Admissible rules and (almost) structural completeness for many-valued logics.

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XIV Congreso Dr. Monteiro
Bahía Blanca, 31 de Mayo, 1 y 2 de Junio de 2017
Given a logic $L$, an $L$-unifier of a formula $\varphi$ is a substitution $\sigma$ such that $\vdash_L \sigma \varphi$.

A rule $\Gamma / \varphi$ is $L$-admissible in $L$ iff every common $L$-unifier of $\Gamma$ is also an $L$-unifier of $\varphi$.

$\Gamma / \varphi$ is passive $L$-admissible in $L$ iff $\Gamma$ has no common $L$-unifier.
Admissibility Theory

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A logic is **structurally complete** iff every admissible rule is a derivable rule.
Admissibility Theory

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A rule $\Gamma/\varphi$ is $L$-admissible in $L$ iff every common $L$-unifier of $\Gamma$ is also an $L$-unifier of $\varphi$.

$\Gamma/\varphi$ is passive $L$-admissible in $L$ iff $\Gamma$ has no common $L$-unifier.

A logic is structurally complete iff every admissible rule is a derivable rule.

A logic is almost structurally complete iff every admissible rule is either derivable rule or a passive admissible.
CPC is structurally complete.
- CPC is structurally complete.
- IPC is not structurally complete.
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- Gödel logic is (hereditarily) structurally complete.
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- Infinite valued Łukasiewicz logic is not structurally complete.
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Gödel logic is (hereditarily) structurally complete.

Infinite valued Łukasiewicz logic is not structurally complete.

$n$-valued Łukasiewicz logic is not structurally complete but almost structurally complete.
CPC is structurally complete.
IPC is not structurally complete.
Gödel logic is (hereditarily) structurally complete.
Infinite valued Łukasiewicz logic is not structurally complete.
$n$-valued Łukasiewicz logic is not structurally complete but almost structurally complete.
Any $n$-contractive extension of Basic logic is almost structurally complete.
Algebraizable logics

\[
\text{Deductive Systems} \quad \leftrightarrow \quad \text{Quasivarieties}
\]

\[L \quad \leftrightarrow \quad K\]
Algebraizable logics

Deductive Systems $\iff$ Quasivarieties

$L \iff K$

$\langle \text{Prop}(X), \vdash_L \rangle \iff \langle \text{Eq}(X), \models_K \rangle$
Algebraizable logics

\[ \text{Deductive Systems} \leftrightarrow \text{Quasivarieties} \]

\[ L \leftrightarrow K \]

\[ \langle \text{Prop}(X), \vdash_L \rangle \leftrightarrow \langle \text{Eq}(X), \models_K \rangle \]

\[ \tau : \text{Prop}(X) \rightarrow \mathcal{P}(\text{Eq}(X)) \]

\[ \sigma : \text{Eq}(X) \rightarrow \mathcal{P}(\text{Prop}(X)) \]
Algebraizable logics

Deductive Systems $\leftrightarrow$ Quasivarieties

$L$ $\leftrightarrow$ $K$

$\langle \text{Prop}(X), \vdash_L \rangle$ $\leftrightarrow$ $\langle \text{Eq}(X), \models_K \rangle$

$\tau : \text{Prop}(X) \to \mathcal{P}(\text{Eq}(X))$

$\sigma : \text{Eq}(X) \to \mathcal{P}(\text{Prop}(X))$

$\Gamma \cup \{ \varphi \} \subseteq \text{Prop}(X)$

$\Sigma \cup \{ p \approx q \} \subseteq \text{Eq}(X)$

$\Gamma \vdash_L \varphi$ iff $\tau[\Gamma] \models_K \tau(\varphi)$

$\Sigma \models_K p \approx q$ iff $\sigma[\Sigma] \vdash_L \sigma(p \approx q)$

$\varphi \vdash_L \sigma(\tau(\varphi))$

$p \approx q \models_K \tau(\sigma(p \approx q))$
Algebraizable logics and Algebraic logic

Finitary Extensions of $L$ $\leftrightarrow$ Quasivarieties of $K$
Algebraizable logics and Algebraic logic

Finitary Extensions of $L$ $\leftrightarrow$ Quasivarieties of $\mathbb{K}$

Axiomatic Extensions $\leftrightarrow$ (Relative) Varieties
Algebraizable logics and Algebraic logic

Finitary Extensions of $L$ $\iff$ Quasivarieties of $K$

Axiomatic Extensions $\iff$ (Relative) Varieties

(Finite) Axiomatization $\iff$ (Finite) Axiomatization
### Algebraizable logics and Algebraic logic

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Given a quasivariety \( K \), we say that a quasiequation
\[
\alpha_1 \approx \gamma_1 & \cdots & \alpha_n \approx \gamma_n \Rightarrow \epsilon \approx \eta
\]
is \( K \)-admissible iff for every term substitution \( \sigma \) if
\( K \models \sigma(\alpha_i) \approx \sigma(\gamma_i) \) for \( i = 1 \div n \), then
\( K \models \sigma(\epsilon) \approx \sigma(\eta) \).

is passive in \( K \) iff there is no term substitution \( \sigma \) such that
\( K \models \sigma(\alpha_i) \approx \sigma(\gamma_i) \) for \( i = 1 \div n \).

\( K \) is structurally complete iff every \( K \)-admissible quasiequation is valid in \( K \).

\( K \) is almost structurally complete iff every admissible quasiequation is either valid in \( K \) or passive in \( K \).
Theorem (Rybakov 1997, Olson et al. 2008)

Let $L$ be an algebraizable logic and $K$ its quasivariety semantics, then $L$ is (almost) structurally complete iff $K$ is (almost) structurally complete.
Theorem (Bergman 1991)

Let $\mathcal{K}$ be a quasivariety, then the following properties are equivalent.

1. $\mathcal{K}$ is structurally complete.
2. Each proper subquasivariety of $\mathcal{K}$ generates a proper subvariety of $\mathcal{V}(\mathcal{K})$.
3. $\mathcal{K}$ is the least $\mathcal{V}(\mathcal{K})$-quasivariety.
4. $\mathcal{K} = Q(\text{Free}_{\mathcal{K}}(\omega)) = Q(\text{Free}_{\mathcal{V}(\mathcal{K})}(\omega))$. 
Theorem (Dzik-Stronkowski 2016)

Let $\mathbb{K}$ be a quasivariety. The following are equivalent

1. $\mathbb{K}$ is almost structurally complete.
2. For every $A \in \mathbb{K}$, $A \times \text{Free}_\mathbb{K}(\omega) \in Q(\text{Free}_\mathbb{K}(\omega))$.
3. For every $A \in \mathbb{K}$, if there is an homomorphism from $A$ into $\text{Free}_\mathbb{K}(\omega)$ then $A \in Q(\text{Free}_\mathbb{K}(\omega))$. 

