A representation for the n-generated free algebra in the subvariety of BL-algebras generated by $[0, 1]_{\mathbf{MV}} \oplus [0, 1]_{\mathbf{G}}$

M. Busaniche, J. L. Castiglioni and N. Lubomirsky

CONICET - IMAL - UNLP

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Examples of BL-algebras

Standard MV-algebra $[0,1]_{\mathbf{MV}}$:

$$\left\langle [0,1], \begin{cases} 0 & \text{if } x+y \leq 1 \\ x+y-1 & \text{oherwise} \end{cases}, \begin{cases} 1 & \text{if } x \leq y \\ 1-x+y & \text{otherwise} \end{cases}, 0 \right\rangle$$

Standard Gödel-algebra $[0,1]_{\mathbf{G\ddot{o}del}}$:

$$\left\langle [0,1], \begin{cases} x & \text{if } x \leq y \\ y & \text{oherwise} \end{cases}, \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}, 0 \right\rangle$$

Examples of free algebras: the case of MV-algebras

Chang's Algebraic Completeness Theorem

The standard MV-algebra

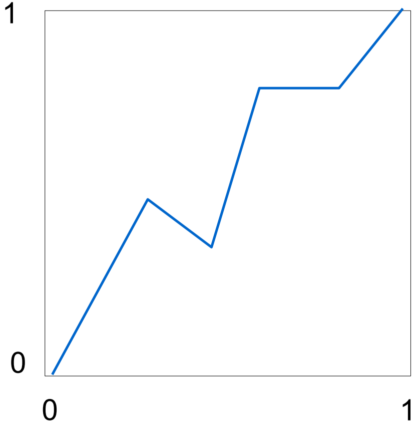
$$\langle [0,1], max(0, x+y-1), min(1, 1-x+y), 0 \rangle$$

is generic for the variety of MV-algebras (BL algebras with $\neg\neg x = x$).

Consider the MV-algebra $\mathcal{M}_n$ of all functions $f : [0,1]^n \to [0,1]$ endowed with the pointwise standard MV-operations:
$(f \cdot g)(x) = max(0, f(x) + g(x) - 1)$,
$(f \to g)(x) = min(1, 1 - f(x) + g(x))$, $\perp(x) = 0$.

McNaughton's Representation Theorem

The free n-generated MV-algebra is the subalgebra of $\mathcal{M}_n$ of all continuous piecewise linear functions $f : [0,1]^n \to [0,1]$ where each one of the finitely many linear pieces has integer coefficients.


1
0
0
1

Examples of free algebras: the case of Gödel hoops

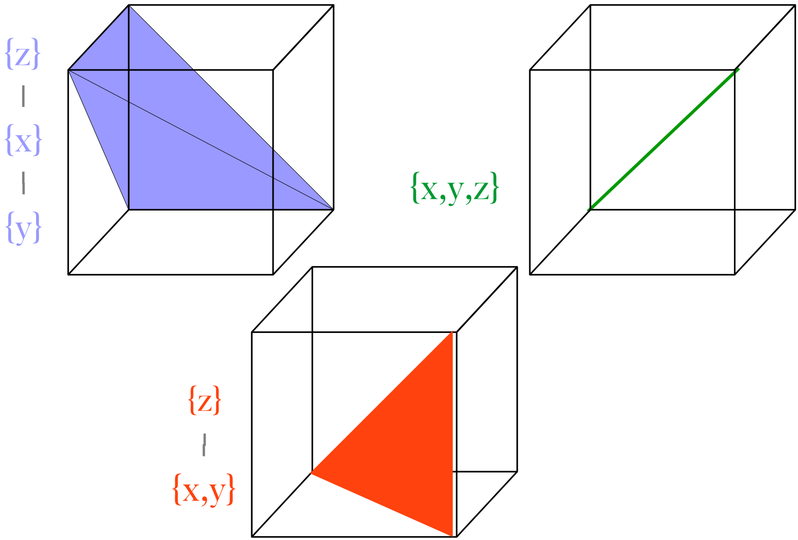
Gödel hoops are the $\perp$-free subreducts of Gödel algebras.
Gödel hoop form a variety $\mathbf{G}$.
The standard Gödel hoop is $[0,1]_{\mathbf{G}}$.

