Lattice-Valued Predicate Logics

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Outline

1 Introduction

2 Lattice-Valued Predicate Logics

3 Formal Fuzzy Mathematics

4 What's next?

Classical first-order logic and its completeness

CFOL: theory of quantification built over classical propositional logic

Formalized (in its present form) by Hilbert and Ackermann (1928)

$\vdash_{\mathrm{CFOL}}$ its Hilbert axiomatization, $\models_{\mathrm{CFOL}}$ the semantical consequence

Its completeness proved by Gödel (1929):

Theorem

For every set of first-order formulas $\Gamma \cup \{\varphi\}$:

$$\Gamma \vdash_{\mathrm{CFOL}} \varphi \text{ if, and only if, } \Gamma \models_{\mathrm{CFOL}} \varphi$$

Altering CFOL

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- strengthened: higher-order, more expressive languages
- reformulated: e.g. algebraizing using cylindric or polyadic algebras
- weakened: changing the propositional part, e.g. to intuitionistic, substructural, fuzzy

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- strengthened: higher-order, more expressive languages
- reformulated: e.g. algebraizing using cylindric or polyadic algebras
- weakened: changing the propositional part, e.g. to intuitionistic, substructural, fuzzy
- and in arbitrary combinations of the above (and probably in thousands of other ways I have never seen or even imagined)

Two aims of this talk

Define: a (natural) first-order variant of arbitrary propositional logic
and prove a variant of Gödel completeness theorem

Demonstrate: the power and usefulness of the resulting theory
using example from formal fuzzy mathematics

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3. Given a semantic, how to define its consequence relation?
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Thus in particular obtaining the Gödel completeness theorem ...

The first part of the talk is based on the paper:

> PC, C. Noguera: *A Henkin-style proof of completeness for first-order algebraizable logics.* J. of Symbolic Logic, 2015

but presented from the first principles and semantics-first

What will be our formulas? (DC 1)

A language is a quadruple $\mathfrak{L} = \langle \mathbf{C}, \mathbf{P}, \mathbf{F}, \mathbf{ar} \rangle$
connectives, predicate and function symbols with their arities

And we build sets of $\mathfrak{L}$-terms Term and $\mathfrak{L}$-formulas Form as usual,
i.e., as the least sets such that:

- object variables ObjVar are terms
- if $t_1, \dots, t_n \in$ Term, $F \in \mathbf{F}$, $\mathbf{ar}(F) = n$, then $F(t_1, \dots, t_n) \in$ Term

- if $t_1, \dots, t_n \in$ Term, $P \in \mathbf{P}$, $\mathbf{ar}(P) = n$, then $P(t_1, \dots, t_n) \in$ Form
- if $\varphi_1, \dots, \varphi_n \in$ Form, $c \in \mathbf{C}$, $\mathbf{ar}(c) = n$, then $c(\varphi_1, \dots, \varphi_n) \in$ Form
- if $\varphi \in$ Form and $x \in$ ObjVar, then $(\forall x)\varphi \in$ Form and $(\exists x)\varphi \in$ Form

More on languages

There is a well-known difference between the role of connectives and other syntactical objects.

Let us fix, for this talk, a set of connectives $\mathbf{C}$ and their arities

Thus we can speak about predicate languages $\mathcal{P} = \langle \mathbf{P}, \mathbf{F}, \mathbf{ar} \rangle$

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Thus we can speak about predicate languages $\mathcal{P} = \langle \mathbf{P}, \mathbf{F}, \mathbf{ar} \rangle$

We also consider a special language: the propositional one

$$\mathcal{L} = \langle \mathbf{C}, \{p_i \mid i \in \mathrm{N}\}, \emptyset, \mathbf{ar} \rangle, \quad \text{where } \mathbf{ar}(p_i) = 0$$

Note that $\mathcal{L}$ can be seen as an algebraic type

i.e., a classical predicate language $\langle \emptyset, \mathbf{C}, \mathbf{ar} \rangle$

How to design the semantics?

We follow the stream of research:

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1961 Mostowski: interpretation of existential (resp. universal) quantifiers as suprema (resp. infima)

1963 Rasiowa and Sikorski: first-order intuitionistic logic

1969 Horn: first-order Gödel–Dummett logic

1974 Rasiowa: first-order implicative logics

1992 Takeuti, Titani: first-order Gödel–Dummett logic with additional connectives

1998 Hájek: first-order axiomatic extensions of HL

Ordered algebra based semantics (DC 2,3)

We want our semantics to assign some 'grades' from a set G to formulas:

$$\|\cdot\|\colon \text{Form} \to G$$

Let us also fix the 'interpretation' of connectives, i.e., operations

$$c^{\boldsymbol{G}}\colon G^n \to G \qquad \text{for each } n\text{-ary } c \in \mathbf{C}$$

Then we simply set (DC 2)

$$\|c(\varphi_1, \dots, \varphi_n)\| = c^{\boldsymbol{G}}(\|\varphi_1\|, \dots, \|\varphi_n\|)$$

The classical structure $\boldsymbol{G} = \langle G, \langle c^{\boldsymbol{G}} \rangle_{c \in \mathbf{C}} \rangle$ is then an algebra of type $\mathcal{L}$