Almost Structural completeness and free algebras
Almost Structural completeness and free algebras

Theorem (Dzik-Stronkowski 2016)

Let $\mathbb{K}$ be a quasivariety. The following are equivalent:

1. $\mathbb{K}$ is almost structurally complete.
2. For every $S \in \mathbb{K}_{SI}$, $S \times \text{Free}_\mathbb{K}(\omega) \in \mathcal{Q}(\text{Free}_\mathbb{K}(\omega))$.
3. For every $P \in \mathbb{K}_{FP}$, if there is an homomorphism from $A$ into $\text{Free}_\mathbb{K}(\omega)$ then $A \in \mathcal{Q}(\text{Free}_\mathbb{K}(\omega))$. 
Let $\mathbb{K}$ be a quasivariety. The following are equivalent

1. $\mathbb{K}$ is almost structurally complete.

2. There is $B$ a subalgebra of $\text{Free}_K(\omega)$, such that for every $S \in \mathbb{K}_{SI}$, $S \times B \in Q(\text{Free}_K(\omega))$.

3. For every $P \in \mathbb{K}_{FP}$, if there is an homomorphism from $A$ into $\text{Free}_K(\omega)$ then $A \in ISP(\text{Free}_K(\omega))$. 
Theorem (Dzik-Stronkowski 2016)

Let $K$ be a quasivariety. If $B_2$ is a subalgebra of $\text{Free}_K(\omega)$, then the following are equivalent

1. $K$ is almost structurally complete.
2. For every $S \in K_{SI}$, $S \times B_2 \in Q(\text{Free}_K(\omega))$.
3. For every $P \in K_{FP}$, if there is an homomorphism from $A$ into $\text{Free}_K(\omega)$ then $A \in ISP(\text{Free}_K(\omega))$. 
To algebraically study (almost) structural completeness of some algebraizable many-valued logics in order to characterize and axiomatize (all) finitary extensions.
To study (almost) structural completeness of some varieties and quasivarieties of (many-valued) algebras in order to characterize and axiomatize (all) subquasivarieties.
• Gödel logics.

• Nilpotent minimum logics.

• Łukasiewicz logics
Gödel-Dummett Logic (G) is the axiomatic extension of the Intuitionistic logic (IPC) given by the axiom

\[ \text{LIN} \ (\varphi \rightarrow \psi) \lor (\psi \rightarrow \varphi) \]
Gödel-Dummett Logic (G) is the axiomatic extension of the Intuitionistic logic (IPC) given by the axiom

\[ \text{LIN } (\varphi \to \psi) \lor (\psi \to \varphi) \]

**Standard semantics:**
Let \([0, 1]_G = \langle \{a \in \mathbb{R} : 0 \leq a \leq 1\}; \land, \lor, \to, \lnot, 0, 1 \rangle\). For every \(a, b \in [0, 1]\), \(a \land b = \min\{a, b\}\) and \(a \lor b = \max\{a, b\}\)

\[ a \to b = \begin{cases} 1, & \text{if } a \leq b; \\ b & \text{otherwise.} \end{cases} \]

and \(\lnot a := a \to 0 = \begin{cases} 1, & \text{if } a = 0; \\ 0 & \text{otherwise.} \end{cases}\)

\[ \Gamma \models_{[0, 1]_G} \varphi \text{ iff for every } h : \text{Prop}(x) \to [0, 1], \]
\[ h(\varphi) = 1 \text{ whenever } h\Gamma = \{1\} \]
Completeness Theorem

Theorem (Dummett 1959)

\[ \Sigma \vdash_G \varphi \text{ iff } \Sigma \models_{[0,1]} \varphi \]
Completeness Theorem

Theorem (Dummett 1959)

\[ \Sigma \vdash_G \varphi \iff \Sigma \models_{[0,1]} \varphi \]

Algebraic logic

The Gödel-Dummett logic is algebraizable with \( G \) the class of all Gödel-algebras as its equivalent quasivariety semantics.
A Gödel-algebra is an algebra \( \langle A, \wedge, \vee, \rightarrow, \neg, \bar{0}, \bar{1} \rangle \) such that

- \( \langle A, \wedge, \vee, \bar{0}, \bar{1} \rangle \) is a bounded distributive lattice.
- For every \( a, b \in A \), \( a \rightarrow b \) is the pseudocomplement of \( a \) relative to \( b \),
  i.e. \( a \rightarrow b = \max \{ c \in A : a \land c \leq b \} \).
- \( \neg a = a \rightarrow \bar{0} \).
- \( \text{(L)} \) For every \( a, b \in A \) \( (a \rightarrow b) \lor (b \rightarrow a) = \bar{1} \).
A **Gödel-algebra** is an algebra $\langle A, \land, \lor, \to, \neg, \bar{0}, \bar{1} \rangle$ such that

- $\langle A, \land, \lor, \bar{0}, \bar{1} \rangle$ is a bounded distributive lattice.
- For every $a, b \in A$, $a \to b$ is the pseudocomplement of $a$ relative to $b$, i.e. $a \to b = \max\{c \in A : a \land c \leq b\}$.
- $\neg a = a \to \bar{0}$.

(L) For every $a, b \in A$ $(a \to b) \lor (b \to a) = \bar{1}$.

A Gödel algebra is a Heyting algebra satisfying (L).
We say that a Gödel-algebra is a **Gödel-chain**, provided that it is totally ordered.
Gödel-chains

We say that a Gödel-algebra is a **Gödel-chain**, provided that it is totally ordered.

Let \( \langle A, \leq, 0, 1 \rangle \) be a totally ordered bounded set, if we define for every \( a, b \in A \),

\[
\begin{align*}
    a \wedge b &= \min \{ a, b \}, \\
    a \vee b &= \max \{ a, b \}, \\
    a \rightarrow b &= \begin{cases} 
        1, & \text{if } a \leq b; \\
        b, & \text{otherwise.}
    \end{cases} \\
    \neg a &= a \rightarrow 0 = \begin{cases} 
        1, & \text{if } a = 0; \\
        0, & \text{if } a \neq 0.
    \end{cases}
\end{align*}
\]

then \( A = \langle A, \wedge, \vee, \rightarrow, \neg, 0, 1 \rangle \) is a Gödel-chain.
We say that a Gödel-algebra is a **Gödel-chain**, provided that it is totally ordered.

Let \( \langle A, \leq, \bar{0}, \bar{1} \rangle \) be a totally ordered bounded set, if we define for every \( a, b \in A \),

\[
    a \land b = \min\{a, b\}, \quad a \lor b = \max\{a, b\},
\]

\[
    a \rightarrow b = \begin{cases} 
        \bar{1}, & \text{if } a \leq b; \\
        b, & \text{otherwise.} 
    \end{cases}, \quad \neg a = a \rightarrow \bar{0} = \begin{cases} 
        \bar{1}, & \text{if } a = 0; \\
        0, & \text{if } a \neq 0. 
    \end{cases}
\]

then \( A = \langle A, \land, \lor, \rightarrow, \neg, \bar{0}, \bar{1} \rangle \) is a Gödel-chain.

Every Gödel-chain is of this form.
Finite Gödel-chains

Therefore up to isomorphism for each natural number $n$, there is only one Gödel-chain $G_n$ with exactly $n$ elements.

$$G_n = \langle \{0, 1, 2, \ldots, n-1\}, \land, \lor, \rightarrow, \neg, 0, n-1 \rangle.$$

Notice that $G_1$ is the trivial algebra and $G_2$ is the 2-element Boolean algebra.

