A canonical representation for free left-dioids

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Motivation

Computational effects $\longrightarrow$ Monads $\longrightarrow$ Monoids

Free structures give syntactic programs modulo expected laws of behavior: we want implementations.

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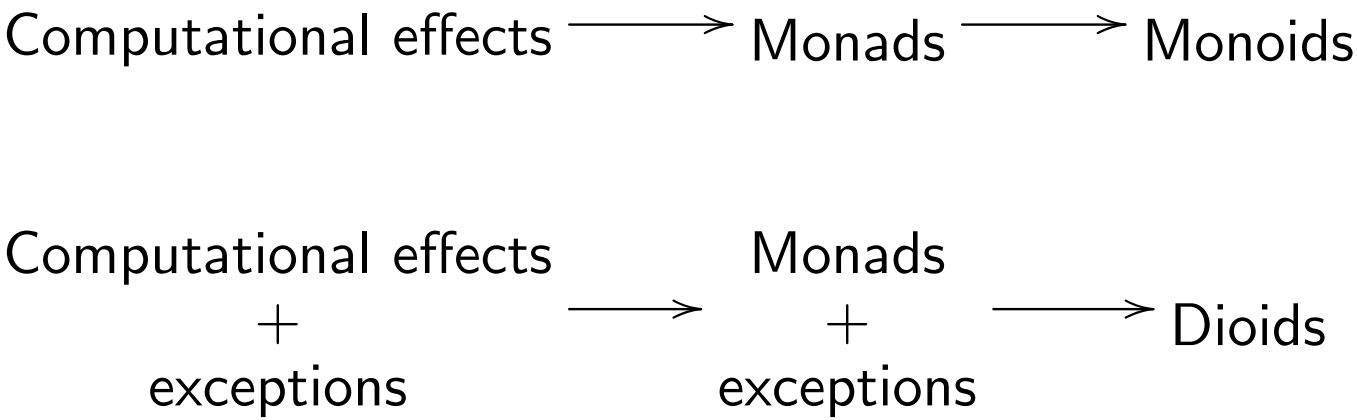

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Left-dioids

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A *(left-)dioid* is a set tuple $(D, \otimes, \oplus, 1_D, 0_D)$, where

- D is a set,
- $\otimes, \oplus : D \times D \to D$ are binary operations,
- $1_D, 0_D \in D$ are constants,

such that the following is satisfied:

- $(D, \otimes, 1_D)$ is a monoid,
- $(D, \oplus, 0_D)$ is a monoid,
- $0_D \otimes x = 0_D$ for all $x \in D$,
- $1_D \oplus x = 1_D$ for all $x \in D$.

Examples of dioids

Bounded lattices

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Let S be any set. Then $\langle S \times S \to S, \diamond, *, \pi_1, \pi_2 \rangle$ is a dioid, where $(f \diamond g)(x, y) = f(g(x, y), y)$ and $(f * g)(x, y) = f(x, g(x, y))$.

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Real unit interval

The real interval $[0, 1]$ has a dioid structure $\langle [0, 1], *, \cdot, 0, 1 \rangle$ where $x \cdot y$ is the usual multiplication and $x * y = x + y - xy$.

Free objects

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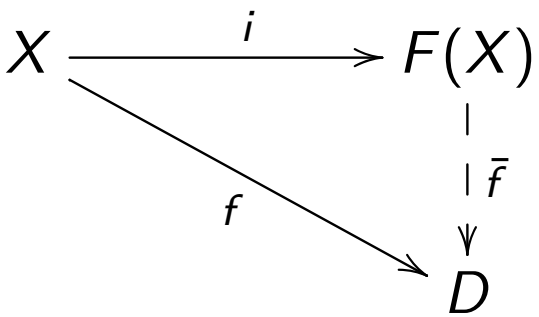
Universality: for any dioid D and function $f : X \to D$, there exists a unique dioid homomorphism $\bar{f} : F(X) \to D$ which make



commute.

Constructing the free algebras

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Alternative plan

We try to find canonical elements for each set, in an uniform manner, obtaining a concrete representation.

An illustrative example: monoids

A concrete representation the free monoid over X is given by the least solution to the following recursive equation of sets:

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$$\{(xy)z, (xe)(yz), e(x(y(ze))), x(yz), \ldots\} \mapsto x(y(ze))$$

This construction can be generalised to obtain free monoids in monoidal categories.

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Don't worry, we have a proof in the proof assistant Agda!

Conclusion and further work

Main conclusion: we have an explicit construction for free dioids which can be generalised to other categories.

We are now mainly interested in finding:

- Another representation for free dioids.
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Thank you!