A algebraic modal logic view on subordination and contact algebras

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This talk was going to be given by Dr. Ramon Jansana, but for personal reasons he is not here today.

Overview

First Part

1. A bit of history
2. Subordination algebras and Pseudo-subordination algebras. The $(0, 1)$-property.
3. Pseudo-subordination algebras as (binary) modal algebras. Congruences and filters.
4. Contact algebras and pseudo-contact algebras. The pseudo-contact algebras with the $(0, 1)$-property.
5. The variety generated by the pseudo-contact algebras with the $(0, 1)$-property. These algebras as the simple elements of that variety.

Second Part

1. Ternary Relational Topological duality for the pseudo-subordination algebras.
2. Correspondence and canonicity results.
3. The $(0,1)$-property is canonical.
4. Connection between the binary and ternary topological duality for subordination algebras (or $(0,1)$-pseudo-subordination algebras).

A bit of history

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In topology, a proximity space, also called a nearness space, is an axiomatization of notions of "nearness" that hold set-to-set, as opposed to the better known point-to-set notions that characterize topological spaces.

A proximity space is a pair $\langle X, \prec \rangle$, where $\prec$ is a relation (called proximity relation) between subsets of X satisfying the following properties.

1. $X \prec X$,
2. $A \prec B$, implies $A \subseteq B$,
3. $A \prec B$ and $A \prec C$, then $A \prec B \cap C$,
4. $A \prec B$, then $X - B \prec X - A$
5. $A \prec B$, then there exists $C \subseteq X$ such that $A \prec C \prec B$.

A proximity space is separated it satisfies the property:

S If $x \not\prec X - \{y\}$, then $x = y$.

Every proximity space is a topological space: a point x belong to the *interior* of A iff $\{x\} \prec A$.

This topology is always completely regular, and is Hausdorff iff the proximity space is separated.

Naimpally S. A. and Warrack D.: *Proximity Spaces,* Cambridge University Press, Cambridge, 1970.

- Riesz (F. Riesz. Stetigkeitsbegriff und abstrakte mengenlehre. Atti IV Congr. Intern. Mat. Roma, II:18–24, 1908. In German) started the discussion of proximities in 1908.

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- In 1939, Alexandroff (*On bicompact extensions of topological spaces. Mat. Sb., 5:403– 423, 1939*) developed the idea of "ends" while studying extensions of topological spaces.
- Y. M. Smirnov (On proximity spaces. Mat. Sb., 31:543574, 1952. In Russian), used the ends of Alexandroff to obtain a compactification of a Tychonoff space. Moreover, Smirnov proves that there is a bijection between the compactifications of a Tychonoff space and the proximities on that space "compatible" with the topology on the space.

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  - de Vries, H.: Compact spaces and compactifications. An algebraic approach. PhD thesis, University of Amsterdam, 1962.
- Deleting some of the conditions of the definition given by de Vries we have the notion of subordination relation.
  The actual definition of subordination was proposed in the paper:
  - Bezhanishvili, G., Bezhanishvili, N., Sourabh, S., and Venema, Y.: *Irreducible equivalence relations, Gleason spaces, and de Vries duality,* Appl. Categ. Structures, 2016.

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  The precontact relations and the subordination relations are dual notions.

- An equivalent concept to those of subordination relation and precontact relation is that of quasi-modal operator.
  - Celani S. A.: *Quasi-Modal algebras*, Mathematica Bohemica Vol. 126, No. 4 (2001), pp. 721-736.
  - Castro J. and Celani S.A: Quasi-Modal Lattices, Order (2004) 21: 107–129

Subordination algebras

Definition

A subordination on a Boolean algebra $\mathbf{B}$ is a binary relation $\prec$ on its domain such that:

(Q1) $0 \prec 0$ and $1 \prec 1$;

(Q2) $a \prec b, c$ implies $a \prec b \wedge c$;

(Q3) $a, b \prec c$ implies $a \vee b \prec c$;

(Q4) $a \leq b \prec c \leq d$ implies $a \prec d$.