$G_n \hookrightarrow G_m$ iff $n \leq m$
G-variety

- $\mathcal{G}$ is a locally finite variety.
- $\mathcal{G} = \mathcal{V}([0, 1]_G) = \mathcal{V}(\{\mathcal{G}_n : n > 1\})$
G- varieties

- $G$ is a locally finite variety.
- $G = \mathcal{V}([0, 1]_G) = \mathcal{V}(\{G_n : n > 1\})$

A variety $\mathcal{V}$ of Gödel-algebras is proper subvariety of $G$ iff
$\mathcal{V} = G_n = \mathcal{V}(G_n)$ for some $n > 0$.

$G_n$ is axiomatizable by $\bigvee_{i < n} ((x_i \leftrightarrow x_{i+1}) \approx \overline{1})$

$G_1 \subsetneq G_2 \subsetneq G_3 \subsetneq \cdots \subsetneq G_n \subsetneq \cdots \subsetneq G$
G-variety

- $\mathcal{G}$ is a locally finite variety.
- $\mathcal{G} = \mathcal{V}([0,1]_G) = \mathcal{V}(\{G_n : n > 1\})$

A variety $\mathcal{V}$ of Gödel-algebras is proper subvariety of $\mathcal{G}$ iff $\mathcal{V} = G_n = \mathcal{V}(G_n)$ for some $n > 0$.

- $G_n$ is axiomatizable by $\bigvee_{i<n} ((x_i \leftrightarrow x_{i+1}) \approx \bar{1})$
- $G_1 \subsetneq G_2 \subsetneq G_3 \subsetneq \cdots \subsetneq G_n \subsetneq \cdots G$

- $\mathcal{G} = \mathcal{Q}([0,1]_G) = \mathcal{Q}(\{G_n : n > 1\})$.
- $G_n = \mathcal{Q}(G_n)$ for every $n > 0$. 
Theorem (Dzik-Wronski 1973)

Gödel logic is structurally complete.
Structural completeness of G

Theorem (Dzik-Wronski 1973)

Gödel logic is structurally complete.

For every $n > 1$, $G_n$ is embeddable into $\text{Free}_G(\omega)$. 
Structural completeness of G

**Theorem (Dzik-Wronski 1973)**

Gödel logic is structurally complete.

For every $n > 1$, $G_n$ is embeddable into $\text{Free}_{G}(\omega)$.

$Q(\text{Free}_{G}(\omega)) = Q(\{G_n : n > 1\}) = G$. 

*J. Gispert*

Structural Completeness for many-valued logics
Structural completeness of G

Theorem (Dzik-Wronski 1973)

Gödel logic is structurally complete.

For every $n > 1$, $G_n$ is embeddable into $\text{Free}_G(\omega)$.

$Q(\text{Free}_G(\omega)) = Q(\{G_n : n > 1\}) = G$.

Let $n > 1$. For every $2 \leq k \leq n$, $G_k$ is embeddable into $\text{Free}_{G_n}(\omega)$.

Theorem

Gödel logic is hereditarily structurally complete.
Every quasivariety of Gödel algebras is a variety.

\[ L_V(\mathcal{G}) = L_Q(\mathcal{G}) \]
Nilpotent Minimum Logic (NML) is the axiomatic extension of the Monoidal t-norm logic (MTL) given by the axioms

- **Inv** \( \neg
\neg \varphi \rightarrow \varphi \)
- **WNM** \( \psi \ast \varphi \rightarrow \bot \) \lor (\psi \land \varphi \rightarrow \psi \ast \varphi) \)
Nilpotent Minimum Logic (NML) is the axiomatic extension of the Monoidal t-norm logic (MTL) given by the axioms

\[ \text{Inv} \quad \neg \neg \varphi \rightarrow \varphi \]
\[ \text{WNM} \quad (\psi \ast \varphi \rightarrow \bot) \lor (\psi \land \varphi \rightarrow \psi \ast \varphi) \]

**Standard Semantics:** \( (\models_{[0,1]}^{NM}) \)
\[ [0,1]_{NM} = \langle [0,1]; \ast, \rightarrow, \land, \lor, \neg, 0, 1 \rangle \]
where for every \( a, b \in [0,1] \),
\[ a \land b = \min\{a, b\}, \quad a \lor b = \max\{a, b\}, \quad \neg a = 1 - a, \]
\[ a \ast b = \begin{cases} \min\{a, b\}, & \text{if } b > 1 - a; \\ 0, & \text{otherwise.} \end{cases} \]
\[ a \rightarrow b = \begin{cases} 1, & \text{if } a \leq b; \\ \max\{1 - a, b\}, & \text{otherwise.} \end{cases} \]
Completeness Theorem

Theorem (Esteva Godo 2001, Noguera et al 2008)

\[ \Sigma \vdash_{NML} \varphi \text{ iff } \Sigma \models_{[0,1]_{NM}} \varphi \]
Completeness Theorem

Theorem (Esteva Godo 2001, Noguera et al 2008)

\[ \Sigma \vdash_{NML} \varphi \iff \Sigma \models_{[0,1]_{NM}} \varphi \]

Algebraic logic

The Nilpotent Minimum Logic NML is algebraizable with \(\mathbf{NM}\) the class of all NM-algebras as its equivalent quasivariety semantics.
A **NM-algebra** is a bounded integral residuated lattice satisfying the following equations:

\[(x \rightarrow y) \lor (y \rightarrow x) \approx \bar{1} \quad (L)\]

\[\neg \neg x \approx x \quad (I)\]

\[\neg (x \ast y) \lor (x \land y \rightarrow x \ast y) \approx \bar{1} \quad (WNM)\]

---

**Example:** 

\([0, 1] \text{NM}\) is a NM-algebra.
A **NM-algebra** is a bounded integral residuated lattice satisfying the following equations:

\[(x \rightarrow y) \lor (y \rightarrow x) \approx \overline{1} \quad (L)\]

\[\neg \neg x \approx x \quad (I)\]

\[\neg(x \ast y) \lor (x \land y \rightarrow x \ast y) \approx \overline{1} \quad (WM)\]

**Example:** \([0, 1]_{NM}\) is a NM-algebra.
We say that a NM-algebra is a **NM-chain**, provided that it is totally ordered.
NM-chains

We say that a NM-algebra is a **NM-chain**, provided that it is totally ordered.

Let $\langle A, \leq, \bar{0}, \bar{1} \rangle$ a totally ordered bounded set equipped with an involutive negation $\neg$, 

\begin{align*}
A & = \langle A, \ast, \rightarrow, \land, \lor, \bar{0}, \bar{1} \rangle \\
\ast & = \{
\bar{0}, \text{ if } b \leq \neg a; \\
\land & = \min\{a, b\} \\
\rightarrow & = \{
\bar{1}, \text{ if } a \leq b; \\
\lor & = \max\{a, b\} \\
\land & = \min\{a, b\} \\
\lor & = \max\{a, b\}
\}
\end{align*}
**NM-chains**

We say that a NM-algebra is a **NM-chain**, provided that it is totally ordered.