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Finally, let us assume that G is partially ordered (DC3)

some grades are ‘better’ than others

Generalized semantics (DC 3′, 4)

Consider a ‘normal’ predicate language: $\mathcal{P} = \langle \mathbf{P}, \mathbf{F}, \mathbf{ar} \rangle$ DC1

Consider an algebra $\boldsymbol{G}$ of type $\mathcal{L}$ with a partial order $\leq$ DC2, DC3

$\boldsymbol{G}$-structure $\mathfrak{M}$ for $\mathcal{P}$ is tuple $\mathfrak{M} = \langle M, \langle f_{\mathfrak{M}} \rangle_{f \in \mathbf{F}}, \langle P_{\mathfrak{M}} \rangle_{P \in \mathbf{P}} \rangle$ where

- $f_{\mathfrak{M}} \colon M^n \to M$ for each n-ary $f \in \mathbf{F}$
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$\mathfrak{M}$-evaluation v: a mapping $\mathrm{v} \colon \mathrm{ObjVar} \to M$; extended to all terms/fle:

$$\begin{array}{rcll} \|f(t_1, \dots, t_n)\|^{\mathfrak{M}}_{\mathrm{v}} & = & f_{\mathfrak{M}}(\|t_1\|^{\mathfrak{M}}_{\mathrm{v}}, \dots, \|t_n\|^{\mathfrak{M}}_{\mathrm{v}}) & \text{for } f \in \mathbf{F} \\ \|P(t_1, \dots, t_n)\|^{\mathfrak{M}}_{\mathrm{v}} & = & P_{\mathfrak{M}}(\|t_1\|^{\mathfrak{M}}_{\mathrm{v}}, \dots, \|t_n\|^{\mathfrak{M}}_{\mathrm{v}}) & \text{for } P \in \mathbf{P} \\ \|c(\varphi_1, \dots, \varphi_n)\|^{\mathfrak{M}}_{\mathrm{v}} & = & c^{\boldsymbol{G}}(\|\varphi_1\|^{\mathfrak{M}}_{\mathrm{v}}, \dots, \|\varphi_n\|^{\mathfrak{M}}_{\mathrm{v}}) & \text{for } c \in \mathbf{C} \end{array}$$

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\|(\forall x)\varphi(x)\|^{\mathfrak{M}}_{\mathrm{v}} & = & \inf_{\leq}\{\|\varphi(x)\|^{\mathfrak{M}}_{\mathrm{v}[x:m]} \mid m \in M\} & \text{DC4} \\
\|(\exists x)\varphi(x)\|^{\mathfrak{M}}_{\mathrm{v}} & = & \sup_{\leq}\{\|\varphi(x)\|^{\mathfrak{M}}_{\mathrm{v}[x:m]} \mid m \in M\} & \text{DC4}
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Consider a 'normal' predicate language: $\mathcal{P} = \langle \mathbf{P}, \mathbf{F}, \mathbf{ar} \rangle$ DC1

Consider an algebra $\boldsymbol{G}$ of type $\mathcal{L}$ with a lattice reduct DC2, DC3′

$\boldsymbol{G}$-structure $\mathfrak{M}$ for $\mathcal{P}$ is tuple $\mathfrak{M} = \langle M, \langle f_{\mathfrak{M}} \rangle_{f \in \mathbf{F}}, \langle P_{\mathfrak{M}} \rangle_{P \in \mathbf{P}} \rangle$ where

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Example

Take the standard MV-algebra $[0,1]_{Ł} = \langle [0,1], \&, \to, \wedge, \vee, 0, 1\rangle$ where

$$x \,\&\, y = \max\{x + y - 1, 0\} \qquad x \to y = \min\{1 - x + y, 1\}$$

$$x \wedge y = \min\{x, y\} \qquad x \vee y = \max\{x, y\}$$

Consider a $[0,1]_{Ł}$-structure with domain $M = \{1, \dots, 6\}$ and binary predicate P: 'x likes y':

$P_{\mathfrak{M}}$	1	2	3	4	5	6
1	1.0	1.0	0.5	0.4	0.3	0.0
2	0.8	1.0	0.4	0.4	0.3	0.0
3	0.7	0.9	1.0	0.8	0.7	0.4
4	0.9	1.0	0.7	1.0	0.9	0.6
5	0.6	0.8	0.8	0.7	1.0	0.7
6	0.3	0.5	0.6	0.4	0.7	1.0

$$\begin{aligned}
\mathrm{Narciss}(R) &\equiv_{\mathrm{df}} (\forall x)Rxx & \|\mathrm{Narciss}(P)\|^{\mathfrak{M}} &= 1\\
\mathrm{Sym}(R) &\equiv_{\mathrm{df}} (\forall x, y)(Rxy \to Ryx) & \|\mathrm{Sym}(P)\|^{\mathfrak{M}} &= 0.4\\
\mathrm{Trans}(R) &\equiv_{\mathrm{df}} (\forall x, y, z)(Rxy \,\&\, Ryz \to Rxz) & \|\mathrm{Trans}(P)\|^{\mathfrak{M}} &= 1
\end{aligned}$$

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5	0.6	0.8	0.8	0.7	1.0	0.7
6	0.3	0.5	0.6	0.4	0.7	1.0