A subordination (Boolean) algebra is a pair $\langle \boldsymbol{B}, \prec \rangle$ where $\boldsymbol{B}$ is a Boolean algebra and $\prec$ is a subordination on $\boldsymbol{B}$.

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A subordination (Boolean) algebra is a pair $\langle \boldsymbol{B}, \prec \rangle$ where $\boldsymbol{B}$ is a Boolean algebra and $\prec$ is a subordination on $\boldsymbol{B}$.
A contact algebra algebra is a subordination algebra $\langle \boldsymbol{B}, \prec \rangle$ such that:

(Q5) if $a \prec b$, then $a \leq b$;

(Q6) if $a \prec b$, then $\neg b \prec \neg a$.

The subordination relations that satisfy conditions **Q5** and **Q6** are called contact relations.

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A precontact relation is a binary relation $C \subseteq B \times B$ such that:

C1 If aCb, then $a \neq 0$ and $b \neq 0$;

C2 $aCb \vee c$ iff aCb or aCc;

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Precontact and subordination relations are interdefinable:

- $a \prec_C b \Leftrightarrow a(-C)\neg b$.
- $aC_\prec b \Leftrightarrow a \not\prec \neg b$.
- $\prec = \prec_{C_\prec}$ and $C = C_{\prec_C}$.

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A quasi-modal operator is a function $\Delta : \mathbf{B} \to \mathrm{Id}(\mathbf{B})$ such that

Q1 $\Delta(a \wedge b) = \Delta a \cap \Delta b$,

Q2 $\Delta 1 = B$.

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Quasi-modal operators and subordinations are interdefinable:

- $\Delta_{\prec}(a) = \{b : b \prec a\}$.
- $a \prec_{\Delta} b \Leftrightarrow a \in \Delta b$.
- $\prec = \prec_{\Delta_{\prec}}$ and $\Delta = \Delta_{\prec_{\Delta}}$.

A subordination $\prec$ on a Boolean algebra **B** can be equivalently described by its *characteristic function:*

$$a \multimap_{\prec} b = \begin{cases} 1 & \text{if } a \prec b \\ 0 & \text{otherwise} \end{cases},$$

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The operation $\multimap_{\prec}: B \times B \to \{0, 1\}$ satisfies the conditions:

(Q1') $0 \multimap_{\prec} 0 = 1$ and $1 \multimap_{\prec} 1 = 1$;

(Q2') if $a \multimap_{\prec} b = a \multimap c_{\prec} = 1$, then $a \multimap_{\prec} (b \wedge c) = 1$;

(Q3') if $a \multimap_{\prec} c = b \multimap c_{\prec} = 1$, then $(a \vee b) \multimap_{\prec} c = 1$

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Hence, subordinations and maps $\multimap: B \times B \to \{0,1\}$ that satisfy the conditions (Q1')–(Q4') are **interdefinable**.

$$a \prec_{\multimap} b \qquad \Leftrightarrow \qquad a \multimap b = 1$$

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Pseudo-subordination algebras

Taking into account the properties of the characteristic function $\multimap_{\prec}$ of a subordination $\prec$ we introduce the following definition.

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Definition

A pseudo-subordination algebra is an algebra $\mathbf{B} = \langle B, \wedge, \vee, \neg, \multimap, 0, 1\rangle$ such that $\langle B, \wedge, \vee, \neg, 0, 1\rangle$ is a Boolean algebra and $\multimap$ is a binary operation satisfying

- E1. $0 \multimap a = 1$ and $a \multimap 1 = 1$
- E2. $(a \multimap b) \wedge (a \multimap c) = a \multimap (b \wedge c)$.
- E3. $(a \multimap c) \wedge (b \multimap c) = (a \vee b) \multimap c$.

With PSB we denote the variety of pseudo-subordination algebras.

If $\mathbf{B}$ is a pseudo-subordination algebra, we will use $\mathbf{B}$ to denote also its Boolean algebra reduct.