Definition

Let $\mathcal{R}$ be the set which contains all the subsets of $[0,1]^n$ given by:

$$R \in \mathcal{R} \text{ iff } R = \{(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) : x_{\sigma(1)} \Box \ldots \Box x_{\sigma(n)}\}$$

for $\Box \in \{=, <\}$ and σ a permutation of $\{1, \ldots, n\}$.


{z}
{x}
{y}
{x,y,z}
{z}
{x,y}

Free n-generated Gödel hoops

Theorem

The algebra of functions $f : [0,1]^n \to [0,1]$ such that for every $R \in \mathcal{R}$

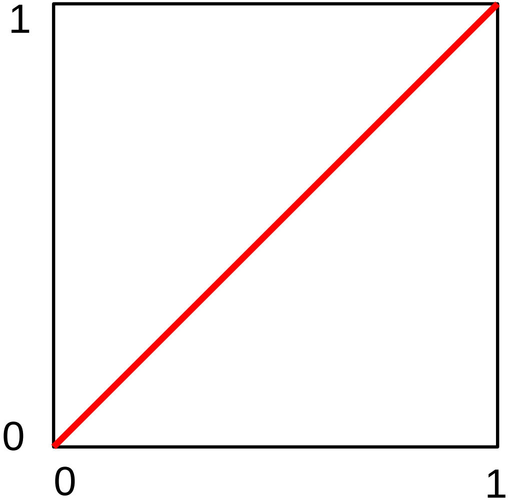
$$f|_R = 1 \text{ } o$$
$$f|_R = x_i \textit{ with } i \in \{1, \ldots, n\}$$

equipped with the pointwise operations $\cdot$ and $\to$ is the free Gödel hoops algebra over n-generators. [1]

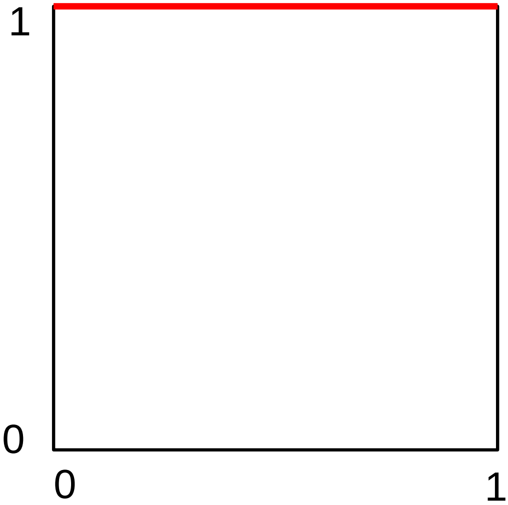
We will write $Free_{\mathcal{G}}(n)$ to refer to this free algebra.

[1] B. Gerla, Many valued Logics of Continuous t-norms and their Functional Representation, PhD thesis, Università di Milano, 2000/2001

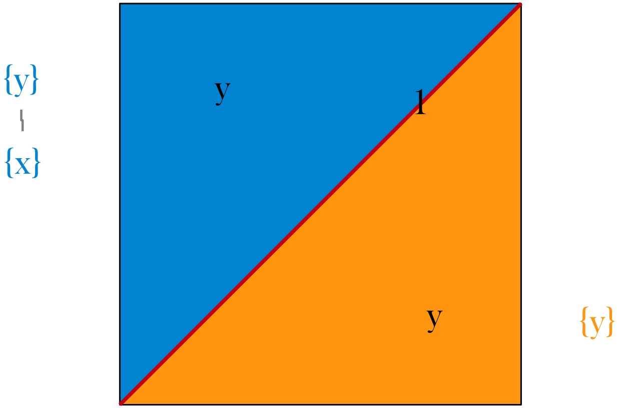
The case of one variable



The case of one variable



The case of two variables



Ordinal sum

Let $\mathbf{R} = (R, *_{\mathbf{R}}, \to_{\mathbf{R}}, \top)$ and $\mathbf{S} = (R, *_{\mathbf{S}}, \to_{\mathbf{S}}, \top)$ be two hoops such that $R \cap S = \{\top\}$. We define the ordinal sum $R \oplus S$ of these two hoops as the hoop given by $(R \cup S, *, \to, \top)$ where the operations $(*, \to)$ are defined as follows:

$$x * y \begin{cases} x *_{\mathbf{R}} y & \text{if } x, y \in R, \\ x *_{\mathbf{S}} y & \text{if } x, y \in S, \\ x & \text{if } x \in R \setminus \{\top\} \text{ and } y \in S, \\ y & \text{if } y \in R \setminus \{\top\} \text{ and } x \in S. \end{cases}$$

$$x \to y \begin{cases} \top & \text{if } x \in R \setminus \{\top\} \text{ and } y \in S, \\ x \to_{\mathbf{R}} y & \text{if } x, y \in R, \\ x \to_{\mathbf{S}} y & \text{if } x, y \in S, \\ y & \text{if } y \in R \setminus \{\top\} \text{ and } x \in S. \end{cases}$$

- $Free_{\mathcal{BL}}(n)$ is generated by the algebra $(n+1)[0,1]_{\mathbf{MV}}$. This fact allows us to characterize the free n-generated BL-algebra $Free_{\mathcal{BL}}(n)$ as the algebra of functions $f : (n+1)[0,1]^n_{\mathbf{MV}} \to (n+1)[0,1]_{\mathbf{MV}}$ generated by the projections.
- S. Bova and S. Aguzzoli gave a representation of the free-n-generated BL-algebra. [2], [3]

We study the subvariety $\mathcal{MG} \subseteq \mathcal{BL}$ generated by the ordinal sum of the algebra $[0,1]_{\mathbf{MV}}$ and the Gödel hoop $[0,1]_{\mathbf{G}}$, that is, generated by $\mathfrak{A} = [0,1]_{\mathbf{MV}} \oplus [0,1]_{\mathbf{G}}$.

[2] S. Bova, PhD thesis, BL-functions and Free BL-algebra,2008

[3] S. Aguzzoli and S. Bova, The free n-generated BL-algebra, Ann. Pure Appl. Logic, Vol. 161, 9, p.1144–1170, 2010

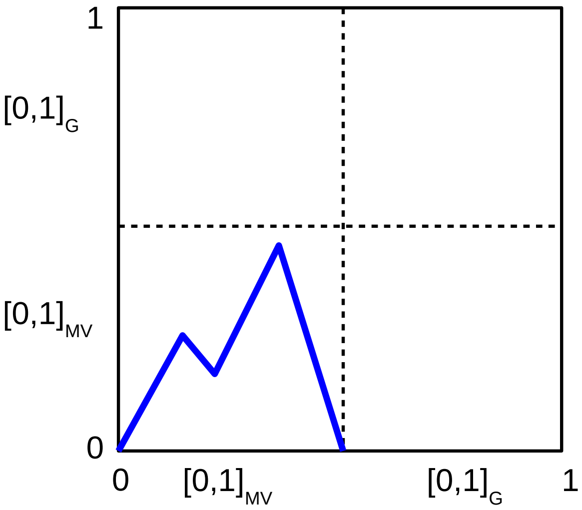
- $[0,1]_{\mathbf{G}}$ is decomposable as an infinite ordinal sum of two-elements Boolean algebra, the idea is to treat it as a whole block (dense elements).
- The elements in $[0,1]_{\mathbf{MV}}$ are usually called regular elements of $\mathfrak{A}$.
- Advantage: The number n of generators of the free algebra does not increase the generating chain.
- That gives an idea of the role of the regular elements and the role of the dense elements.
- To give a functional representation for the free algebra $Free_{\mathcal{MG}}(n)$ we decompose the domain $[0,1]_{\mathbf{MV}} \oplus [0,1]_{\mathbf{G}}$ in a finite number of pieces. In each piece a function $F \in Free_{\mathcal{V}}(n)$ coincides either with McNaughton functions or functions on the free algebra in the variety of Gödel hoops.
- $\mathcal{MG}$: $\mathcal{BL} + (\neg\neg x \to x)^2 = (\neg\neg x \to x)$.

