Demiquantifiers on MV-algebras

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Let X be a nonvoid set and let $A = [0,1]^X$ be the MV-algebra of all functions from X to the unit real interval $[0,1]$. Let $\exists$ be the classical functional existential quantifier defined on A given by $\exists f = \bigvee_{x \in X} f(x)$. Then the operator $\exists$ satisfies the following condition:

$\exists f = 1$ if and only if $f^{-1}(1-\varepsilon, 1] \neq \emptyset$ for every $\varepsilon > 0$.
Analogously the universal quantifier $\forall$ defined by $\exists f = \bigwedge_{x \in X} f(x)$ satisfies
$\forall f = 0$ if and only if $f^{-1}[0, \varepsilon) \neq \emptyset$ for every $\varepsilon > 0$.
Notation: we write $\exists_1 = \exists$ and $\exists_0 = \forall$.

General Problem: for each constant $c \in [0,1]$ find an operator $\exists_c$ defined on A satisfying the analog conditions for the operators $\exists_0$ and $\exists_1$, i.e.
(P) $\exists_c f = c$ if and only if $f^{-1}(c-\varepsilon, c+\varepsilon) \neq \emptyset$ for every $\varepsilon > 0$, or equivalently, the inverse image of every neighborhood of c is non-empty.
expressed by means of the infinite connectives $\bigvee, \bigwedge$ and the usual connectives of MV-algebras.

In this talk we give a positive answer provided $c = \frac{1}{2}$.

Definition

Given $f : X \longrightarrow [0,1]$ we define the operator $\exists_{\frac{1}{2}}$ on A given by the formula:

$\exists_{\frac{1}{2}} f = [\bigvee_{x\in X} f(x)] \wedge [\bigwedge_{x\in X}(f(x) \vee \neg f(x))]$

Then the operator satisfies condition (P) when $c = \frac{1}{2}$. The proof is immediate consequence of the identity:

$a \vee \neg a = |a - \frac{1}{2}| + \frac{1}{2}$ for all $a \in [0,1]$.

Definition

Let $\langle A, \oplus, \neg, 0, \rangle$ be an MV-algebra. An operator $\exists_{\frac{1}{2}} : A \to A$ is said to be a *demiquantifier* provided it satisfies the following conditions:

(DM1) $\exists_{\frac{1}{2}} 0 = 0$

(DM2) $x \wedge \neg x \leq \exists_{\frac{1}{2}}(x \wedge \neg x))$

(DM3) $\exists_{\frac{1}{2}}(x \wedge \exists_{\frac{1}{2}} y) = \exists_{\frac{1}{2}} x \wedge \exists_{\frac{1}{2}} y$

(DM4) $\exists_{\frac{1}{2}} 2x = 2\exists_{\frac{1}{2}} x \wedge \neg\exists_{\frac{1}{2}}(2x \wedge \neg 2x)$

(DM5) $\exists_{\frac{1}{2}} x \leq x \vee \neg x$

(DM6) $\exists_{\frac{1}{2}}(\exists_{\frac{1}{2}} x \oplus \exists_{\frac{1}{2}} y) = \exists_{\frac{1}{2}} x \oplus \exists_{\frac{1}{2}} y$

(DM7) $\exists_{\frac{1}{2}}(\neg\exists_{\frac{1}{2}} x) = \neg\exists_{\frac{1}{2}} x$

(DM8) $x \wedge \neg\exists_{\frac{1}{2}}(x \wedge \neg x) \leq \exists_{\frac{1}{2}} x$

(DM9) $\exists_{\frac{1}{2}}(2x \odot \exists_{\frac{1}{2}}(y \wedge \neg y)) = 2\exists_{\frac{1}{2}} x \odot \exists_{\frac{1}{2}}(y \wedge \neg y)$

It is plain that the class of MV-algebras endowed with a demiquantifier determines a variety which is denoted by $\mathcal{D}$. Note that axioms (DM1), (DM6) and (DM7) imply the image of the operator $\exists_{\frac{1}{2}}$ is a subalgebra of A; while axioms (DM1) and (DM7) imply $\exists_{\frac{1}{2}} 1 = 1$.

Recall that a *monadic MV-algebra* is an algebra $\mathbf{A} = \langle A, \oplus, \neg, \exists, 0, \rangle$ of type (2,1,1,0) where $\langle A, \oplus, \neg, 0, \rangle$ is an MV-algebra and $\exists$ satisfies the following equations:

(MV1) $x = x \wedge \exists x$,

(MV2) $\exists(x \vee y) = \exists x \vee \exists y$,

(MV3) $\exists(\exists x \oplus \exists y) = \exists x \oplus \exists y$,

(MV4) $\exists(\neg\exists x) = (\neg\exists x$,

(MV5) $\exists(x \oplus x) = \exists x \oplus \exists x$,

(MV6) $\exists(x \odot x) = \exists x \odot \exists x$.