Note: The negation-less reducts of the pseudo-subordination algebras are the elements of the variety of bounded distributive lattice with an implication whose lattice reduct is a Boolean lattice.

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- Celani, S.A.: *Bounded distributive lattices with fusion and implication*, Southest Asian Bulletin of Mathematics, 28 (2004), 999-1010.
- Cabrer, L. M. & Celani, S. A.: *Priestley dualities for some lattice-ordered algebraic structures, including MTL, IMTL and MV-algebra*s, Central European Journal of Mathematics, Versita, 2006, 4, 600-623.

A pseudo-subordination algebra $\mathbf{B}$ has the $(0,1)$-property, or is a $(0,1)$-pseudo-subordination algebra, if for every $a, b \in B$,

$$a \multimap b = 1 \quad \text{or} \quad a \multimap b = 0.$$

Therefore we have the following equivalences:

A pseudo-subordination algebra $\mathbf{B}$ has the $(0,1)$-property, or is a $(0,1)$-pseudo-subordination algebra, if for every $a, b \in B$,

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$$\iff \text{precontact algebras}$$

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$\iff$ quasi-modal algebras

Pseudo-subordination algebras as (binary) modal algebras

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Let $\mathbf{B} = \langle B, \wedge, \vee, \neg, \multimap, 0, 1\rangle$ be a pseudo-subordination algebra. We define two binary operations $\diamond$ and $\boxdot$ on B as follows:

$$\diamond(a, b) := \neg(a \multimap \neg b)$$

$$\boxdot(a, b) := \neg a \multimap b.$$

Pseudo-subordination algebras as (binary) modal algebras

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$$\diamond(a, b) := \neg(a \multimap \neg b)$$

$$\boxdot(a, b) := \neg a \multimap b.$$

The operation $\diamond$ is a binary modal operator, i.e.,

1. $\diamond(a, 0) = \diamond(0, b) = 0$ (by E1)
2. $\diamond$ distributes over joins in both coordinates (by E2 and E3).

- $\boxdot$ is the dual operator of $\diamond$, i.e.,

$$\boxdot(a, b) = \neg \diamond (\neg a, \neg b).$$

The algebra $\langle B, \wedge, \vee, \neg, \diamond, 0, 1\rangle$ will be called the modal version of $\boldsymbol{B}$.

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$$a \multimap b := \neg \diamond (a, \neg b)$$

is such that $\langle B, \wedge, \vee, \neg, \multimap, 0, 1\rangle$ is a pseudo-subordination algebra (whose modal version is $\langle B, \wedge, \vee, \neg, \diamond, 0, 1\rangle$).

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CONSEQUENCE: We can exploit the theory of Boolean algebras with operators (BAO's) in the study of the varieties of pseudo-subordination algebras.

Congruences and filters

Well known:

• If **B** is a Boolean algebra, the map $\Theta : \mathrm{Fi}(\mathbf{B}) \to \mathrm{Co}(\mathbf{B})$ defined by

$$\langle a, b\rangle \in \Theta(F) \text{ iff } a \leftrightarrow b \in F$$

is an isomorphism between the lattice of filters of **B** and the lattice of the congruences of **B**.

• Θ has the property that for every filter F, $1/\Theta(F) = F$.

• The inverse of Θ is the map that sends every congruence $\theta \in \mathrm{Co}(\mathbf{B})$ to $1/\theta$.

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• The inverse of Θ is the map that sends every congruence $\theta \in \mathrm{Co}(\mathbf{B})$ to $1/\theta$.

If we expand $\boldsymbol{B}$ with modal operators to $\boldsymbol{B}'$, then Θ restricted to the congruences of the expanded algebra is an isomorphism between the lattice of these congruences and the sublattice of the lattice of filters of $\mathbf{B}$ whose elements are the modal filters of $\boldsymbol{B}'$.

- Since every pseudo-subordination algebra **B** is equivalent to a Boolean algebra with a binary modal operator, we have that a Boolean congruence on B is a congruence of $\boldsymbol{B}$ if and only if it is a congruence of the modal version of $\boldsymbol{B}$.