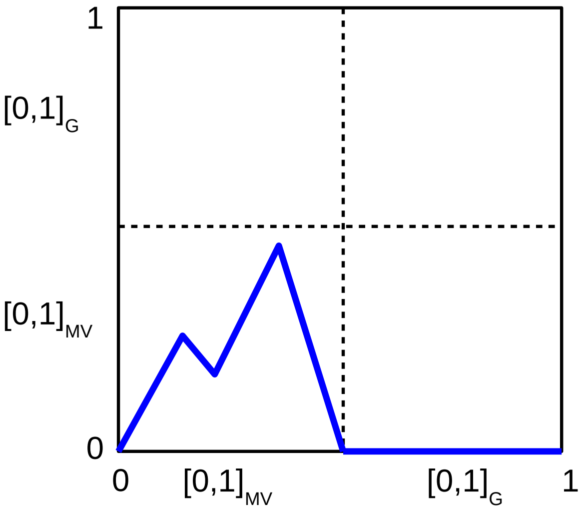
$Free_{\mathcal{MG}}(1)$

Proposition

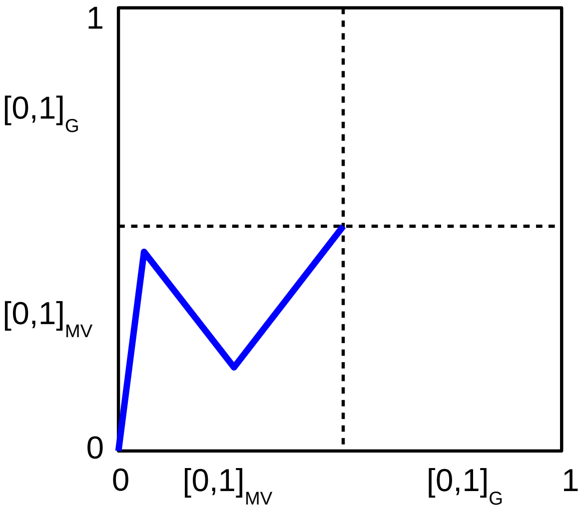
Let $\alpha(x)$ be a BL-term in one variable that we evaluate in $\mathfrak{A}$. Then:

- *If $\alpha_{\mathfrak{A}}(1) = 1$ then $\alpha_{\mathcal{V}}(x)$ is a function of $Free_{\mathcal{G}}(1)$ for each $x \in [0,1]_{\mathbf{G}}$.*
- *If $\alpha_{\mathfrak{A}}(1) = 0$ then $\alpha_{\mathcal{V}}(x) = 0$ for each $x \in [0,1]_{\mathbf{G}}$.*


1
[0,1]G
[0,1]MV
0
0
[0,1]MV
[0,1]G
1



1
[0,1]G
[0,1]MV
0
0
[0,1]MV
[0,1]G
1


1
[0,1]G
[0,1]MV
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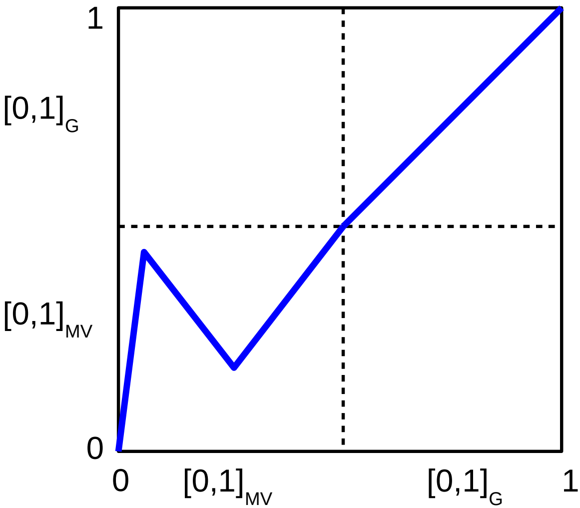
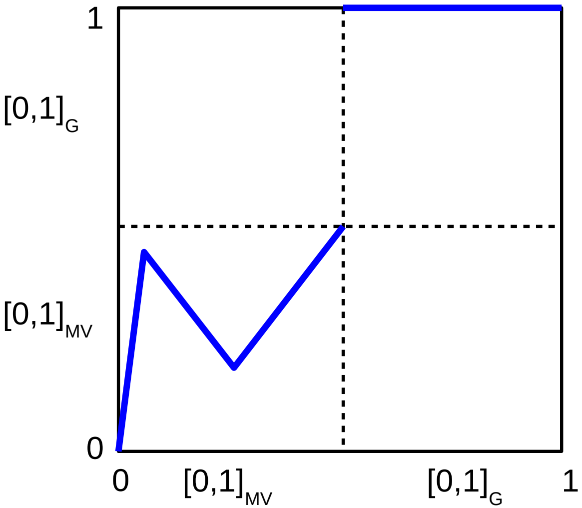
[0,1]MV
[0,1]G
1



