On Kalman's functor for bounded hemiimplicative semilattices and hemiimplicative lattices

Ramon Jansana [1] and Hernán Javier San Martín [2]

[1] Departament de Filosofia, Universitat de Barcelona.
[2] CONICET and Departamento de Matemática, FCE, UNLP.

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Kalman's functor

A *De Morgan* algebra is an algebra $\langle A, \vee, \wedge, \sim, 0, 1\rangle$ of type $(2, 2, 1, 0, 0)$ such that satisfies the following conditions:

- $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice,
- $\sim\sim x = x$,
- $\sim (x \vee y) = \sim x \wedge \sim y$, $\sim (x \wedge y) = \sim x \vee \sim y$.

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$$x \wedge \sim x \leq y \vee \sim y.$$

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A Kleene algebra is *centered* if it has a center. That is, an element c such that $\sim \mathrm{c} = \mathrm{c}$ (it is necessarily unique).

Kalman's functor

In 1958 Kalman proved that if L is a bounded distributive lattice, then

$$\mathrm{K}(L) = \{(a, b) \in L \times L : a \wedge b = 0\}$$

is a centered Kleene algebra by defining

$$\begin{aligned} (a, b) \vee (d, e) &:= (a \vee d, b \wedge e), \\ (a, b) \wedge (d, e) &:= (a \wedge d, b \vee e), \\ \sim (a, b) &:= (b, a), \end{aligned}$$

$(0, 1)$ as the zero, $(1, 0)$ as the top and $(0, 0)$ as the center.

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- Kalman J.A, *Lattices with involution*. Trans. Amer. Math. Soc. 87, 485–491, 1958.

Kalman's functor

For $(a, b) \in \mathrm{K}(L)$ we have that

$$(a, b) \wedge (0, 0) = (a \wedge 0, b \vee 0) = (0, b),$$

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Therefore, the center give us the coordinates of (a, b).

Kalman's functor

Later, in 1986 Cignoli proved the following facts:

1. K can be extended to a functor from the category of bounded distributive lattices BDL to the category of centered Kleene algebras.
2. There exists an equivalence between BDL and the full subcategory of centered Kleene algebras which satisfy (IP) (a topological condition).

- Cignoli R., *The class of Kleene algebras satisfying an interpolation property and Nelson algebras*. Algebra Universalis 23, 262–292, 1986.

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1. In an unpublished manuscript (2004) M. Sagastume proved that if T is a centered Kleene algebra then

   T satisfies (IP) iff T satisfies (CK) (an algebraic condition).

Kalman's functor

1. If T is a centered Kleene algebra then $\mathrm{C}(T) = \{x : x \geq \mathrm{c}\} \in \mathrm{BDL}$.
2. C can be extended to a functor.

Theorem

There is a categorical equivalence $\mathrm{K} \dashv \mathrm{C}$ between BDL and the full subcategory of centered Kleene algebras whose objects satisfy (CK).

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- Sagastume, M. Categorical equivalence between centered Kleene algebras with condition (CK) and bounded distributive lattices, 2004.

Kalman's functor for Heyting algebras

1. A quasi-Nelson algebra is a Kleene algebra such that
$$x \to y := x \to_{\mathrm{HA}} (\sim x \vee y)$$
exists, where $\to_{\mathrm{HA}}$ is the Heyting implication.
2. A Nelson algebra is a quasi-Nelson algebra such that
$$(x \wedge y) \to z = x \to (y \to z).$$
3. A Nelson lattice is an involutive bounded conmutative residuated lattice which satisfies an additional equation. The varieties of Nelson algebras and Nelson lattices are term equivalent.

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Kalman's functor for Heyting algebras

Let H be a Heyting algebra where $\to$ is the Heyting implication.
In $\mathrm{K}(H)$ the implication as Nelson algebra is given by

$$(a, b) \Rightarrow_{\mathrm{NA}} (d, e) = (a \to d, a \wedge e)$$

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Remark

Notice that the well definition of $\Rightarrow_{\mathrm{NA}}$ and $\Rightarrow$ follows from the equation

$$a \wedge (a \to b) \leq b$$

Hemiimplicative semilattices (lattices)

Definition

An algebra $(H, \wedge, \to, 1)$ of type $(2, 2, 0)$ is a hemiimplicative semilattice if:

1. $(H, \wedge, 1)$ is a bounded semilattice.
2. For every $a, b \in H$, $a \wedge (a \to b) \leq b$.
3. For every $a \in H$, $a \to a = 1$.

We say that an algebra $(H, \wedge, \vee, \to, 0, 1)$ of type $(2, 2, 2, 0, 0)$ is a hemiimplicative lattice if $(H, \wedge, \vee, 0, 1)$ is a bounded distributive lattice and $(H, \wedge, \to, 1)$ is a hemiimplicative semilattice.

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An algebra $(H, \wedge, \rightarrow, 1)$ of type $(2, 2, 0)$ is a hemiimplicative semilattice if:

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Item 2 is equivalent to the following condition:

$$\text{For every } a, b, c \in H, \text{ if } a \leq b \rightarrow c \text{ then } a \wedge b \leq c.$$

Example 1: Implicative semilattices

Definition

An implicative semilattice is an algebra $(H, \wedge, \to)$ of type $(2, 2)$ such that $(H, \wedge)$ is semilattice, and for every $a, b, c \in H$ we have that $a \wedge b \leq c$ if and only if $a \leq b \to c$.

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Every implicative semilattice has a greatest element.

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Every implicative semilattice has a greatest element.