$\mathrm{ELS}(R) \equiv_{\mathrm{df}} (\forall x)(\exists y)(Rxy)$ $\qquad \|\mathrm{ELS}(P)\|^{\mathfrak{M}} = 1$

$\mathrm{EILS}(R) \equiv_{\mathrm{df}} (\forall x)(\exists y)(Ryx)$ $\qquad \|\mathrm{EILS}(P)\|^{\mathfrak{M}} = 1$

$\mathrm{ELSE}(R) \equiv_{\mathrm{df}} (\forall x)(\exists y)(x \neq y \wedge Rxy)$ $\qquad \|\mathrm{ELSE}(P)\|^{\mathfrak{M}} = 0.7$

How to define the consequence? (DC5)

Definition ((Sentential) consequence relation)

Let $\boldsymbol{G}$ be an $\mathcal{L}$-algebra with lattice reduct.
Let $T \cup \{\varphi\}$ be a set of $\mathcal{P}$-formulas.
Then φ is a semantical consequence of T w.r.t. $\boldsymbol{G}$, $T \models_{\boldsymbol{G}} \varphi$, if

each $\boldsymbol{G}$-model of T is $\boldsymbol{G}$-model of φ

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Assume, from now on, that $\mathcal{L}$ contains a nullary connective $\overline{1}$ DC5

$\mathfrak{M}$ is a $\boldsymbol{G}$-model of T if for each $\mathfrak{M}$-evaluation v:

- $\|\chi\|^{\mathfrak{M}}_{\mathrm{v}}$ is defined for each formula χ and
- $\|\varphi\|^{\mathfrak{M}}_{\mathrm{v}} \geq \overline{1}^{\boldsymbol{G}}$ for each formula $\varphi \in T$ DC5

$\overline{1}^{\boldsymbol{G}}$ is the least ‘good’ grade

How to define the consequence? (DC5)

Definition ((Sentential) consequence relation)

Let $\mathbb{K}$ be a class of $\mathcal{L}$-algebras with lattice reduct.
Let $T \cup \{\varphi\}$ be a set of $\mathcal{P}$-formulas.
Then φ is a semantical consequence of T w.r.t. $\mathbb{K}$, $T \models_{\mathbb{K}} \varphi$, if

for each $\boldsymbol{G} \in \mathbb{K}$, each $\boldsymbol{G}$-model of T is $\boldsymbol{G}$-model of φ

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Examples

$\mathbb{K}$	logic
$\{\mathbf{2}\}$	classical FOL
(complete) Boolean algebras	classical FOL
(complete) Heyting algebras	intuitionistic FOL
(complete) SI Heyting algebras	intuitionistic FOL + CD
(complete) Heyting chains	int. FOL + $(\varphi \to \psi) \vee (\psi \to \varphi)$ + CD
Gödel algebras	int. FOL + $(\varphi \to \psi) \vee (\psi \to \varphi)$
(complete) $\mathrm{FL}_{\mathrm{ew}}$-algebras	affine FOL (w/o expon.)
MV-algebras	Łukasiewicz FOL

- SI Heyting algebras = Heyting algebras with a coatom
- CD: $(\forall x)(\chi \vee \varphi) \to \chi \vee (\forall x)\varphi$ (x not free in χ)
- Gödel algebras = variety generated by Heyting chains
- MV-algebras = variety generated by $[0,1]_{Ł}$

How the propositional logics look like?

Recall propositional language $\mathcal{L} = \langle \mathbf{C}, \{p_i \mid i \in \mathrm{N}\}, \emptyset, \mathbf{ar}\rangle$, $\mathbf{ar}(p_i) = 0$

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- If $\varphi \in T$, then $T \models_{\mathbb{K}} \varphi$ (Reflexivity)
- If $S \models_{\mathbb{K}} \psi$ for each $\psi \in T$ and $T \models_{\mathbb{K}} \varphi$, then $S \models_{\mathbb{K}} \varphi$ (Cut)
- If $T \models_{\mathbb{K}} \varphi$, then $\sigma[T] \models_{\mathbb{K}} \sigma(\varphi)$ for all substitutions σ (Structurality)

where substitution is any mapping from $\{p_i \mid i \in \mathrm{N}\}$ to Form

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But $\models_{\mathbb{K}}$ need not be finitary, i.e., we do not have

$$T \vdash \varphi \text{ implies } T' \vdash \varphi \text{ for some finite } T' \subseteq T$$

This is the case e.g. for $\mathbb{K} = \{[0,1]_{\text{Ł}}\}$.