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- Thus, the lattice of congruences of $\boldsymbol{B}$ is isomorphic to the lattice of the modal filters of the modal version of $\boldsymbol{B}$.

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- Thus, the lattice of congruences of $\boldsymbol{B}$ is isomorphic to the lattice of the modal filters of the modal version of $\boldsymbol{B}$.
- These filters can be characterized in terms of the operation $\multimap$ as the Boolean filters that are closed under the following two rules:

  (Pre) if $a \to b \in F$, then $(c \multimap a) \to (c \multimap b) \in F$.
  (Su) if $a \to b \in F$, then $(b \multimap c) \to (a \multimap c) \in F$.

  We call these filters strong.

Theorem

Let **B** *be a pseudo-subordination algebra.*

1. *For every filter* F *of* **B**,

   $$\Theta(F) \text{ is a congruence of } \mathbf{B} \text{ if and only if } F \text{ is strong.}$$

2. *The map* $\Theta(.)$ *establishes an isomorphism between the lattice of the strong filters of* **B** *and the lattice of the congruences of* **B**.

Pseudo-contact algebras

We recall that a contact algebra is a subordination algebra such that

(Q5) if $a \prec b$, then $a \leq b$;

(Q6) if $a \prec b$, then $\neg b \prec \neg a$.

The conditions (Q5) and (Q6) lead naturally to the following definition.

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Definition

A pseudo-contact algebra is a pseudo-subordination algebra that satisfies the conditions

E5. $a \multimap b \leq a \to b$.

E6. $a \multimap b = \neg b \multimap \neg a$.

We denote by PsC the variety of pseudo-contact algebras.

We note that

pseudo-contact algebras with the $(0, 1)$-property are equivalent to contact algebras.

Strong filters on Pseudo-contact algebras

Let **B** be a pseudo-subordination algebra. A Boolean filter F of **B** will be called open if

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The map $\Box : B \to B$ defined by

$$\Box a := 1 \multimap a$$

is a unary modal operator and thus $\langle B, \vee, \wedge, \Box, \neg, 0, 1\rangle$ is a modal algebra. The open filters of the modal algebra obtained taking the Boolean reduct of **B** and the operation $\Box$ are then our open filters.

Theorem

Let **B** *be a pseudo-contact algebra. Then*

1. *the strong filters of* **B** *are exactly the open filters.*
2. *the map* $\Theta(.)$ *establishes an isomorphism between the lattice of the open filters of* **B** *and the lattice of the congruences of* **B**.

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Theorem

Let **B** *be a pseudo-contact algebra. Then the following conditions are equivalent:*

1. **B** has the $(0,1)$-property.
2. **B** is simple and satisfies the condition E4 $a \multimap b \leq c \multimap (a \multimap b)$.

This theorem tells us that the pseudo-contact algebras with the $(0,1)$-property are the simple algebras of the subvariety of the variety of pseudo-contact algebras axiomatized by E4.

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So, we have the following:
PROBLEM: Find an axiomatization of the variety generated by the simple elements of the variety pseudo-contact algebras + E4 = PsCE4.

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Theorem

The variety generated by the simple elements of the variety PsCE4 *is the subvariety of* PsCE4 *axiomatized by the condition E7.*
$\neg(a \multimap b) \leq c \multimap \neg(a \multimap b)$.

Second Part

Topological Dualities

Ternary Relational Topological duality for pseudo-subordination algebras

To obtain a topological duality for pseudo-subordination algebras we can relay on B. Jónsson and A. Tarski duality for BAO's, or the duality given in the paper: Celani, S. A.: *Bounded distributive lattices with fusion and implication*, Southest Asian Bulletin of Mathematics, 28 (2004), 999-1010, for bounded distributive lattices with an implication.
Both approaches give the same results in our setting.