- Nemitz W., *Implicative semi-lattices*. Trans. Amer. Math. Soc. 117, 128–142 (1965).

Example 2: Hilbert algebras with infimum

Definition

A *Hilbert algebra with infimum* is an algebra $(H, \to, \wedge, 1)$ of type $(2, 2, 0)$ such that satisfies the following conditions:

1. $a \to (b \to a) = 1$.
2. $a \to (b \to d) = (a \to b) \to (a \to d)$.
3. If $a \to b = b \to a = 1$, then $a = b$.
4. $(H, \wedge, 1)$ is a bounded semilattice.
5. For every $a, b \in H$, $a \leq b$ if and only if $a \to b = 1$, where $\leq$ is the semilattice order.

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- Diego A., *Sobre Algebras de Hilbert*. Notas de Lógica Matemática. Instituto de Matemática, UNS, Bahía Blanca (1965).
- Figallo A.V., Ramon G. and Saad S., *A note on the Hilbert algebras with infimum*. Math. Contemp. 24, 23–37 (2003).

Example 3: Semi-Heyting algebras

Definition

An algebra $(H, \wedge, \vee, \rightarrow, 0, 1)$ is a semi-Heyting algebra if the following conditions hold for every $a, b, c \in H$:

1. $(H, \wedge, \vee, 0, 1)$ is a bounded distributive lattice,
2. $a \wedge (a \rightarrow b) = a \wedge b$,
3. $a \wedge (b \rightarrow c) = a \wedge [(a \wedge b) \rightarrow (a \wedge c)]$,
4. $a \rightarrow a = 1$.

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- Sankappanavar H.P., *Semi-Heyting algebras: an abstraction from Heyting algebras*. Proceedings of the 9th Congreso "Dr. Antonio A. R.", 33–66, Actas del Congreso "Dr. Antonio A. R. Monteiro", UNS, Bahía Blanca, Argentina (2008).
- Cornejo J.M., *Subvariedades de álgebras de semi-Heyting*. Tesis Doctoral, UNS, Bahía Blanca, Argentina (2011).

Extending Kalman's functor

Consider the algebraic category KhBDL whose objects are algebras $(T, \wedge, \vee, \rightarrow, \sim, \mathrm{c}, 0, 1)$ of type $(2, 2, 2, 1, 0, 0, 0)$ such that

1. $(T, \wedge, \vee, \sim, \mathrm{c}, 0, 1)$ is a centered Kleene algebra,
2. $\rightarrow$ is a binary operation on T which satisfies certain equations involving the other operations.

Theorem

There is an equivalence $\mathrm{K} \dashv \mathrm{C}$ between the full subcategory of KhBDL whose objects satisfy (CK) and the algebraic category of hemiimplicative lattices.

A similar result is obtained for the category of semi-Heyting algebras (the equivalent category to the category of semi-Heyting algebras is a variety)

Extending Kalman's functor

Consider the category KhIS whose objects are structures $(T, \leq, \rightarrow, \sim, \mathrm{c}, 0, 1)$ such that

1. $(T, \leq)$ is a poset with first element 0 and last element 1,
2. $(T, \rightarrow, \sim, \mathrm{c}, 0, 1)$ is an algebra which satisfies certain conditions involving the order and the operations.

Theorem

There is an equivalence $\mathrm{K} \dashv \mathrm{C}$ between the full subcategory of KhIS whose objects satify the condition (CK) (over the corresponding existing infima and suprema) and the algebraic category of hemiimplicative lattices with first element.

A similar result is obtained for the category of Hilbert algebras with infimum and first element, and for the category of implicative semilattices with first element.

Extending Kalman's functor

In the paper

- *Semi-Nelson Algebras*, from Juan Manuel Cornejo and Ignacio Viglizzo (to appear in Order),

it was introduced and studied the variety of semi-Nelson algebras.

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it was introduced and studied the variety of semi-Nelson algebras.

- The varieties of Nelson algebras and Nelson lattices are term equivalent. Is there any relation between the variety of semi-Nelson algebras and the variety whose algebras are the objects of our equivalent category of the category of semi-Heyting algebras?

Extending Kalman's functor

- If $T \in \mathsf{KhBDL}$ then

$$\mathrm{Con}(T) \cong \mathrm{Con}(\mathrm{C}(T)).$$

In particular, if $\mathrm{C}(T)$ is a semi-Heyting algebra then we characterize the principal congruences of T.

- Let $T \in \mathsf{KhIS}$. We say that an equivalence relation θ of T is a *well-behaved congruence of T* if it satisfies the following conditions:
  1. $\theta \in \mathrm{Con}((T, \rightarrow, \sim))$.
  2. For $x, y \in T$, $(x, y) \in \theta$ if and only if $(x \vee \mathrm{c}, y \vee \mathrm{c}) \in \theta$ and $(\sim x \vee \mathrm{c}, \sim y \vee \mathrm{c}) \in \theta$.
  3. For x, y, z and w in $\mathrm{C}(T)$, if $(x, y) \in \theta$ and $(z, w) \in \theta$, then $(x \wedge z, y \wedge w) \in \theta$.

Let $\mathrm{Con}_{\mathrm{wb}}(T)$ be the set of well-behaved congruence of T. Then,

$$\mathrm{Con}_{\mathrm{wb}}(T) \cong \mathrm{Con}(\mathrm{C}(T)).$$

In particular, if $\mathrm{C}(T)$ is an implicative semilattice then we characterize the principal well-behaved congruences of T.