We want propositional logics to be a bit ‘better’ (DC6)

Assume, from now on, that there is a binary operation $\to$ in $\mathcal{L}$ st: DC6

$$x \to^{\boldsymbol{G}} y \geq \overline{1}^{\boldsymbol{G}} \quad \text{iff} \quad x \leq y \qquad\qquad \text{for each } \boldsymbol{G}$$

y is ‘better’ than x IFF $x \to^{\boldsymbol{G}} y$ is ‘good’

Then $\models_{\mathbb{K}}$ is algebraically implicative à la C and Noguera

and, if finitary, algebraizable logic à la Blok and Pigozzi

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i.e., we will always have:

$$\begin{array}{lll}
\models_{\mathbb{K}} \varphi \to \varphi & \varphi, \varphi \to \psi \models_{\mathbb{K}} \psi & \varphi \to \psi, \psi \to \chi \models_{\mathbb{K}} \varphi \to \chi \\
\varphi \models_{\mathbb{K}} \overline{1} \to \varphi & \overline{1} \to \varphi \models_{\mathbb{K}} \varphi & \\
\models_{\mathbb{K}} \varphi \wedge \psi \to \varphi & \models_{\mathbb{K}} \varphi \wedge \psi \to \psi & \chi \to \varphi, \chi \to \psi \models_{\mathbb{K}} \chi \to \varphi \wedge \psi \\
\models_{\mathbb{K}} \varphi \to \varphi \vee \psi & \models_{\mathbb{K}} \psi \to \varphi \vee \psi & \varphi \to \chi, \psi \to \chi \models_{\mathbb{K}} \varphi \vee \psi \to \chi
\end{array}$$

and for each n-ary $c \in \mathbf{C}$, formulas $\varphi, \psi, \chi_1, \ldots, \chi_n$, and each $i < n$:

$$\varphi \to \psi, \psi \to \varphi \models_{\mathbb{K}} c(\chi_1, \ldots, \chi_i, \varphi, \ldots, \chi_n) \leftrightarrow c(\chi_1, \ldots, \chi_i, \psi, \ldots, \chi_n)$$

How to axiomatize $\models_{\mathbb{K}}$?

Lets us first restrict to propositional languages

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A quasivariety is a class of algebras axiomatized by quasiidentities
formulas of the form $(\bigwedge_{i \leq n} \alpha_i \approx \beta_i) \to \varphi \approx \psi$

By $\mathbf{Q}(\mathbb{K})$ we denote the quasivariety generated by $\mathbb{K}$ i.e.,
the smallest class of algebras satisfying all quasiidentites valid in $\mathbb{K}$

Theorem (For propositional logic only)

$\models_{\mathbb{K}}$ *is finitary iff* $\models_{\mathbb{K}} = \models_{\mathbf{Q}(\mathbb{K})}$.

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Theorem (For propositional logic only)

$\models_{\mathbb{K}}$ *is finitary iff* $\models_{\mathbb{K}} = \models_{\mathbf{Q}(\mathbb{K})}$.

If $\models_{\mathbb{K}}$ is finitary logic, than

- $\mathbf{Q}(\mathbb{K})$ is its equivalent algebraic semantics
- $\models_{\mathbb{K}}$ is axiomatized 'using' the quasiidentities axiomatizing $\mathbf{Q}(\mathbb{K})$

1st axiomatizability result

Theorem (PC, C. Noguera. JSL 2015)

Let $\mathbb{K}$ be a quasivariety of $\mathcal{L}$-algebras satisfying DC3', DC4, DC5 and $\mathcal{AX}$ an arbitrary axiomatization of the propositional logic of $\mathbb{K}$.

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- $T \models_{\mathbb{K}} \varphi$
- *there is a proof of φ from T in the axiomatic system:*

(P)	*first-order substitutions of axioms and rules of $\mathcal{AX}$*	
$(\forall 1)$	$\vdash (\forall x)\varphi(x, \vec{z}) \to \varphi(t, \vec{z})$	*t substitutable for x in φ*
$(\exists 1)$	$\vdash \varphi(t, \vec{z}) \to (\exists x)\varphi(x, \vec{z})$	*t substitutable for x in φ*
$(\forall 2)$	$\chi \to \varphi \vdash \chi \to (\forall x)\varphi$	*x not free in χ*
$(\exists 2)$	$\varphi \to \chi \vdash (\exists x)\varphi \to \chi$	*x not free in χ*

1st axiomatizability result

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Let L *be a finitary algebraically implicative logic with axiomatization* $\mathcal{AX}$ *such that its equivalent algebraic semantics* $\mathbb{K}$ *satisfies DC3', DC4, DC5. Then the following are equivalent:*

- $T \models_{\mathbb{K}} \varphi$
- *there is a proof of* φ *from* T *in the axiomatic system:*

(P)	*first-order substitutions of axioms and rules of* $\mathcal{AX}$	
$(\forall 1)$	$\vdash (\forall x)\varphi(x,\vec{z}) \to \varphi(t,\vec{z})$	t *substitutable for* x *in* φ
$(\exists 1)$	$\vdash \varphi(t,\vec{z}) \to (\exists x)\varphi(x,\vec{z})$	t *substitutable for* x *in* φ
$(\forall 2)$	$\chi \to \varphi \vdash \chi \to (\forall x)\varphi$	x *not free in* χ
$(\exists 2)$	$\varphi \to \chi \vdash (\exists x)\varphi \to \chi$	x *not free in* χ

Why only for such big $\mathbb{K}$s?

