle \to \langle \boldsymbol{B}, G\rangle$: a morphism of algebras s.t. $f[F] \subseteq G$

- $\mathbf{S}(\mathbb{K})$: the class of all submatrices of elements of $\mathbb{K}$
- $\mathbf{S}^*(\mathbb{K}) = \{\langle \boldsymbol{A}, F\rangle/\Omega_{\boldsymbol{A}}(F) \mid \langle \boldsymbol{A}, F\rangle \in \mathbf{S}(\mathbb{K})\}$
- $\mathbf{I}(\mathbb{K})$: the class of all isomorphic images of elements of $\mathbb{K}$
- $\mathbf{H}(\mathbb{K})$: the class of all homomorphic images of elements of $\mathbb{K}$
- $\mathbf{P}(\mathbb{K})$: the class of all products of elements of $\mathbb{K}$
- $\mathbf{P}_{\mathrm{U}}(\mathbb{K})$: the class of all ultraproducts of elements of $\mathbb{K}$
- $\mathbf{P}_{\sigma\text{-}f}(\mathbb{K})$: the class of reduced products of $\mathbb{K}$ using filters closed under countable intersections

General characterization

Theorem

Let L *be a logic and* $\mathbb{K} \subseteq \mathbf{MOD}^*(\mathrm{L})$. *Then:*

1. L *has the* $\mathrm{FS}\mathbb{K}\mathrm{C}$ *if, and only if,* $\mathbf{MOD}^*(\mathrm{L}) \subseteq \mathbf{IS}^*\mathbf{PP}_{\mathrm{U}}(\mathbb{K})$.
2. L *has the* $\mathrm{S}\mathbb{K}\mathrm{C}$ *if, and only if,* $\mathbf{MOD}^*(\mathrm{L}) \subseteq \mathbf{IS}^*\mathbf{P}_{\sigma\text{-}f}(\mathbb{K})$.

Corollary

Let L *be a protoalgebraic logic and* $\mathbb{K} \subseteq \mathbf{MOD}^*(\mathrm{L})$. *Then:*

1. L *has the* $\mathbb{K}\mathrm{C}$ *if, and only if,* $\mathbf{H}(\mathbf{MOD}^*(\mathrm{L})) = \mathbf{HS}^*\mathbf{P}(\mathbb{K})$.
2. *If* L *is finitary then it has a finite p-implication and has the* $\mathrm{FS}\mathbb{K}\mathrm{C}$ *if, and only if,* $\mathbf{MOD}^*(\mathrm{L}) = \mathbf{IS}^*\mathbf{PP}_{\mathrm{U}}(\mathbb{K})$.
3. L *has the* $\mathrm{S}\mathbb{K}\mathrm{C}$ *if, and only if,* $\mathbf{MOD}^*(\mathrm{L}) = \mathbf{IS}^*\mathbf{P}_{\sigma\text{-}f}(\mathbb{K})$.

Characterization using RFSI-matrices

Theorem

Let L *be an* RFSI*-complete logic and* $\mathbb{K} \subseteq \mathbf{MOD}^*(\mathrm{L})$*. Then:*

1. *If* $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}} \subseteq \mathbf{HSP}_{\mathrm{U}}(\mathbb{K})$*, then* L *has the* $\mathbb{K}\mathrm{C}$*.*
2. *If* $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}} \subseteq \mathbf{IS}^*\mathbf{P}_{\mathrm{U}}(\mathbb{K})$*, then* L *has the* $\mathrm{FS}\mathbb{K}\mathrm{C}$*.*

If L *is p-disjunctional or finitary protoalgebraic, then the reverse implications also hold.*

Characterization using RFSI-matrices

Let us denote by $\mathbb{K}^\sigma$ the class of countable elements of $\mathbb{K}$

Theorem

Let L *be an* RSI*-complete logic and* $\mathbb{K} \subseteq \mathbf{MOD}^*(\mathrm{L})$. *Then:*

- *If* $\mathbf{MOD}^*(\mathrm{L})^\sigma_{\mathrm{RSI}} \subseteq \mathbf{IS}^*(\mathbb{K})$, *then* L *has the* $\mathrm{S}\mathbb{K}\mathrm{C}$