Let \( \langle A, \leq, \bar{0}, \bar{1} \rangle \) a totally ordered bounded set equipped with an involutive negation \( \neg \), if we define for every \( a, b \in A \),

\[
a \ast b = \begin{cases} 
\bar{0}, & \text{if } b \leq \neg a; \\
 a \land b, & \text{otherwise.}
\end{cases}
\]

\[
a \rightarrow b = \begin{cases} 
\bar{1}, & \text{if } a \leq b; \\
 \neg a \lor b, & \text{otherwise.}
\end{cases}
\]

\[
a \land b = \min\{a, b\} \quad \quad \quad a \lor b = \max\{a, b\},
\]

then \( A = \langle A, \ast, \rightarrow, \land, \lor, \bar{0}, \bar{1} \rangle \) is a NM-chain.
We say that a NM-algebra is a **NM-chain**, provided that it is totally ordered.

Let $\langle A, \leq, \overline{0}, \overline{1} \rangle$ a totally ordered bounded set equipped with an involutive negation $\neg$, if we define for every $a, b \in A$,

\[
\begin{align*}
    a \ast b &= \begin{cases} 
        \overline{0}, & \text{if } b \leq \neg a; \\
        a \land b, & \text{otherwise.}
    \end{cases} \\
    a \rightarrow b &= \begin{cases} 
        \overline{1}, & \text{if } a \leq b; \\
        \neg a \lor b, & \text{otherwise.}
    \end{cases}
\end{align*}
\]

\[
\begin{align*}
    a \land b &= \min\{a, b\} \\
    a \lor b &= \max\{a, b\},
\end{align*}
\]

then $A = \langle A, \ast, \rightarrow, \land, \lor, \overline{0}, \overline{1} \rangle$ is a NM-chain.

Every NM-chain is of this form.
Finite NM-chains

Therefore up to isomorphism for each finite \( n \in \mathbb{N} \), there is only one NM-chain \( A_n \) with exactly \( n \) elements.

\[
A_{2n+1} = \langle [-n, n] \cap \mathbb{Z}, *, \to, \land, \lor, -n, n \rangle.
\]

\[
A_{2n} = \langle A_{2n+1} \setminus \{0\}, *, \to, \land, \lor, -n, n \rangle.
\]

For every \( n, k > 0 \),

- \( A_{2n} \hookrightarrow A_{2k+1} \) iff \( A_{2n} \hookrightarrow A_{2k} \) iff \( A_{2n+1} \hookrightarrow A_{2k+1} \) iff \( n \leq k \).
- \( A_{2n+1} \not\hookrightarrow A_{2k} \).
Let \( A \) be an NM-algebra, 
\( a \in A \) is a **negation fixpoint** (or just **fixpoint**, for short) iff 
\( \neg a = a \).

**Let \( C \) be an NM-chain. Then \( C \setminus \{c\} \) is the universe of a subalgebra of \( C \) which we denote by \( C^- \).**

\[ A_{2n} = A_{2n+1}^- \]
NM-varieties

Let

\[ S_n(x_0, \ldots, x_n) = \bigwedge_{i<n} ((x_i \rightarrow x_{i+1}) \rightarrow x_{i+1}) \rightarrow \bigvee_{i<n+1} x_i \]

\[ \nabla(x) = \neg(\neg x^2)^2 \quad \Delta(x) = (\neg(\neg x)^2)^2 \]

where \( x^2 \) is an abbreviation of \( x \ast x \).

**Lemma**

*Let \( A \) be an NM-chain. Then we have*

1. \( A \) does not have a fixpoint iff \( \nabla(x) \approx \Delta(x) \) holds in \( A \).
2. \( A \) has less than \( 2n + 2 \) elements if and only if \( S_n(x_0, \ldots, x_n) \approx 1 \) holds in \( A \).
NM is a locally finite variety.

\[ \text{NM} = \mathcal{V}([0, 1]_{NM}) = \mathcal{V}(\{A_n : n > 1\}) \]
**NM-varieties**

\[ \text{NM is a locally finite variety.} \]

\[ \text{NM} = \mathcal{V}([0, 1]_{NM}) = \mathcal{V}\left(\{A_n : n > 1\}\right) \]

\[ \text{NM}^- = \text{NM} + \nabla(x) \approx \Delta(x) \]

\[ \text{NM}^- = \mathcal{V}\left(\{A_{2n} : n > 0\}\right) \]
Theorem (Gispert 03)

Every nontrivial variety of NM-algebras is of one of the following types:

1. \( \text{NM} = \mathcal{V}([0, 1]) = \mathcal{V}({A_n : n > 1}) \)
2. \( \text{NM}^- = \mathcal{V}([0, 1]^-) = \mathcal{V}({A_{2n} : n > 0}) \)
3. \( \text{NM}_{2m+1} = \mathcal{V}(A_{2m+1}) \text{ for some } m > 0 \)
4. \( \text{NM}_{2n} = \mathcal{V}(A_{2n}) \text{ for some } n > 0 \)
5. \( \text{NM}_{2n2m+1} = \mathcal{V}({A_{2n}, A_{2m+1}}) \text{ for some } n > m > 0 \)
6. \( \text{NM}^-_{2m+1} = \mathcal{V}({[0, 1]^-, A_{2m+1}}) = \mathcal{V}({A_{2n} : n > 0} \cup \{A_{2m+1}\}) \)
NM-varieties as quasivarieties

Theorem (Noguera et al. 08)

Every nontrivial variety of NM-algebras is of one of the following types:

1. $\text{NM} = \mathcal{Q}([0, 1]) = \mathcal{Q}({\mathcal{A}_n : n > 1})$
2. $\text{NM}_{\neg} = \mathcal{Q}([0, 1]^\neg) = \mathcal{Q}({\mathcal{A}_{2n} : n > 0})$
3. $\text{NM}_{2m+1} = \mathcal{Q}(\mathcal{A}_{2m+1}) \text{ for some } m > 0$
4. $\text{NM}_{2n} = \mathcal{Q}(\mathcal{A}_{2n}) \text{ for some } n > 0$
5. $\text{NM}_{2n2m+1} = \mathcal{Q}({\mathcal{A}_{2n}, \mathcal{A}_{2m+1}}) \text{ for some } n > m > 0$
6. $\text{NM}_{\neg 2m+1} = \mathcal{Q}([0, 1]^\neg, \mathcal{A}_{2m+1}) = \mathcal{Q}({\mathcal{A}_{2n} : n > 0} \cup \{\mathcal{A}_{2m+1}\})$
Lattice of NM-varieties

Subvarieties of NM
Proposition

\textbf{NM is not structurally complete.}
Proposition

\( \text{NM is not structurally complete.} \)

Proof:

\( \neg x \approx x \Rightarrow \bar{0} \approx \bar{1} \) is \( \text{NM-} \)admissible (passive) but not valid in \( \text{NM} \).
Theorem

\[ \text{NM} - \textit{is hereditarily structurally complete.} \]
Structural completeness of NM-

**Theorem**

\(\text{NM}^-\) is hereditarily structurally complete.