Especially if we know that for finitary $\models_{\mathbb{K}}$ and propositional languages:

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Consider $\mathbb{K}$ being the class of Heyting chains:

$$\text{Then } \varphi \vee \psi \models_{\mathbb{K}} ((\forall x)\varphi) \vee \psi \text{ but } \varphi \vee \psi \not\models_{\mathbf{Q}(\mathbb{K})} ((\forall x)\varphi) \vee \psi$$

Other example: the set $\{\varphi \mid \models_{[0,1]_{Ł}} \varphi\}$ is coNP-complete for propositional languages but Π_2-complete in general while $\{\varphi \mid \models_{\mathbf{Q}([0,1]_{Ł})} \varphi\}$ is Σ_1-complete

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But at least we will have soundness:

$$\models_{\mathbb{K}} \quad \supseteq \quad \models_{\mathbf{Q}(\mathbb{K})} \quad = \quad \vdash$$

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Theorem

Let $\mathbb{K}$ be a class of $\mathcal{L}$-algebras and for each countable $\boldsymbol{A} \in \mathbf{Q}(\mathbb{K})$ there is a σ-embedding of $\boldsymbol{A}$ into some $\boldsymbol{B} \in \mathbb{K}$. Then

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This condition is not necessary, only sufficient.

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A function $f \colon A \to B$ is a σ-embedding if:

- f is one-one
- $f(c^{\boldsymbol{A}}(a_1, \dots, a_n)) = c^{\boldsymbol{B}}(f(a_1), \dots, f(a_n))$ for each n-ary $c \in \mathbf{C}$
- for each $X \subseteq A$, if $\inf_{\leq^A} X$ exists, then $f(\inf_{\leq^A} X) = \inf_{\leq^B} f[A]$.
- for each $X \subseteq A$, if $\sup_{\leq^A} X$ exists, then $f(\sup_{\leq^A} X) = \sup_{\leq^B} f[A]$.

2nd axiomatizability result

Theorem (PC, C. Noguera. JSL 2015)

Let $\mathbb{K}'$ be a quasivariety of $\mathcal{L}$-algebras satisfying DC3', DC4, DC5 such that $\mathbb{K}' = \mathbf{Q}(\mathbb{K})$, where $\mathbb{K}$ is the class of all chains in $\mathbb{K}'$ and $\mathcal{AX}$ an arbitrary axiomatization of the propositional logic of $\mathbb{K}$.

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$(\forall 2)^{\vee}$	$(\chi \to \varphi) \vee \psi \vdash (\chi \to (\forall x)\varphi) \vee \psi$	*x not free in χ and ψ*
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Let L *be a semilinear finitary alg. impl. logic with axiomatization* $\mathcal{AX}$*, such that its equivalent algebraic semantics* $\mathbb{K}'$ *satisfies DC3', DC4, DC5 and let* $\mathbb{K}$ *be the class of its all chains in* $\mathbb{K}'$
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When can we axiomatize a logic based on a ‘smaller’ class of chains? (let us restrict to countable predicate languages)

Theorem

Let $\mathbb{K}$ be a class of $\mathcal{L}$-chains and for each countable chain $\boldsymbol{A} \in \mathbf{Q}(\mathbb{K})$ there is a σ-embedding of $\boldsymbol{A}$ into some $\boldsymbol{B} \in \mathbb{K}$. Then

$$\models_{\mathbb{K}} = \models_{\mathbf{Q}(\mathbb{K})}$$

Again, this condition is not necessary, only sufficient.

A summary of this section

We have designed 'lattice-valued predicate logics' based on design choices:

- DC1 the syntax is almost classical; we only consider an arbitrary set $\mathcal{L}$ of propositional connectives
- DC2 connectives have truth-functional interpretations
- DC3′ some grades are better than others and for each two grades there is the best (worst) grade worse (better) than both of them $\wedge, \vee \in \mathcal{L}$
- DC4 quantifiers are interpreted using infima and suprema over the set of instances of the formulas quantified
- DC5 some grades are 'good'; the logic/consequence is the transition of 'goodness'; and there is the least 'good' grade $\overline{1} \in \mathcal{L}$
- DC6 the order of grades and the set of good grades are mutually definable using implication $\rightarrow \in \mathcal{L}$

We have axiomatized, in some cases, the resulting logics

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Outline

1 Introduction

2 Lattice-Valued Predicate Logics

3 Formal Fuzzy Mathematics

4 What's next?

Recall our example

Take the standard MV-algebra $[0,1]_{Ł} = \langle [0,1], \&, \to, \wedge, \vee, 0, 1\rangle$ where

$$x \mathbin{\&} y = \max\{x + y - 1, 0\} \qquad x \to y = \min\{1 - x + y, 1\}$$

$$x \wedge y = \min\{x, y\} \qquad x \vee y = \max\{x, y\}$$

Consider a $[0,1]_{Ł}$-structure with domain $M = \{1, \dots, 6\}$ and binary predicate P: 'x likes y':

$P_{\mathfrak{M}}$	1	2	3	4	5	6
1	1.0	1.0	0.5	0.4	0.3	0.0
2	0.8	1.0	0.4	0.4	0.3	0.0
3	0.7	0.9	1.0	0.8	0.7	0.4
4	0.9	1.0	0.7	1.0	0.9	0.6
5	0.6	0.8	0.8	0.7	1.0	0.7
6	0.3	0.5	0.6	0.4	0.7	1.0

$\mathrm{Refl}(R) \equiv_{\mathrm{df}} (\forall x)Rxx \qquad \|\mathrm{Refl}(P)\|^{\mathfrak{M}} = 1$