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1. $\langle X, \tau\rangle$ is a Stone space,
2. T is a ternary relation on X such that:
    1. For every $x \in X$, $T(x)$ is a closed set in the product space $X \times X$.
    2. For all clopen sets U, V, $\diamond(U, V) := \{x \in X : (\exists y, z \in X)(\langle x, y, z\rangle \in T, y \in U, z \in V)\}$ is a clopen.

The last condition is equivalent to:

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Definition

A pseudo-subordination space is a the triple $\langle X, \tau, T\rangle$ where

1. $\langle X, \tau\rangle$ is a Stone space,
2. T is a ternary relation on X such that:
   1. For every $x \in X$, $T(x)$ is a closed set in the product space $X \times X$.
   2. For all clopen sets U, V, $\diamond(U, V) := \{x \in X : (\exists y, z \in X)(\langle x, y, z\rangle \in T, y \in U, z \in V)\}$ is a clopen.

The last condition is equivalent to:

(2') for all clopen sets U, V, the set

$$U \multimap V = \{x \in X : (\forall y, z \in X)(\langle x, y, z\rangle \in T \;\&\; y \in U \Rightarrow z \in V)\}$$

is clopen.

Using this last condition we obtain the characterization of the dual space of a pseudo-subordination algebra that follows from the duality for distributive lattices with an implication.

Dual space of a pseudo-subordination algebra

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Let $\mathbf{B} = \langle B, \wedge, \vee, \neg, \diamond, 0, 1\rangle$ be a Boolean algebra with a binary operator $\diamond$. The dual space of $\mathbf{B}$ is the triple

$$\langle \mathrm{Ult}\mathbf{B}, \tau, T\rangle$$

where $\langle \mathrm{Ult}\mathbf{B}, \tau\rangle$ is the Stone space of the Boolean reduct of $\mathbf{B}$ and T is the ternary relation on $\mathrm{Ult}\mathbf{B}$ defined by

$$\langle x, y, z\rangle \in T \quad \text{iff} \quad (\forall a, b \in B)(a \in y \ \& \ b \in z \Rightarrow \diamond(a, b) \in x)$$

for all $x, y, z \in \mathrm{Ult}\mathbf{B}$. Thus, we take as the dual space of a pseudo-subordination algebra $\mathbf{B} = \langle B, \wedge, \vee, \neg, \multimap, 0, 1\rangle$ the dual space $\langle \mathrm{Ult}\mathbf{B}, \tau, T\rangle$ of the modal version of $\mathbf{B}$

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- Note: It is easy to see that the relation T can be defined as

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In this way we obtain exactly the definition of the ternary relation on the set of ultrafilters that we obtain when we apply duality for bounded distributive lattices with an implication to a pseudo-subordination algebra.

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In this way we obtain exactly the definition of the ternary relation on the set of ultrafilters that we obtain when we apply duality for bounded distributive lattices with an implication to a pseudo-subordination algebra.

- The ternary relational structure $\langle \mathrm{Ult}\mathbf{B}, T\rangle$ is known as the discrete dual of $\mathbf{B}$.

Theorem

The following categories are dually equivalent:

1. PsA: *Pseudo-subordination algebras + Boolean homomorphisms* $h : B_1 \to B_2$ *such that* $h(a \multimap_1 b) \leq h(a) \multimap_2 h(b)$ *(semi-homomorphisms).*
2. PsSpace: *Pseudo-subordination spaces + continuous maps* $f : X_1 \to X_2$ *such that* $\langle x, y, z \rangle \in T_1 \implies \langle f(x), f(y), f(z) \rangle \in T_2$ *(stable maps).*

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Theorem

The following categories are dually equivalent:

1. PsA^h : *Pseudo-subordination algebras + Boolean homomorphisms* $h : B_1 \to B_2$ *such that* $h(a \multimap_1 b) = h(a) \multimap_2 h(b)$ *(homomorphisms).*
2. $\mathrm{PsSpace}^b$: *Pseudo-subordination spaces + stable maps satisfying the additional condition:*

$$\langle f(x), y', z'\rangle \in T_2 \implies (\exists y, z \in X_1)(f(y) = y' \;\&\; f(z) = z' \;\&\; \langle x, y, z\rangle \in T_1) \text{ (bounded morphisms)}$$