**Proposition**

For every \(n > 0\), \(A_{2n}\) is embeddable into \(\text{Free}_{\text{NM}^-}(\omega)\).
Structural completeness of NM-

Theorem

$\text{NM} -$ is hereditarily structurally complete.

Proposition

For every $n > 0$, $A_{2n}$ is embeddable into $\text{Free}_{\text{NM} -} (\omega)$.

\[
Q(\text{Free}_{\text{NM} -} (\omega)) = Q(\{A_{2n} : n > 0\}) = \text{NM} -
\]

For every $n > 0$, $Q(\text{Free}_{\text{NM}2n} (\omega)) = Q(A_{2n}) = \text{NM}2n$
Almost structural completeness of NM

If $M \not\subseteq \text{NM}$, then

**Proposition**

For every $k > 1$,

$A_2 \times A_k$ is embeddable into $\text{Free}_M(\omega)$ if and only if $A_k \in M$
Almost structural completeness of NM

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**Proposition**

For every $k > 1$, $A_2 \times A_k$ is embeddable into $\text{Free}_M(\omega)$ if and only if $A_k \in M$

$$Q(\text{Free}_M(\omega)) = Q(\{A_2 \times A_k : A_k \in M\})$$
Almost structural completeness of NM

If $M \not\subseteq \text{NM}$, then

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For every $k > 1$, $A_2 \times A_k$ is embeddable into $\text{Free}_M(\omega)$ if and only if $A_k \in M$

**Theorem**

$Q(\text{Free}_M(\omega)) = Q(\{A_2 \times A_k : A_k \in M\})$

$M$ is almost structurally complete
Almost structural completeness of NM

Theorem

\( \text{NM} \) is almost structurally complete and all their subvarieties are almost structurally complete.
Axiomatization of admissible quasiequations

Theorem

For every variety of NM-algebras the quasiequation

\[ \neg x \approx x \Rightarrow \bar{0} \approx \bar{1} \]

axiomatizes all passive admissible quasiequations.
Axiomatization of admissible quasiequations

**Theorem**

For every variety of NM-algebras the quasiequation
\[ \neg x \approx x \Rightarrow \tilde{0} \approx \tilde{1} \]
axiomatizes all passive admissible quasiequations.

**Proof:**

(Jeřábek 2010)

The rule \[ \neg (p \vee \neg p)^n / \bot \] axiomatizes all passive rules for every \(n\)-contractive axiomatic extension of MTL.

\(\text{NM}\) is 2 contractive (\(x^2 \approx x^3\))

\[ \neg p \leftrightarrow p \vdash_{\text{NML}} \neg (p \vee \neg p)^2 \]
Proposition

Let $\mathbb{M}$ be a non trivial variety of NM-algebras and $\mathbb{K}$ be an $\mathbb{M}$-quasivariety. Then $\mathbb{K}$ is a proper $\mathbb{M}$-quasivariety iff there is $\mathbb{A}_{2n+1} \in \mathbb{M} \setminus \mathbb{K}$ for some $n > 1$. 
Theorem

Let $\mathcal{M}$ be a non trivial NM-variety. If $\mathbf{K}$ is proper $\mathcal{M}$-quasivariety and $k = \max \{ n \in \mathbb{N} : A_{2n+1} \in \mathbf{K} \}$, then

$$\mathbf{K} = \mathcal{Q}(\{A_{2n} : A_{2n} \in \mathcal{M}\} \cup \{A_2 \times A_{2m+1} : A_{2m+1} \in \mathcal{M}\} \cup \{A_{2k+1}\})$$

Moreover, $\mathbf{K}$ is axiomatized relative to $\mathcal{M}$ by the quasiequation

$$x \approx \neg x \Rightarrow S_k(x_0, \ldots, x_k) \approx 1 \text{ if } k > 0$$

or

$$x \approx \neg x \Rightarrow 0 \approx 1 \text{ if } k = 0.$$
Quasivarieties of $\text{NM}$

**Theorem**

$$L_Q(\text{NM}) \cong \langle \{(n, m, k) \in (\omega^+)^3 : n \geq m \geq k \}, \leq^3 \rangle$$

where

$$(n_1, m_1, k_1) \leq^3 (n_2, m_2, k_2) \text{ iff } n_1 \leq n_2, m_1 \leq m_2 \text{ and } k_1 \leq k_2$$
Quasivarieties of NM

$\text{Lq}(\text{NM})$

- NM3 = MV3
- (0,0,0)
- (1,0,0)
- (2,0,0)
- NM2 = B
- (1,1,1)
- NM4
- (2,0,0)
- (0,0,0)
- NM5
- (2,2,2)
- NM12,9
- (6,4,2)
- (6,4,4)
- NM11
- (ω,ω,ω)
- (ω,ω,2)
- (ω,ω,1)
- (ω,ω,0)

J. Gispert
Structural Completeness for many-valued logics
Łukasiewicz logics

The Infinite valued Łukasiewicz Calculus, Ł∞

Axioms:

Ł1. $\varphi \rightarrow (\psi \rightarrow \varphi)$
Ł2. $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \nu) \rightarrow (\varphi \rightarrow \nu))$
Ł3. $((\varphi \rightarrow \psi) \rightarrow \psi) \rightarrow ((\psi \rightarrow \varphi) \rightarrow \varphi)$
Ł4. $(\neg \psi \rightarrow \neg \varphi) \rightarrow (\varphi \rightarrow \psi)$

Rules:

Modus Ponens. $\{\varphi, \varphi \rightarrow \psi\}/\psi$. 
Original logic semantics

\[ [0, 1]_L = \langle \{ a \in \mathbb{R} : 0 \leq a \leq 1 \}; \rightarrow, \neg \rangle \]

For all \( a, b \in [0, 1], \)

\[ a \rightarrow b = \begin{cases} 1, & \text{if } a \leq b; \\ 1 - a + b, & \text{otherwise.} \end{cases} \]

\[ \neg a = 1 - a. \]
Original logic semantics

\[ [0, 1]_L = \langle \{ a \in \mathbb{R} : 0 \leq a \leq 1 \}; \rightarrow, \neg \rangle \]

For all \( a, b \in [0, 1] \),
\[ a \rightarrow b = \begin{cases} 1, & \text{if } a \leq b; \\ 1 - a + b, & \text{otherwise}. \end{cases} \]
\[ \neg a = 1 - a. \]

Let \( \Gamma \cup \{ \varphi \} \subseteq \text{Prop}(X) \), then

\[ \Gamma \models_{[0,1]} \varphi \text{ iff } \]
\[ \text{for every } h : \text{Prop}(x) \rightarrow [0, 1], \ h(\varphi) = 1 \text{ whenever } h\Gamma = \{1\} \]
Weak Completeness Theorem

Theorem (Rose-Rosser 1958, Chang 1959)

\[ \vdash_{L_\infty} \varphi \iff \models_{[0,1]} \varphi \]
Completeness Theorems

