Synonymy and categories of logics

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- $F(\Sigma)[0]$ set of formulas without variables.

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- For every $f : \Sigma_1 \to \Sigma_2$ and $g : \Sigma_2 \to \Sigma_3$, the composition $(gf)[n] : \Sigma_1[n] \to F(\Sigma_3)[n]$ is given by $(gf)[n] := \hat{g} f[n]$.

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- Category S_f.

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- $(\Sigma, \vdash)$ is a **logic**.

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- A **flexible translation of logics** $f : (\Sigma_1, \vdash_1) \to (\Sigma_2, \vdash_2)$ is a $f : \Sigma_1 \to \Sigma_2$, such that $\hat{f}(\Gamma) \vdash_2 \hat{f}(\phi)$ for all $\Gamma \subseteq F(\Sigma)$ and $\phi \in F(\Sigma)$ such that $\Gamma \vdash_1 \phi$.

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- L_f does not have terminal object or non-empty products.

Interdemonstrability

- [Caleiro 2007][2] $f \dashv\vdash f'$ if and only if $f, f' : (\Sigma_1, \vdash_1) \to (\Sigma_2, \vdash_2)$ and $\hat{f}(\alpha) \dashv\vdash_2 \hat{g}(\alpha)$ for all $\alpha \in F(\Sigma_1)$.

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Congruential logics

- [Mendes 2016][3] $(\Sigma, \vdash)$ is **congruential** if for every $\alpha(p_{i_1}, ..., p_{i_n}), \beta(p_{i_1}, ..., p_{i_n}) \in F(\Sigma)$ such that $\alpha \dashv\vdash \beta$, and for all $\{(\phi_1, \psi_1), ..., (\phi_n, \psi_n)\}$ such that $\phi_1 \dashv\vdash \psi_1, ..., \phi_n \dashv\vdash \psi_n$, then $\alpha(\phi_1, ..., \phi_n) \dashv\vdash \beta(\psi_1, ..., \psi_n)$.

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- We ended up restricting the morphisms.

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Synonymy

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- $\lhd\rhd$ is it not a congruence on L_f.

Translations of synonymous

- $f : (\Sigma_1, \vdash_1) \to (\Sigma_2, \vdash_2)$ is a **translation of synonymous** if f is a flexible translation of logics such that $\hat{f}(\alpha) \lhd\rhd_2 \hat{f}(\beta)$ for all $\alpha, \beta \in F(\Sigma)$ such that $\alpha \lhd\rhd_1 \beta$.

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- L_s the category whose objects are logics and whose morphisms are translations of synonymous.
- $\lhd\rhd$ is a congruence on L_s.
- Quotient category QL_s whose objects are logics and whose morphisms are equivalence classes of synonymous morphisms $[f] = \{g : f \lhd \rhd g\}$.

Synonymous morphisms

- If $(\Sigma_1, \vdash_1)$ and $(\Sigma_2, \vdash_2)$ are congruential logics and $f, g : (\Sigma_1, \vdash_1) \to (\Sigma_2, \vdash_2)$ flexible translations of logics,
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- f are g translations of synonymous.
- $f \lhd \rhd g$ if and only if $f \dashv\vdash g$.
- The presentation of classical logics are congruentials.
- Any flexible translation between them is a synonymous morphism.

Equivalents logics

- $(\Sigma_1, \vdash_1)$ and $(\Sigma_2, \vdash_2)$ are equivalents on QL_s if there exists $f : (\Sigma_1, \vdash_1) \to (\Sigma_2, \vdash_2)$ and $g : (\Sigma_2, \vdash_2) \to (\Sigma_1, \vdash_1)$ translation of synonymous such that $gf \lhd \rhd id_{(\Sigma_1, \vdash_1)}$ and $fg \lhd \rhd id_{(\Sigma_2, \vdash_2)}$.

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- The presentations of classical logics are equivalents.

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  - Σ_T such that $\Sigma_T[0] = \{c_0\}$, $\Sigma_T[2] = \{c_2\}$ and $\Sigma_T[n] = \varnothing$ for all $2 \neq n \in \mathbb{N}$.
  - $\Gamma \vdash_T \phi$ for all $\Gamma \subseteq F(\Sigma_T)$ and $\phi \in F(\Sigma_T)$.
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- If QL_s has weak coproducts, it is not with the coproduct signature.

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- If $\{(\Sigma_i, \vdash_i)\}$ has weak coproduct with the coproduct signature then it has coproduct.
- Looking for a sufficient condition for existence of weak coproducts with coproduct signatures.

THANKS!