Correspondence and canonicity results

- A property Φ of pseudo-subordination algebras corresponds to a first-order condition Π (that we express in the first-order language with a ternary relation symbol T) if for every pseudo-subordination algebra $\mathbf{B}$,

$$\mathbf{B} \text{ has } \Phi \iff \langle \mathrm{Ult}\mathbf{B}, T\rangle \text{ has } \Pi.$$

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In particular, an equation $\varphi \approx \psi$ corresponds to a first-order condition if the property "$\varphi \approx \psi$ is valid" has a first-order correspondent.

- A property Φ of pseudo-subordination algebras is canonical if for every pseudo-subordination algebra $\mathbf{B}$

$$\mathbf{B} \text{ has } \Phi \implies \langle \mathcal{P}(\mathrm{Ult}(\mathbf{B}), \multimap_T\rangle \text{ has } \Phi$$

  This is equivalent to say:

$$\mathbf{B} \text{ has } \Phi \implies \langle \mathrm{Ult}(\mathbf{B}), T\rangle \text{ has } \Pi$$

  where Π is the first-order correspondent of Φ.

First-order correspondents

	Property	Modal equation
E4	$a \multimap b \leq c \multimap (a \multimap b)$	$\boldsymbol{z} \diamond (\boldsymbol{x} \diamond \boldsymbol{y}) \leq \boldsymbol{x} \diamond \boldsymbol{y}$
E5	$a \multimap b \leq a \to b$	$\boldsymbol{x} \wedge \neg \boldsymbol{y} \leq \boldsymbol{x} \diamond \boldsymbol{y}$
E6	$a \multimap b = \neg b \multimap \neg a$	$\boldsymbol{x} \diamond \boldsymbol{y} = \boldsymbol{y} \diamond \boldsymbol{x}$
E7	$\neg(a \multimap b) \leq c \multimap \neg(a \multimap b)$	$\boldsymbol{z} \diamond (\boldsymbol{x} \boxdot \boldsymbol{y}) \leq \boldsymbol{x} \boxdot \boldsymbol{y}$

	Property	First-order correspondent
E4	$a \multimap b \leq c \multimap (a \multimap b)$	$\forall xyzuv(Txyz \,\&\, Tzuv \to Txuv)$
E5	$a \multimap b \leq a \to b$	$\forall x Txxx$
E6	$a \multimap b = \neg b \multimap \neg a$	$\forall xyz(Txyz \to Txzy)$
E7	$\neg(a \multimap b) \leq c \multimap \neg(a \multimap b)$	$\forall xyzuv(Txyz \,\&\, Txuv \to Tzuv)$

Theorem

Let $\mathbf{B}$ *be a pseudo-subordination algebra. If* $\mathbf{B}$ *satisfies one of the properties E4–E7, then its discrete dual* $\langle \mathrm{Ult}\mathbf{B}, T\rangle$ *satisfies the first-order correspondent.*

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Corollary

The properties E4–E7 are canonical.

We can obtain topological dualities for the varieties of pseudo-subordination algebras obtained by combinations of the properties E4–E7 by requiring of the ternary relation of the dual spaces to satisfy the suitable first-order correspondents.

The canonicity of the $(0, 1)$-property

We recall that subordination algebras can be described as (0-1)-pseudo-subordination algebras. We prove that he $(0, 1)$-property has a first-order correspondent.

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If $\forall yz(\exists x Txyz \to \forall w Twyz)$ ir true in $\langle X, T\rangle$, then the powerset pseudo-subordination algebra $\langle \mathcal{P}(X), \multimap_T\rangle$ has the $(0,1)$-property.

