# A categorical equivalence for Nelson algebras

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# Kalman´s construction

J. Kalman [*Lattices with involution*. Trans. Amer. Math. Soc., 87:485–491, 1958.]
Given a bounded distributive lattice $(L, \wedge, \vee, 0, 1)$, and an element $a \in L$, let

$$L(a) = \{(x, y) \in L \times L : x \wedge y \leq a \leq x \vee y\}.$$

Operations on $L(a)$:

K1) $(0, 1), (1, 0) \in L(a)$.

K2) $(x_1, x_2) \cup (y_1, y_2) = (x_1 \vee y_1, x_2 \wedge y_2)$;

K3) $(x_1, x_2) \cap (y_1, y_2) = (x_1 \wedge y_1, x_2 \vee y_2)$;

K4) $\sim (x_1, x_2) = (x_2, x_1)$;

$(L(a), \cap, \cup, (1, 0))$ is a Kleene algebra.

# Vakarelov´s construction

Dimiter Vakarelov. *Notes on N-lattices and constructive logic with strong negation.* Studia Logica, 36(1–2):109–125, 1977.
Given a Heyting algebra $(H, \wedge, \vee, \Rightarrow, 0, 1)$, let $V(H)$ be the set:

$$V(H) = \{(a, b) \in H \times H : a \wedge b = 0\}.$$

In addition to the operations defined on $V(H)$ by K2)-K4) Vakarelov defines $\rightarrow$ by:

K5) $(x_1, x_2) \rightarrow (y_1, y_2) = (x_1 \Rightarrow y_1, x_1 \wedge y_2)$,

and proves it is a binary operation on $V(H)$, so $V(H)$ is a Nelson algebra.

# D. Brignole and M. Fidel

M. Fidel and D. Brignole define in [*Algebraic study of some nonclassical logics using product algebras*. In Proceedings of the First "Dr. Antonio A. R. Monteiro" Congress on Mathematics (Spanish) (Bahía Blanca, 1991), pages 23–38. Univ. Nac. del Sur, Bahía Blanca, 1991.] the following notion:

**Definition**

*A* P-De Morgan algebra *is a non-empty subset $M$ of $L \times L^*$ such that:*

P1) $(0,1),(1,0) \in M$,

P2) *If* $(x_1, x_2) \in M$ *then* $(x_2, x_1) \in M$,

P3) *If* $(x_1, x_2), (y_1, y_2) \in M$, *then* $(x_1 \wedge y_1, x_2 \vee y_2) \in M$,

*in which the operations* $\cap, \cup$ *and* $\sim$ *are defined by V2)-V4).*

# Applications

- Andrzej Sendlewski. *Nelson algebras through Heyting ones.* Studia Logica 49 (1990), no. 1, 105–126.
- Luiz Monteiro and I.V, *Construction of Nelson algebras*, IV congreso A. Monteiro, 1997.
- Umberto Rivieccio. *Implicative twist-structures.* Algebra Universalis 71 (2014), no. 2, 155–186.
- Marta Sagastume, Hernán San Martín, *A categorical equivalence motivated by Kalman's construction.* Studia Logica 104 (2016), no. 2, 185–208.
- Juan Manuel Cornejo, Ignacio Viglizzo. *Semi-Nelson algebras*. Order (2017).
- S. Aguzzoli, M. Busaniche, B. Gerla, Miguel Andrés Marcos. *NPc-algebras and Gödel hoops*. Yesterday!

# $M(L, I, F)$

For a given ideal $I$ and a filter $F$ of a bounded distributive lattice $(L, \wedge, \vee, 0, 1)$, we consider the set:

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$M(L, I, F)$ with the operations defined by Kalman is a De Morgan algebra.

# Results about the construction $M(L, I, F)$

- It is more general than Kalman's:

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*$M(L, I, F)$ is a Kleene algebra if and only if for all $i \in I$ and $f \in F$, $i \leq f$ is verified.*

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**Lemma**

*$M(L, I, F)$ is a Kleene algebra if and only if for all $i \in I$ and $f \in F$, $i \leq f$ is verified.*

**Lemma**

*If $I_1, I_2$ are ideals of $L$ and $I_1 \subseteq I_2$ then $M(L, I_1, F)$ is a subalgebra of $M(L, I_2, F)$.*
*If $F_1, F_2$ are filters of $L$ and $F_1 \subseteq F_2$ then $M(L, I, F_1)$ is a subalgebra of $M(L, I, F_2)$.*

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**Lemma**

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**Lemma**

*Let $H$ be a Heyting algebra and $(a, b), (c, d) \in M(H, I, F)$. Then $(a, b) \to (c, d) = (a \Rightarrow c, a \wedge d) \in M(H, I, F)$ if and only if $(a \Rightarrow c) \vee a \in F$.*

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## Lemma

*Let $H$ be a Heyting algebra with an ideal $I$ and a filter $F$ such that $\pi_1(M(H, I, F)) = H$. Then K5) defines a binary operation on $M(H, I, F)$ if and only if $Ds(H) = \{y \in H : y = x \vee \neg x\} \subseteq F$.*

## $N(H, F)$

**Theorem**

*Let $H$ be a Heyting algebra. Then $(M(H, I, F), \cap, \cup, \to, \sim, (1, 0))$ is a Nelson algebra if and only if $I = \{0\}$.*