If L *is protoalgebraic, then the reverse implication also holds.*

Characterization using RFSI-matrices

Let us denote by $\mathbb{K}^\sigma$ the class of countable elements of $\mathbb{K}$

Theorem

Let L *be an* RFSI*-complete logic and* $\mathbb{K} \subseteq \mathbf{MOD}^*(\mathrm{L})$. *Then:*

- *If* $\mathbf{MOD}^*(\mathrm{L})^\sigma_{\mathrm{RFSI}} \subseteq \mathbf{IS}^*(\mathbb{K}^+)$, *then* L *has the* $\mathrm{S}\mathbb{K}\mathrm{C}$

If L *is finitary protoalgebraic logic with a lattice disjunction* $\vee$, *then the reverse implication also holds.*

A p-disjunction $\nabla = \{p \vee q\}$ is *lattice disjunction* in a protoalgebraic logic L with a weak p-implication $\Rightarrow$ if:

$$\begin{array}{ll} (\vee 1) & \vdash_{\mathrm{L}} \varphi \Rightarrow \varphi \vee \psi \\ (\vee 2) & \vdash_{\mathrm{L}} \psi \Rightarrow \varphi \vee \psi \\ (\vee 3) & \varphi \Rightarrow \chi, \psi \Rightarrow \chi \vdash_{\mathrm{L}} \varphi \vee \psi \Rightarrow \chi \end{array}$$

Back to our motivating example

Theorem

Ł_∞ *has the* $\mathrm{S}[0,1]_{\text{Ł}}\mathrm{C}$.

As Ł_∞ has the IPEP, it is RFSI-complete and so we only need to show:

$$\mathbf{MOD}^*(\text{Ł}_\infty)^\sigma_{\mathrm{RFSI}} \subseteq \mathbf{IS}^*([0,1]_{\text{Ł}}) = \mathbf{IS}([0,1]_{\text{Ł}})$$

We know that $\mathbf{MOD}^*(\text{Ł}_\infty)^\sigma_{\mathrm{RFSI}} = \mathbf{MOD}^\ell_\Rightarrow(\text{Ł}_\infty)^\sigma$

Fact: If $\mathbf{A} \in \mathbf{MOD}^\ell_\Rightarrow(\text{Ł}_\infty)^\sigma$, then $\boldsymbol{A}$ is simple

Back to our motivating example

Theorem

$Ł_\infty$ *has the* $\mathrm{S}[0,1]_{Ł}\mathrm{C}$.

As $Ł_\infty$ has the IPEP, it is RFSI-complete and so we only need to show:

$$\mathbf{MOD}^*(Ł_\infty)^\sigma_{\mathrm{RFSI}} \subseteq \mathbf{IS}^*([0,1]_{Ł}) = \mathbf{IS}([0,1]_{Ł})$$

We know that $\mathbf{MOD}^*(Ł_\infty)^\sigma_{\mathrm{RFSI}} = \mathbf{MOD}^\ell_\Rightarrow(Ł_\infty)^\sigma$

Fact: If $\mathbf{A} \in \mathbf{MOD}^\ell_\Rightarrow(Ł_\infty)^\sigma$, then A is simple

Proof of the fact: we know that if $\neg x \leq x \& \overset{n}{\ldots} \& x$ for each n, then $x = 1$

Take $x < 1$, then $\neg x > x \& \overset{n}{\ldots} \& x$ for some n and so $0 = x \& \neg x \geq x \& \overset{n+1}{\ldots} \& x$

Back to our motivating example

Theorem

Ł_∞ *has the* $\mathrm{S}[0,1]_{\text{Ł}}\mathrm{C}$.

As Ł_∞ has the IPEP, it is RFSI-complete and so we only need to show:

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We know that $\mathbf{MOD}^*(\text{Ł}_\infty)^\sigma_{\mathrm{RFSI}} = \mathbf{MOD}^\ell_\Rightarrow(\text{Ł}_\infty)^\sigma$

Fact: If $\mathbf{A} \in \mathbf{MOD}^\ell_\Rightarrow(\text{Ł}_\infty)^\sigma$, then $\boldsymbol{A}$ is simple

Algebraic fact: simple MV-chains are embeddable into $[0,1]_{\text{Ł}}$

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