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Connection with the binary relational topological dualities for subordination algebras

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Binary relational dualities for subordination algebras

Algebraic Category

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Algebraic Category	Topological Category
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Sub	StR
homomorphisms $h : \boldsymbol{B}_1 \to \boldsymbol{B}_2$	Continuos maps $f : X_1 \to X_2$
(S) $a \prec_1 b \Rightarrow h(a) \prec_2 h(b)$	**(E)** $xRy \Rightarrow f(x) R' f(y)$
subordination algebra morphism	stable morphism

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Sub homomorphisms $h : \boldsymbol{B}_1 \to \boldsymbol{B}_2$ **(S)** $a \prec_1 b \Rightarrow h(a) \prec_2 h(b)$ subordination algebra morphism	StR Continuos maps $f : X_1 \to X_2$ **(E)** $xRy \Rightarrow f(x) R' f(y)$ stable morphism
Sub^{s} $a \prec_1 b \Leftrightarrow h(a) \prec_2 h(b)$ strong subordination algebra morphism	StR^{s} **(E)** $+ [uR'z \Rightarrow \exists x, y \in X$ $(xRy \text{ and } f(x) = u \text{ and } f(y) = z)]$ strict and stable morphism

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$\mathsf{Sub^s}$ $a \prec_1 b \Leftrightarrow h(a) \prec_2 h(b)$ strong subordination algebra morphism	$\mathsf{StR^s}$ **(E)** $+ [uR'z \Rightarrow \exists x, y \in X$ $(xRy \text{ and } f(x) = u \text{ and } f(y) = z)]$ strict and stable morphism
$\mathsf{Sub^c}$ **(S)** $+$ if $c \prec_2 h(a)$, then $\exists b \in B_1$ $b \prec_1 a$ and $c \leq h(b)$. c-subordination algebra morphism	$\mathsf{StR^c}$ **(E)** $+ [f(x)R'z \Rightarrow \exists y \in X$ $(xRy \text{ and } f(y) = z)]$ p-morphism

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Sub	is equivalent to	PsA^*	(0,1)-pseudo-subordination algebras $h(a \multimap_1 b) \leq h(a) \multimap_2 h(b)$
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We show that the isomorphisms in the next table also hold

StR		$\cong$		$\mathsf{PsSpace}^*$
$\mathsf{StR}^{\mathsf{s}}$		$\cong$		$\mathsf{PsSpace^b}^*$

Where $\mathsf{PsSpace}^*$ and $\mathsf{PsSpace^b}^*$ are the categories whose objects are the ternary relational topological spaces $\langle X, \tau, T \rangle$ that satisfy the first-order condition $\forall yz(\exists x Txyz \to \forall w Twyz)$.

Subordination spaces to $(0,1)$-pseudo-subordination spaces

Let $\langle X, \tau, R \rangle \in$ StR. We define a ternary relation T_R on X by setting for every $x, y, z \in X$

$$\langle x, y, z \rangle \in T_R \text{ iff } yRz.$$

Then

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3. The sentence $\forall yz(\exists x Txyz \to \forall w Twyz)$ is true in $\langle X, T_R\rangle$.
4. Hence, the powerset subordination algebra of $\langle X, T_R\rangle$ has the $(0,1)$-property, that is, $U \multimap V = X$ or $U \multimap V = \emptyset$, for all $U, V \subseteq X$.

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6. Thus, $\langle X, \tau, T_R\rangle$ is a $(0,1)$-pseudo-subordination space.

Subordination spaces to $(0, 1)$-pseudo-subordination spaces

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Then

1. $T_R(x) = R$, for every $x \in X$.
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3. The sentence $\forall yz(\exists x Txyz \to \forall w Twyz)$ is true in $\langle X, T_R\rangle$.
4. Hence, the powerset subordination algebra of $\langle X, T_R\rangle$ has the $(0, 1)$-property, that is, $U \multimap V = X$ or $U \multimap V = \emptyset$, for all $U, V \subseteq X$.
5. Hence, for all clopen sets U, V, the set $U \multimap V$ is clopen.
6. Thus, $\langle X, \tau, T_R\rangle$ is a $(0, 1)$-pseudo-subordination space.