Weak Completeness Theorem

Theorem (Rose-Rosser 1958, Chang 1959)

$$\vdash_{L_\infty} \varphi \text{ iff } \models_{[0,1]} \varphi$$

Strong Finite Completeness Theorem

Theorem (Hay 1963)

$$\varphi_1, \ldots, \varphi_n \vdash_{L_\infty} \varphi \text{ iff } \varphi_1, \ldots, \varphi_n \models_{[0,1]} \varphi$$
The infinite valued Łukasiewicz calculus $L_\infty$ is algebraizable with $MV$ the class of all MV-algebras as its equivalent quasivariety semantics.
An **MV-algebra** is an algebra $\langle A, \oplus, \neg, 0 \rangle$ satisfying the following equations:

- **MV1** \((x \oplus y) \oplus z \approx x \oplus (y \oplus z)\)
- **MV2** \(x \oplus y \approx y \oplus x\)
- **MV3** \(x \oplus 0 \approx x\)
- **MV4** \(\neg(\neg x) \approx x\)
- **MV5** \(x \oplus \neg 0 \approx \neg 0\)
- **MV6** \(\neg(\neg x \oplus y) \oplus y \approx \neg(\neg y \oplus x) \oplus x.\)
\begin{itemize}
\item $1 =_{\text{def}} \neg 0$.
\item $x \rightarrow y =_{\text{def}} \neg x \oplus y$.
\item $x \lor y =_{\text{def}} (x \rightarrow y) \rightarrow y$.
\item $x \land y =_{\text{def}} \neg (\neg x \lor \neg y)$.
\item $x \odot y =_{\text{def}} \neg (\neg x \oplus \neg y)$.
\end{itemize}

For any MV-algebra $A$, $a \leq b$ iff $a \rightarrow b = 1$ endows $A$ with a distributive lattice-order $\langle A, \lor, \land \rangle$, called the \textit{natural order} of $A$.

An MV-algebra whose natural order is total is said to be an MV-chain.
Let $\langle G, +, -, 0, \leq \rangle$ be a totally ordered abelian group and an element $0 < u \in G$, if we define $\Gamma(G, u) = \langle [0, u], \oplus, \neg, 0 \rangle$ by

$[0, u] = \{ a \in G \mid 0 \leq a \leq u \}, \quad a \oplus b = u \land (a + b), \quad \neg a = u - a,$

then $\langle [0, u], \oplus, \neg, 0 \rangle$ is an MV-chain.

Moreover every MV-chain is of this form.
Examples

- \([0, 1]_L = \Gamma(\mathbb{R}, 1)\),

- \([0, 1]_L \cap \mathbb{Q} = \Gamma(\mathbb{Q}, 1)\),

For every \(0 < n < \omega\)

- \(L_n = \Gamma(\mathbb{Z}, n) = \langle\{0, 1, \ldots, n\}, \oplus, \neg, 0\rangle\),

- \(L_n^\omega = \Gamma(\mathbb{Z} \times \text{lex} \ Z, (n, 0)) = \langle\{(k, i) : (0, 0) \leq (k, i) \leq (n, 0)\}, \oplus, \neg, 0\rangle\).

- \(L_n^s = \Gamma(\mathbb{Z} \times \text{lex} \ Z, (n, s)) = \langle\{(k, i) : (0, 0) \leq (k, i) \leq (n, s)\}, \oplus, \neg, 0\rangle\), where \(0 \leq s < n\).

- \(S_n = \Gamma(T, n)\) where \(T\) is a totally ordered dense subgroup of \(\mathbb{R}\) such that \(T \cap \mathbb{Q} = \mathbb{Z}\).
For every $0 < n < \omega$, every $n + 1$ element MV-chain is isomorphic to $L_n = \Gamma(\mathbb{Z}, n) = \langle \{0, 1, \ldots, n\}, \oplus, \neg, 0 \rangle$

Let $0 < n, k < \omega$. $L_n \leftrightarrow L_k$ if and only if $n | k$. 
The class $\mathbf{MV}$ of all MV-algebras is a (not locally finite) variety.

$\mathbf{MV} = \mathcal{V}([0,1]) = \mathcal{V}({\mathbb{L}_n} : n > 0)$.

For every $n > 0$, $\mathbf{MV}_n = \mathcal{V}(\mathbb{L}_n)$ is a locally finite variety. $\mathbf{MV}_n$ is the equivalent quasivariety semantics of $\mathbb{L}_{n+1}$ the $n + 1$-valued Łukasiewicz logic.

Moreover if $\mathcal{V}$ is a variety of MV-algebras, $\mathcal{V}$ is locally finite iff $\mathcal{V} \subseteq \mathbf{MV}_n$ for some $n > 0$.
Theorem (Komori, 1981)

\[ \forall \text{ is a proper subvariety of } \mathbb{MV} \text{ iff there exist two finite sets } I \text{ and } J \text{ (in a reduced form) of integers } \geq 1, \text{ not both empty, such that} \]

\[ \forall = \forall_{I,J} := \forall(\{L_m \mid m \in I\} \cup \{L^\omega_n \mid n \in J\}). \]
**MV-varieties**

**Theorem (Komori, 1981)**

$\mathcal{V}$ is a proper subvariety of $\mathbb{MV}$ iff there exist two finite sets $I$ and $J$ (in a reduced form) of integers $\geq 1$, not both empty, such that

$$\mathcal{V} = \mathcal{V}_{I,J} := \mathcal{V}(\{L_m \mid m \in I\} \cup \{L^n_\omega \mid n \in J\}).$$

- Every proper subvariety of $\mathbb{MV}$ is finitely axiomatizable.
- The lattice of all varieties of MV-algebras is a Pseudo-Boolean algebra.
MV-varieties as quasivarieties

\[ MV = \mathcal{Q}([0, 1] \cap \mathbb{Q}) = \mathcal{Q}([0, 1]) = \mathcal{Q}(\{L_n : n > 0\}). \]

\[ \mathcal{V}_{I,J} := \mathcal{Q}(\{L_m \mid m \in I\} \cup \{L_n^\omega \mid n \in J\}). \]
Theorem (Pogorzelski, Torzak, Wojtylak 1970’s, Dzik 2008, Jerabek 2010)

- $L_\infty (\text{MV})$ is not structurally complete.
- $L_\infty (\text{MV})$ is not almost structurally complete.
- $L_{n+1} (\text{MV}_n)$ is not structurally complete.
- $L_{n+1} (\text{MV}_n)$ is hereditarily almost structurally complete.
Structural completeness of Łukasiewicz logics

Theorem

- $\mathcal{V}_{\emptyset,\{1\}} = \mathcal{V}(L_1^\omega)$ is structurally complete.
- $\mathcal{V}_{\emptyset,\{1\}}$ and $\mathbb{B}$ are the only structurally complete varieties of MV-algebras.
- $\mathcal{V}$ is almost structurally complete iff $\mathcal{V}$ is locally finite or $\mathcal{V} = \mathcal{V}_{I,\{1\}}$ for some reduced pair $(I,\{1\})$. 
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For every reduced pair $(I, J)$,

$Q(Free_{\mathcal{V}_{I,J}}) = Q(\{L_1 \times L_n : n \in I\} \cup \{L_1 \times L_m^1 : m \in J\})$. 