$\mathrm{Sym}(R) \equiv_{\mathrm{df}} (\forall x, y)(Rxy \to Ryx) \qquad \|\mathrm{Sym}(P)\|^{\mathfrak{M}} = 0.4$

$\mathrm{Trans}(R) \equiv_{\mathrm{df}} (\forall x, y, z)(Rxy \mathbin{\&} Ryz \to Rxz) \qquad \|\mathrm{Trans}(P)\|^{\mathfrak{M}} = 1$

Recall our example

Take the standard MV-algebra $[0,1]_{Ł} = \langle [0,1], \&, \to, \wedge, \vee, 0, 1\rangle$ where

$$x \,\&\, y = \max\{x+y-1, 0\} \qquad x \to y = \min\{1-x+y, 1\}$$

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Consider a $[0,1]_{Ł}$-structure with domain $M = \{1,\dots,6\}$ and binary predicate P: 'x likes y':

$P_{\mathfrak{M}}$	1	2	3	4	5	6
1	1.00	1.00	0.56	0.40	0.30	0.00
2	0.87	1.00	0.33	0.44	0.26	0.02
3	0.67	0.92	0.93	0.87	0.70	0.39
4	0.93	1.00	0.64	1.00	0.97	0.67
5	0.52	0.79	0.82	0.71	1.00	0.59
6	0.27	0.50	0.61	0.41	0.72	1.00

$\mathrm{Refl}(R) \equiv_{\mathrm{df}} (\forall x)Rxx \qquad \|\mathrm{Refl}(P)\|^{\mathfrak{M}} = 0.93$

$\mathrm{Sym}(R) \equiv_{\mathrm{df}} (\forall x,y)(Rxy \to Ryx) \qquad \|\mathrm{Sym}(P)\|^{\mathfrak{M}} = 0.41$

$\mathrm{Trans}(R) \equiv_{\mathrm{df}} (\forall x,y,z)(Rxy \,\&\, Ryz \to Rxz) \qquad \|\mathrm{Trans}(P)\|^{\mathfrak{M}} = 0.93$

What we want to study?

Fuzzy equivalence relation a.k.a. similarity

$$\begin{aligned}
\mathrm{Refl}(R) &\equiv_{\mathrm{df}} (\forall x)Rxx\\
\mathrm{Sym}(R) &\equiv_{\mathrm{df}} (\forall x, y)(Rxy \to Ryx)\\
\mathrm{Trans}(R) &\equiv_{\mathrm{df}} (\forall x, y, z)(Rxy \mathbin{\&} Ryz \to Rxz)\\
\mathrm{Sim}(R) &\equiv_{\mathrm{df}} \mathrm{Refl}(R) \mathbin{\&} \mathrm{Sym}(R) \mathbin{\&} \mathrm{Trans}(R)
\end{aligned}$$

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We need a unary connective $\triangle$ interpreted as $\triangle 1 = 1$ and $\triangle x = 0$ for $x < 1$

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We need a unary connective $\triangle$ interpreted as $\triangle 1 = 1$ and $\triangle x = 0$ for $x < 1$

Fuzzy partitions

$$\begin{aligned}
\mathrm{Cover}(\mathcal{A}) &\equiv_{\mathrm{df}} (\forall x)(\exists A \in \mathcal{A})\triangle(x \in A) \\
\mathrm{Disj}(\mathcal{A}) &\equiv_{\mathrm{df}} (\forall A, B \in \mathcal{A})((\exists x)(x \in A \mathbin{\&} x \in B) \to A \subseteq B) \\
\mathrm{Crisp}(\mathcal{A}) &\equiv_{\mathrm{df}} (\forall A)\triangle(A \in \mathcal{A} \vee \neg(A \in \mathcal{A})) \\
\mathrm{NormM}(\mathcal{A}) &\equiv_{\mathrm{df}} (\forall A \in \mathcal{A})(\exists x)\triangle(x \in A) \\
\mathrm{Part}(\mathcal{A}) &\equiv_{\mathrm{df}} \mathrm{Crisp}(\mathcal{A}) \mathbin{\&} \mathrm{NormM}(\mathcal{A}) \mathbin{\&} \mathrm{Cover}(\mathcal{A}) \mathbin{\&} \mathrm{Disj}(\mathcal{A})
\end{aligned}$$

This part of the talk is based on: L. Běhounek, U. Bodenhofer, PC: *Relations in Fuzzy Class Theory.* Fuzzy Sets and Systems, 2008

Setting the stage—language

Consider three-sorted predicate language with sorts for

- O objects
- C classes of objects
- $\mathcal{C}$ classes of classes of objects

binary predicates

- $\in \subseteq O \times C$ and $\in \subseteq C \times \mathcal{C}$
- $= \subseteq O \times O$ and $= \subseteq C \times C$ and $= \subseteq \mathcal{C} \times \mathcal{C}$

and terms:

- $\langle \cdot, \cdot \rangle \colon O^2 \to O$, we write Rxy for '$\langle x, y \rangle \in X$'
- $\{x \mid \varphi\}$ gives a class and $\{X \mid \varphi\} \in \mathcal{C}$ a class of classes

We shall also use defined binary predicate:

- $A \subseteq B \equiv_{\mathrm{df}} (\forall x)(x \in A \to x \in B)$

Setting the stage—models

Intended models:

- Object variables range over a set (universe) U
- Class variables range over $[0,1]^U$
- Class-class variables range over $[0,1]^{[0,1]^U}$

'General' models for an MV-chain $\boldsymbol{A}$:

- Object variables range over a set (universe) U
- Class variables range over a subset of A^U
- Class-class variables range over a subset of A^{A^U}

Setting the stage—axiomatization?