The variety of monadic MV-algebras will be denoted by $\mathcal{M}$.

Theorem

Let $\mathbf{A} = \langle A, \oplus, \neg, \exists, 0, \rangle$ *be an algebra in* $\mathcal{M}$*. Then the operator* $\exists_{\frac{1}{2}}$ *defined by:*

$$\exists_{\frac{1}{2}} x = \exists x \wedge \neg\exists(x \wedge \neg x)$$

for all $x \in A$ *is a demiquantifier. Moreover,* $\exists_{\frac{1}{2}} x = \exists x$ *for all* $x \in A^{-}$ *and* $\exists_{\frac{1}{2}} x = \neg\exists\neg x = \forall x$ *for all* $x \in A^{+}$*.*

Given an MV-algebra A, a *fixed point* of A is an element c of A (necessarily unique) such that $\neg c = c$.

Lemma

Let A be an MV-algebra having a fixed point. Then the following conditions hold for every $x \in A$.

(i) $x = (x \wedge c) \oplus (x \odot c)$.

(ii) $x^2 = 2(x \odot c)$.

Proposition

Let $(A, \exists_{\frac{1}{2}})$ *be an algebra in* $\mathcal{D}$ *and assume that* c *is a fixed point of* A*. Then the operator* $\exists_{\frac{1}{2}}$ *satisfies the following identities:*

(a) $\exists_{\frac{1}{2}} c = c.$

(b) $c \odot \exists_{\frac{1}{2}} x = \exists_{\frac{1}{2}} (c \odot x) \wedge (c \odot \neg \exists_{\frac{1}{2}} (x \wedge \neg x)).$

(c) $\exists_{\frac{1}{2}} (x \wedge \neg x) = \exists_{\frac{1}{2}} x \wedge \neg \exists_{\frac{1}{2}} x.$

(d) $\exists_{\frac{1}{2}} ((x \wedge \neg x) \odot \exists_{\frac{1}{2}} y) = \exists_{\frac{1}{2}} (x \wedge \neg x) \odot \exists_{\frac{1}{2}} y$

Let $\langle A, \oplus, \neg, 0, \rangle$ be an MV-algebra. Recall that an MV-ideal of A is a lattice ideal of A closed under the sum $\oplus$. It is well known that the correspondence $\equiv \mapsto \{x \in A : x \equiv 0\}$ is a bijection between $Con(A)$ and the MV-ideals of A. Our next result will be to extend this result to the algebras in $\mathcal{D}$ provided the underlying MV-algebra has a fixed point.

Theorem

Let $(A, \exists_{\frac{1}{2}})$ be an algebra in $\mathcal{D}$ and let c be a fixed point of A. Then the correspondence $\equiv \mapsto \{x \in A : x \equiv 0\}$ establishes a bijection between $Con(A, \exists_{\frac{1}{2}})$ and the MV-ideals of A which are closed under the operator $\exists_{\frac{1}{2}}$. Moreover, if $(A, \exists_{\frac{1}{2}})$ is subdirectly irreducible then $\exists_{\frac{1}{2}}(A)$ is a subdirectly irreducible MV-algebra. In particular it is an MV-chain.

We know that every algebra in $\mathcal{M}$ induces an algebra in $\mathcal{D}$. Our next task will be to prove that the converse holds provided the underlying MV-algebras have a fixed point.

Theorem

Let $\mathbf{A} = \langle A, \oplus, \neg, \exists_{\frac{1}{2}}, 0, \rangle$ *be an algebra in* $\mathcal{D}$. *The the operator* $\exists : A \to A$ *defined by:*

$$\hat{\exists} x = \exists_{\frac{1}{2}}(x \wedge c) \oplus \exists_{\frac{1}{2}}(x \odot c)$$

for all $x \in A$, *is an existential quantifier where* c *is a fixed point of* A.

THE INTERDEFINABILITY THEOREM

Theorem

Let A be an MV-algebra having a fixed point c. Then the following conditions hold:

(a) If $\exists : A \longrightarrow A$ is an existential quantifier then $\hat{\exists}_{\frac{1}{2}} = \exists$.

(b) If $\exists_{\frac{1}{2}} : A \longrightarrow A$ is a demiquantifier, then $(\hat{\exists}_{\frac{1}{2}})_{\frac{1}{2}} = \exists_{\frac{1}{2}}$.