J.Gispert Structural Completeness for many-valued logics
Structural completeness of Łukasiewicz logics

**Theorem**

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For every reduced pair $(I,J)$,
- $Q(\text{Free}_{\mathcal{V}_{I,J}}) = Q(\{L_1 \times L_n : n \in I\} \cup \{L_1 \times L_1^m : m \in J\})$.
- $Q(\text{Free}_{\mathcal{V}_{\emptyset,\{1\}}}) = Q(L_1 \times L_1^1) = Q(L_1^1) = Q(L_1^\omega) = \mathcal{V}(L_1^\omega)$. 
Structural completeness of Łukasiewicz logics

**Theorem**

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- $Q(\text{Free}_{\mathcal{V}_{I,\{1\}}}) = Q(\{L_1 \times L_n : n \in I\} \cup \{L_1 \times L_1^1\})$.
- $\mathcal{V}_{I,\{1\}} = Q(\{L_n : n \in I\} \cup \{L_1^1\})$. 
Theorem (Adams-Dziobiak)

The class $\mathbb{MV}$ is $Q$-universal, in the sense that, for every quasivariety $\mathbb{K}$ of algebras of finite type (not necessarily MV-algebras), the lattice of all quasivarieties of $\mathbb{K}$ is the homomorphic image of a sublattice of the lattice of all quasivarieties of $\mathbb{MV}$.

$$L_Q(\mathbb{K}) \in HS(L_Q(\mathbb{MV}))$$
Locally finite MV-quasivarieties

In the case of MV-algebras:
Locally finite MV-quasivarieties

In the case of MV-algebras:

The following conditions are equivalent:

1. $K$ is a locally finite quasivariety.
2. $K \subseteq MV_n$ for some $n \in \mathbb{N}$.
3. $K$ is subquasivariety contained in a discriminator variety.

Vaggione et al: "The subquasivariety lattice of a discriminator variety"

Structural Completeness for many-valued logics
Locally finite MV-quasivarieties

In the case of MV-algebras:

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Vaggione et al: "The subquasivariety lattice of a discriminator variety"
Locally finite MV-quasivarieties

Every locally finite quasivariety of MV-algebras is generated by a finite set of critical algebras.

A critical algebra is a finite algebra not belonging to the quasivariety generated by all its proper subalgebras. From the characterization of critical MV-algebras (Gispert-Torrens)
Every locally finite quasivariety of MV-algebras is generated by a finite set of critical algebras.

A critical algebra is a finite algebra not belonging to the quasivariety generated by all its proper subalgebras. From the characterization of critical MV-algebras (Gispert-Torrens)

Let $\mathbb{V}$ be a locally finite MV-variety. Then

- $L_Q(\mathbb{V})$ is finite
- Every member of $L_Q(\mathbb{V})$ is finitely based.

Moreover for any $\mathbb{K} \in L_Q(\mathbb{V})$,

$\mathbb{K}$ is a variety iff $\mathbb{K}$ is generated by MV-chains.
Quasivarieties generated by MV-chains

**Theorem**

Two MV-chains generate the same quasivariety iff they generate the same variety and they contain the same rational elements.

Given $A = \Gamma(G, b)$, $a$ is a **rational element** of $A$ iff there exist $m, n \in \omega$, $0 \leq n \leq m \neq 0$ and $c \in G$ such that $b = mc$ and $a = nc$. In that case, we say that $a = \frac{n}{m}$. 
Quasivarieties generated by MV-chains

**Theorem**

$K$ is a quasivariety generated by MV-chains if and only if there are $\alpha, \gamma, \kappa$ subsets of positive integers, not all of them empty, and for every $i \in \gamma$, a nonempty subset $\gamma(i) \subseteq \text{Div}(i)$ such that

$$K = \mathcal{Q}(\{L_n : n \in \alpha\} \cup \{L_i^{d_i} : i \in \gamma, \ d_i \in \gamma(i)\} \cup \{S_k : k \in \kappa\}).$$
Quasivarieties generated by MV-chains

**Theorem**

\( K \) is a quasivariety generated by MV-chains if and only if there are \( \alpha, \gamma, \kappa \) subsets of positive integers, not all of them empty, and for every \( i \in \gamma \), a nonempty subset \( \gamma(i) \subseteq \text{Div}(i) \) such that

\[
K = Q(\{L_n : n \in \alpha\} \cup \{L_i^{d_i} : i \in \gamma, \ d_i \in \gamma(i)\} \cup \{S_k : k \in \kappa\}).
\]

- Every quasivariety generated by MV-chains contained in a proper subvariety of \( \text{MV} \) is finitely axiomatizable.
- The lattice of all quasivarieties generated by MV-chains is a bounded distributive lattice.
From the characterization of quasivarieties generated by MV-chains it can be deduced:
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- $Q(L_n^1)$ is the least $V(L_n^\omega)$-quasivariety generated by chains.
From the characterization of quasivarieties generated by MV-chains it can be deduced:

- $Q(L_n^1)$ is the least $\mathcal{V}(L_n^\omega)$-quasivariety generated by chains.

- $Q(L_n)$ is the least $\mathcal{V}(L_n)$-quasivariety generated by chains.
From the characterization of quasivarieties generated by MV-chains it can be deduced:

- $Q(L_{1}^{n})$ is the least $\mathcal{V}(L_{n}^{\omega})$-quasivariety generated by chains.

- $Q(L_{n})$ is the least $\mathcal{V}(L_{n})$-quasivariety generated by chains.

- For every reduced pair $(I, J)$, $Q(\{L_{n} : n \in I\} \cup \{L_{m}^{1} : m \in J\})$ is the least $\mathcal{V}_{I,J}$-quasivariety generated by chains.
Structurally complete quasivarieties and least $\mathcal{V}$-quasivarieties.

**Theorem**

For every reduced pair $(I, J)$,

\[ Q(\{L_1 \times L_n : n \in I\} \cup \{L_1 \times L^1_m : m \in J\}) = Q(\text{Free}_{\mathcal{V}_{I,J}}) \]

and therefore it is the least $\mathcal{V}_{I,J}$-quasivariety.
For every reduced pair \((I, J)\),

- \(Q(\{L_1 \times L_n : n \in I\} \cup \{L_1 \times L^1_m : m \in J\})\) is the least \(\mathcal{V}_{I,J}\)-quasivariety.
- \(Q(\{L_n : n \in I\} \cup \{L^1_m : m \in J\})\) is the least \(\mathcal{V}_{I,J}\)-quasivariety generated by chains.
(Almost) structural completeness again

For every reduced pair \((I, J)\),

- \(Q(\{L_1 \times L_n : n \in I\} \cup \{L_1 \times L_m^1 : m \in J\})\) is the least \(V_{I,J}\)-quasivariety.

- \(Q(\{L_n : n \in I\} \cup \{L_m^1 : m \in J\})\) is the least \(V_{I,J}\)-quasivariety generated by chains.

Thus,

**Theorem**

*Let \((I, J)\) be a reduced pair. Then \(Q(\{L_n : n \in I\} \cup \{L_m^1 : m \in J\})\) is almost structurally complete.*
Axiomatization of admissible rules.