W.r.t. intended models: not a nice one (it contains second-order logic)

W.r.t. general models: yes (due to the completeness theorem)

But even soundness w.r.t. intended models is very usefull

Setting the stage—axiomatization

First-order axioms: those of $\models_{\mathbb{K}}$ where $\mathbb{K}$ is the class all MV-chains with $\triangle$

Equality axioms: as usual plus $(\forall x, y)(\triangle(x = y) \leftrightarrow x = y)$

Additional axioms:

- *Comprehension axioms:*

$$(\forall y)(y \in \{x \mid \varphi(x)\} \leftrightarrow \varphi(y)) \text{ and } (\forall Y)(Y \in \{X \mid \varphi(X)\} \leftrightarrow \varphi(Y))$$

- *Extensionality:*

$$(\forall x)\triangle(x \in A \leftrightarrow x \in B) \to A = B$$

$$(\forall X)\triangle(X \in \mathcal{A} \leftrightarrow X \in \mathcal{B}) \to \mathcal{A} = \mathcal{B}$$

- *Axioms for tuples:* tuples equal iff all components equal, etc.

From similarities to partitions

Definitions

$$[x]_R =_{\mathrm{df}} \{y \mid Ryx\}$$

$$\mathrm{V}/R \ =_{\mathrm{df}} \ \{A \mid (\exists x)(A = [x]_R)\}$$

Results

- $\mathrm{Crisp}(\mathrm{V}/R)$
- $\triangle\,\mathrm{Refl}(R) \to \mathrm{Cover}(\mathrm{V}/R)\ \&\ \mathrm{NormM}(\mathrm{V}/R)$ $\qquad$ $\mathrm{Refl}(R) \to (\forall x)(x \in [x]_R)$
- $\mathrm{Trans}^2(R)\ \&\ \mathrm{Sym}(R) \to \mathrm{Disj}(\mathrm{V}/R)$

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1. $Rzx\ \&\ Rxy \to Rzy$ — $\mathrm{Trans}(R)$ and $(\forall 1)$
2. $Rxy \to (Rzx \to Rzy)$ — 1., residuation, MP
3. $Rxy \to (z \in [x]_R \to z \in [y]_R)$ — 2. and comprehension axioms
4. $Rxy \to [x]_R \subseteq [y]_R$ — 3. and $(\forall 2)$
5. $\mathrm{Trans}(R) \to (Rxy \to [x]_R \subseteq [y]_R)$ — 4. and deduction theorem
6. $\mathrm{Trans}(R) \to (\forall x, y)(Rxy \to [x]_R \subseteq [y]_R)$ — 5. and $(\forall 2)$

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1. $Rxy \,\&\, Rxz \to Ryz$ $\qquad$ $\mathrm{Trans}(R), \mathrm{Sym}(R)$, and $(\forall 1)$
2. $x \in [y]_R \,\&\, x \in [z]_R \to Ryz$ $\qquad$ 1. and comprehension axioms
3. $x \in [y]_R \,\&\, x \in [z]_R \to [y]_R \subseteq [z]_R$ $\qquad$ 2. and $\mathrm{Trans}(R)$
4. $(\exists x)(x \in [y]_R \,\&\, x \in [z]_R) \to [y]_R \subseteq [z]_R$ $\qquad$ 3. and $(\exists 2)$
5. $\mathrm{Trans}^2(R) \,\&\, \mathrm{Sym}(R) \to ((\exists x)(x \in [y]_R \,\&\, x \in [z]_R) \to [y]_R \subseteq [z]_R)$ $\qquad$ 4. and DT
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So together we proved:

$$\mathrm{Trans}^2(R)\ \&\ \mathrm{Sym}(R)\ \&\ \triangle\,\mathrm{Refl}(R) \to \mathrm{Part}(\mathrm{V}/R)$$

$$\triangle\,\mathrm{Sim}(R) \to \triangle\,\mathrm{Part}(\mathrm{V}/R)$$

Semantical content

$$\mathrm{Trans}^2(R) \mathbin{\&} \mathrm{Sym}(R) \to \mathrm{Disj}(\mathrm{V}/R)$$

Thus for fuzzy relation $R\colon U^2 \to [0,1]$ st. $\|\mathrm{Trans}(R)\| = \|\mathrm{Sym}(R)\| = 0.9$:

$$0.7 \leq \|\mathrm{Disj}(\mathrm{V}/R)\|$$

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$$0.7 \leq \|\mathrm{Disj}(\mathrm{V}/R)\|$$

This says that:

$$0.7 \leq \inf_{A,B \in \mathrm{V}/R} (\sup_{z \in U}(z \in A \,\&\, z \in B) \to \inf_{z \in U}(z \in A \to z \in B))$$

where

$$x \,\&\, y = \max\{x + y - 1, 0\} \qquad x \to y = \min\{1 - x + y, 1\}$$

From partitions to similarities

Definition

$$R^{\mathcal{A}} \quad =_{\mathrm{df}} \quad \{\langle x, y\rangle \mid (\exists A \in \mathcal{A})(x \in A \mathbin{\&} y \in A)\}$$