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$$V(H) = N(H, H).$$

# $N(H, F)$

**Theorem**

*Let* $(H, \wedge, \vee, \Rightarrow, 0, 1)$ *be an algebra of type* $(2, 2, 2, 0, 0)$. *If* $(N(H, F), (1, 0), \sim, \cap, \cup, \rightarrow)$ *is a Nelson algebra, where the operations are defined by K2)-K5) and* $\pi_1(N(H, F)) = H$, *then* $H$ *is a Heyting algebra.*

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**Lemma**

*Let $H$ be a Heyting algebra and $S$ a subalgebra of the Nelson algebra $V(H)$ such that $\pi_1(S) = H$. The filter $F = \{x \in H : x = a \vee b, (a, b) \in S\}$ in $H$ such that $S = N(H, F)$.*

# The equivalence

Andrzej Sendlewski. *Nelson algebras through Heyting ones.* Studia Logica 49 (1990), no. 1, 105–126.

# The equivalence

Given a Nelson algebra $A$, one can find a Heyting algebra $H(A)$ by finding the quotient through the relation

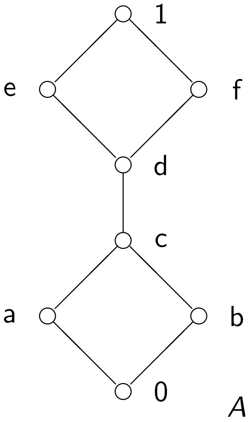
$$x \cong y \quad \textit{iff} \quad x \to y = 1 = y \to x,$$

defining $|x| \Rightarrow |y|$ as $|x \to y|$.
There is an injection $h : A \to V(H(A))$.

$$h(a) = (|a|, |\sim a|).$$

For any Heyting algebra $H$, $H(V(H))$ is isomorphic to $H$.
Different Nelson algebras can have the same associated Heyting algebra.

# An example



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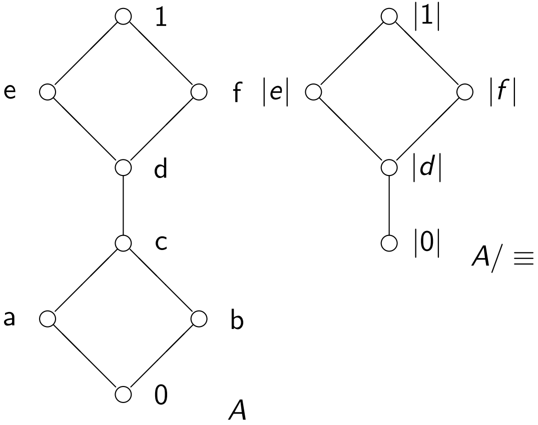

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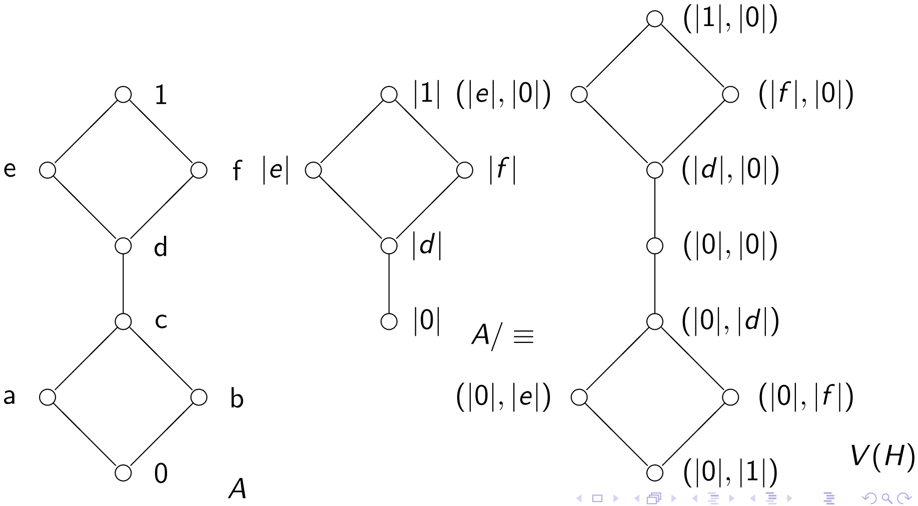

# The equivalence

We define the category **HF**:

- Objects: pairs $(H, F)$, where $H$ is a Heyting algebra and $F$ one of its filters such that $Ds_H(H) \subseteq F$.
- Morphisms: $f : (H, F) \to (H', F')$ , where $f : H \to H'$ is a Heyting algebra morphism such that $f(F) \subseteq F'$.

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The functor NH: **HF** $\to$ **N** is given by $\mathrm{NH}(\mathrm{H}, \mathrm{F}) = \mathrm{N}(\mathrm{H}, \mathrm{F})$ and if

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$\mathrm{NHf((x, y)) = (f(x), f(y))}$.

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The filter is $E = \{|a| \vee | \sim a| : a \in A\}$.

Going in the other direction, given a a Nelson algebra $A$, let $F(A)$ be the filter $\{|a \vee \sim a| : a \in A\}$.
The functor

$$\mathrm{HN} : \mathbf{N} \to \mathbf{HF}.$$

$$\mathrm{HN(A) = (H(A), F(A))}$$

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The functors NH and HN establish the equivalence between **N** and **HF**.