Let F : StR $\to$ PsSpace* be given by:

$$F(\langle X, \tau, R\rangle) := \langle X, \tau, T_R\rangle.$$

$(0,1)$-pseudo-subordination spaces to pseudo-subordination spaces

Let $\langle X, \tau, T \rangle \in \text{PsSpace}^*$. We define the binary relation R_T on X by setting for every $x, y \in X$

$$xR_T y \ \text{ iff } \ (\exists u \in X)\langle u, x, y \rangle \in T.$$

Since $\forall yz(\exists x Txyz \to \forall w Twyz)$ is true in $\langle X, T \rangle$, it follows

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Let G : PsSpace* $\to$ StR be given by:

$$G(\langle X, \tau, T\rangle) := \langle X, \tau, R_T\rangle.$$

The composition $F \circ G$ and the composition $G \circ F$ are the identity maps.

The maps F and G can be turned into functors using the next lemma.

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Lemma

Let $\mathbb{X} = \langle X, \tau, R\rangle$ *and* $\mathbb{X}' = \langle X', \tau', R'\rangle$ *in* StR *and* $f : X \to X'$. *Then*

1. f *is a stable morphism from* $\mathbb{X}$ *to* $\mathbb{X}'$ *if and only if it is a stable morphism from* $F(\mathbb{X})$ *to* $F(\mathbb{X}')$.
2. f *is a strict and stable morphism from* $\mathbb{X}$ *to* $\mathbb{X}'$ *if and only if it is a bounded morphism from* $F(\mathbb{X})$ *to* $F(\mathbb{X}')$.

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Let $\mathbb{X} = \langle X, \tau, T\rangle$ and $\mathbb{X}' = \langle X', \tau', T'\rangle$ in PsSpace* *and $f : X \to X'$. Then*

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2. *f is a bounded morphism from $\mathbb{X}$ to $\mathbb{X}'$ if and only if it is strict and stable morphism from $G(\mathbb{X})$ to $G(\mathbb{X}')$.*

We characterize the morphisms in between $(0,1)$-subordination spaces correspond to the p-morphisms between subordination spaces.

Lemma

Let $\mathbb{X} = \langle X, \tau, R\rangle$ and $\mathbb{X}' = \langle X', \tau', R'\rangle$ in StR *and $f : X \to X'$. The following are equivalent:*

1. *f is a p-morphism from $\mathbb{X}$ to $\mathbb{X}'$*
2. *f is a morphism in* PsSpace* *from $F(\mathbb{X})$ to $F(\mathbb{X}')$ such that for every $x \in X$ and every $u, v \in X'$, if $\langle u, f(x), v\rangle \in T_{R'}$, then there exists $y, z \in X$ such that $f(y) = v$ and $\langle z, x, y\rangle \in T_R$.*

Still open: Which morphisms in PsA when taken for $(0,1)$-pseudo-subordination algebras correspond to the p-morphisms?

Bezhanishvili, G., Bezhanishvili, N., Sourabh, S., and Venema, Y.: *Irreducible equivalence relations, Gleason spaces, and de Vries duality,* Appl. Categ. Structures, 2016.

Celani S. A.: *Quasi-Modal algebras,* Mathematica Bohemica Vol. 126, No. 4 (2001), pp. 721-736.

Castro J. and Celani S.A, *Quasi-modal Lattices*, Order (2004) 21: 107–129.

Dimov G. and Vakarelov D.: *Topological representation of precontact algebras*, in Lecture Notes Comp. Sci., 3929, W. MacCaull et al. (eds.), Springer-Verlag, Berlin (2006), pp. 1-16.

Düntsch, I. & Vakarelov, D.: *Region-based theory of discrete spaces: a proximity approach,* Annals of Mathematics and Artificial Intelligence, 49, (2007), 5-14.

Efremovič V. A.: The geometry of proximity. I, Mat. Sb. (N.S.) 31 (73) (1952), 189–200 (in Russian).

de Vries, H.: *Compact spaces and compactifications. An algebraic approach,* PhD thesis, University of Amsterdam, 1962.

THANK YOU