\( \text{MV-} \)admissible quasiequations.

(Jeřábek)

- An infinite basis of non passive admissible rules in order to axiomatize all admissible \( \mathcal{L}_\infty \)-rules. Infinite axiomatization of \( \text{MV-} \)admissible quasiequations.
- \( \text{MV-} \)admissible quasiequations are not finitely axiomatizable.
- Let \( \mathbb{V} \) be a variety of \( \text{MV-} \)algebras. Then
  \[ \{(x \lor \neg x)^n \approx 0 \Rightarrow 0 \approx 1 : n \in \omega\} \text{ is a basis for passive } \lor\text{-admissible quasiequations.} \]
Axiomatization of admissible rules.

\( \text{MV}\)-admissible quasiequations.

\((\text{Jeřábek})\)

- An infinite basis of non passive admissible rules in order to axiomatize all admissible \( \mathcal{L}_\infty \)-rules. Infinite axiomatization of \( \text{MV}\)-admissible quasiequations.
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- Let \( \mathcal{V} \) be a variety of \( \text{MV}\)-algebras. Then 
  \( \{ (x \lor \neg x)^n \approx 0 \Rightarrow 0 \approx 1 : n \in \omega \} \) is a basis for passive \( \lor \)-admissible quasiequations.

\( \mathcal{Q}(\text{Free}_{\text{MV}}) = \mathcal{Q}(\mathcal{M}([0, 1])) \) is the only almost structurally complete \( \text{MV}\)-quasivariety
Axiomatization of admissible rules.

Admissible quasiequations in locally finite MV-varieties

- Let $\forall$ be an MV-variety. Then $\forall$ is locally finite iff $\forall$ is $n$-contractive for some $n \in \omega$.

- Every locally finite MV-variety is almost structurally complete. (Dzik)

- $(x \lor \neg x)^n \approx 0 \Rightarrow 0 \approx 1$ is a basis of passive admissible quasiequations for every $n$-contractive subvariety of $\text{MV}$. (Jeřábek)
Let $\mathcal{V}_{I,J}$ be a proper subvariety of $\text{MV}$.

- $Q_{I,J}^1 := Q(\{L_n : n \in I\} \cup \{L^1_m : m \in J\})$ is almost structurally complete.
Basis for admissible quasiequations for proper subvarieties of $\mathbb{MV}$

Let $\mathcal{V}_{I,J}$ be a proper subvariety of $\mathbb{MV}$.

- $Q^1_{I,J} := Q(\{L_n : n \in I\} \cup \{L^1_m : m \in J\})$ is almost structurally complete.

- $Q^1_{I,J}$ is a $\mathcal{V}_{I,J}$-quasivariety ($\mathcal{V}(Q^1_{I,J}) = \mathcal{V}_{I,J}$).
Basis for admissible quasiequations for proper subvarieties of $\mathbb{MV}$

Let $\mathcal{V}_{I,J}$ be a proper subvariety of $\mathbb{MV}$.

- $Q_{I,J}^1 := Q(\{L_n : n \in I\} \cup \{L_m^1 : m \in J\})$ is almost structurally complete.
- $Q_{I,J}^1$ is a $\mathcal{V}_{I,J}$-quasivariety ($\mathcal{V}(Q_{I,J}^1) = \mathcal{V}_{I,J}$)
- $Q(\{L_n : n \in I\} \cup \{L_m^1 : m \in J\})$ is finitely axiomatizable.
Let $\mathcal{V}_{I,J}$ be a proper subvariety of $\mathbb{MV}$.

- $Q_{I,J}^1 := Q(\{L_n : n \in I\} \cup \{L_m^1 : m \in J\})$ is almost structurally complete.

- $Q_{I,J}^1$ is a $\mathcal{V}_{I,J}$-quasivariety ($\mathcal{V}(Q_{I,J}^1) = \mathcal{V}_{I,J}$)

- $Q(\{L_n : n \in I\} \cup \{L_m^1 : m \in J\})$ is finitely axiomatizable.

- $(x \lor \neg x)^n \approx 0 \Rightarrow 0 \approx 1$ is a basis for passive $\mathcal{V}_{I,J}$-admissible quasiequations where $n = \max\{I \cup \{\max J + 1\}\}$
Let $\mathcal{V}_{I,J}$ be a proper subvariety of $\mathbb{MV}$.

1. $Q^1_{I,J} := \mathcal{Q}(\{L_n : n \in I\} \cup \{L^1_m : m \in J\})$ is almost structurally complete.

2. $Q^1_{I,J}$ is a $\mathcal{V}_{I,J}$-quasivariety ($\mathcal{V}(Q^1_{I,J}) = \mathcal{V}_{I,J}$).

3. $\mathcal{Q}(\{L_n : n \in I\} \cup \{L^1_m : m \in J\})$ is finitely axiomatizable.

4. $(x \lor \neg x)^n \approx 0 \Rightarrow 0 \approx 1$ is a basis for passive $\mathcal{V}_{I,J}$-admissible quasiequations where $n = \max\{I \cup \{\max J + 1\}\}$

**Theorem**

All $\mathcal{V}_{I,J}$-admissible quasiequations are finitely axiomatizable.
Basis for admissible quasiequations for proper subvarieties of MV

**Theorem**

Let \((I, J)\) be a reduced pair, then a base for admissible quasiequations of \(V_{I,J}\) is given by

\[
\Delta'(Q_m) := [(-x)^{m-1} \leftrightarrow x] \lor y \approx 1 \Rightarrow y \approx 1 \text{ for every } m \in \text{Div}(J) \setminus \text{Div}(I) \text{ minimal with respect the divisibility.}
\]

\[
\Delta'(U_k) := [(-x)^{k-1} \leftrightarrow x] \lor y \approx 1 \Rightarrow \alpha_{I_k,\emptyset}(z) \lor y \approx 1 \text{ for every } 1 < k \in \text{Div}(I), \text{ where } I_k = \{n \in I : k | n\}.
\]

\[
CC_{n}^1 := (\varphi \lor \neg \varphi)^n \approx 0 \Rightarrow 0 \approx 1 \text{ where } n = \max\{I \cup \{\max J + 1\}\}.
\]
Results on admissibility theory allow to characterize and axiomatize the lattice of subquasivarieties (finitary extensions).
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Results on certain quasivarieties (locally finite, generated by chains) allow to obtain axiomatization of admissible quasiequations.
Conclusions

- Results on admissibility theory allow to characterize and axiomatize the lattice of subquasivarieties (finitary extensions).

- Results on certain quasivarieties (locally finite, generated by chains) allow to obtain axiomatization of admissible quasiequations.

- There is a relation among least $V$-quasivarieties generated by chains and (almost) structural completeness.
Future Work

- Similar algebraic approach to admissible rules for other many-valued logics: BL, MTL, FL...
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- Study the relation among almost structural completeness and least $V$-quasivarieties generated by (finite) subdirectly irreducible algebras.
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- Study the relation among almost structural completeness and least $V$-quasivarieties generated by (finite) subdirectly irreducible algebras.

- Multiple conclusion admissible rules and universal classes.
Thank you for your attention