Results

- $\mathrm{Sym}(R^{\mathcal{A}})$
- $\mathrm{Crisp}(\mathcal{A}) \mathbin{\&} \mathrm{Cover}(\mathcal{A}) \to \triangle\, \mathrm{Refl}(R^{\mathcal{A}})$
- $\mathrm{Disj}(\mathcal{A}) \to \mathrm{Trans}(R^{\mathcal{A}})$

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- $\mathrm{Disj}(\mathcal{A}) \to \mathrm{Trans}(R^{\mathcal{A}})$

1. $(y \in X \mathbin{\&} y \in Y) \to X \subseteq Y$ — $\mathrm{Disj}\,(\mathcal{X})$, $(\forall 1)$, and $(\exists 1)$
2. $(\exists X \in \mathcal{X})(x \in X \mathbin{\&} y \in X)$ — $R^{\mathcal{X}}xy$
3. $(\exists Y \in \mathcal{X})(y \in Y \mathbin{\&} z \in Y)$ — $R^{\mathcal{X}}yz$
4. $(\exists X, Y \in \mathcal{X})(x \in X \mathbin{\&} y \in X \mathbin{\&} y \in Y \mathbin{\&} z \in Y)$ — 2., 3., and ...
5. $(\exists X, Y \in \mathcal{X})(x \in X \mathbin{\&} X \subseteq Y \mathbin{\&} z \in Y)$ — 1., 4., and ...
6. $(\exists X, Y \in \mathcal{X})(x \in Y \mathbin{\&} z \in Y)$ — 5. and ...

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- $\mathrm{Sym}(R^{\mathcal{A}})$
- $\mathrm{Crisp}(\mathcal{A}) \mathbin{\&} \mathrm{Cover}(\mathcal{A}) \to \triangle\,\mathrm{Refl}(R^{\mathcal{A}})$
- $\mathrm{Disj}(\mathcal{A}) \to \mathrm{Trans}(R^{\mathcal{A}})$

So together we proved:

$$\mathrm{Part}(\mathcal{A}) \to \triangle\,\mathrm{Sym}(R^{\mathcal{A}}) \mathbin{\&} \triangle\,\mathrm{Refl}(R^{\mathcal{A}}) \mathbin{\&} \mathrm{Trans}(R^{\mathcal{A}})$$

$$\mathrm{Part}(\mathcal{A}) \to \mathrm{Sim}(R^{\mathcal{A}})$$

$$\triangle\,\mathrm{Part}(\mathcal{A}) \to \triangle\,\mathrm{Sim}(R^{\mathcal{A}})$$

There and back again . . .

Results

$$\mathrm{Sim}(R) \to (R^{\mathrm{V}/R} \approx R)$$

$$\triangle\, \mathrm{Part}(\mathcal{A}) \to \mathrm{V}/R^{\mathcal{A}} = \mathcal{A}$$

Proof of $R \subseteq R^{\mathrm{V}/R}$:

1. Rxy
2. $[y]_R = [y]_R \;\&\; x \in [y]_R \;\&\; y \in [y]_R$ $\qquad$ $\mathrm{Refl}(R)$
3. $(\exists z)([z]_R = [z]_R \;\&\; x \in [z]_R \;\&\; y \in [z]_R)$
4. $(\exists Z)(\exists z)([z]_R = Z \;\&\; x \in Z \;\&\; y \in Z)$
5. $(\exists Z)((\exists z)([z]_R = Z) \;\&\; x \in Z \;\&\; y \in Z)$
6. $(\exists Z \in \mathrm{V}/R)(x \in Z \;\&\; y \in Z)$
7. $R^{\mathrm{V}/R}\, xy$

Outline

1. Introduction
2. Lattice-Valued Predicate Logics
3. Formal Fuzzy Mathematics
4. What's next?

Future work: the ‘logical’ part

- Extending the scope of our results
- Generalizing other usual classical results
  - Developing model-theory of our structures
  - Studying the usual strengthenings of classical FO
- Studying genuinely ‘non-classical’ aspects of our approach:
  - Safe structures
  - Unusual forms of Skolemization, Herbrand theorem etc.
  - Witnessed structures
  - Generalized quantifiers
- Exploring connections to other approaches to non-classical FOL:
  - Those close in spirit to ours; e.g. Ono’s treatment of first-order substructural logics
  - Those based on some kind of Kripke semantics
  - Those based on polyadic and cylindric algebras
  - Categorial approaches
  - Game-theoretic semantics
  - Continuous model theory

Future work: the ‘fuzzy mathematics’ part

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Future work: the ‘fuzzy mathematics’ part

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http://www.cs.cas.cz/fct

Future work: The big picture

$$\infty$$

Future work: The big picture

$$\infty$$

Find a ‘market’ for all this work

Future work: The big picture

$$\infty$$

Find a ‘market’ for all this work
not only in mathematics

Future work: The big picture

$$\infty$$

Find a ‘market’ for all this work
not only in mathematics
but also in
computer science, linguistic, philosophy, etc.