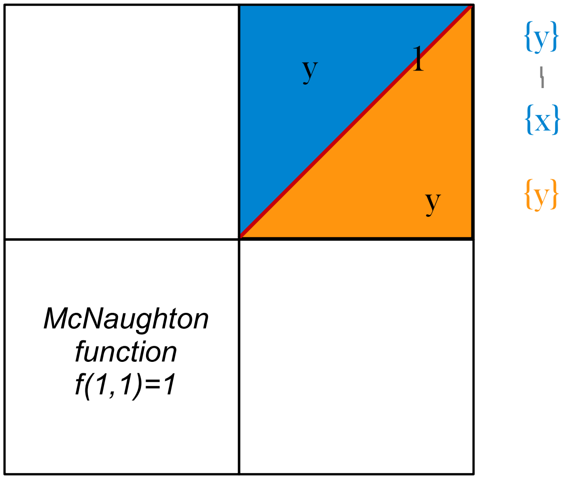
1
[0,1]G
[0,1]MV
0
0
[0,1]MV
[0,1]G
1

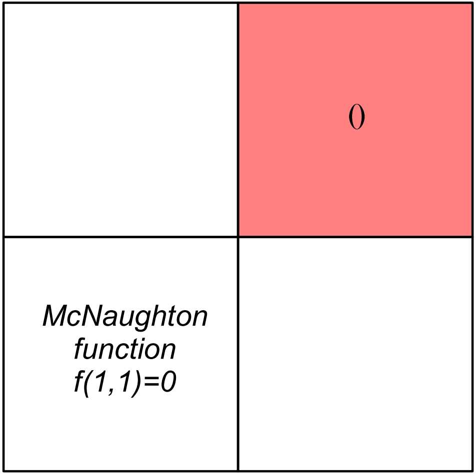

1
[0,1]G
[0,1]MV
0
0
[0,1]MV
[0,1]G
1

$Free_{\mathcal{MG}}(2)$

As before, if $\alpha(x, y)$ is a BL-term and we evaluate it in $\mathfrak{A}$ we have:

- If $\alpha_{\mathfrak{A}}(1, 1) = 1$ then there is a function $g \in Free_{\mathcal{G}}(2)$ such that $\alpha_{\mathfrak{A}}(x, y) = g(x, y)$ for every $(x, y) \in [0, 1]^2_{\mathbf{G}}$.
- If $\alpha_{\mathfrak{A}}(1, 1) = 0$ then $\alpha_{\mathfrak{A}}(x, y) = 0$ for every $(x, y) \in [0, 1]^2_{\mathbf{G}}$.


y
1
y
McNaughton
function
f(1,1)=1
{y}
{x}
{y}


0
McNaughton function f(1,1)=0

Proposition

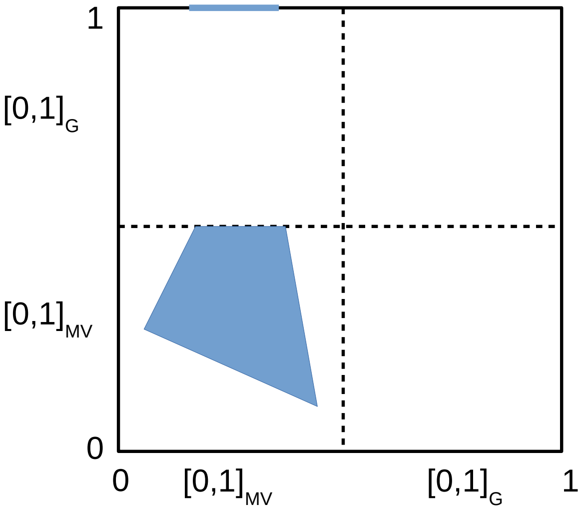
Let $\alpha(x, y)$ *and* $a \in [0, 1]_{MV} \setminus \{1\}$. *Then, if we evaluate* α *on* $\mathcal{MG}$, *it holds:*

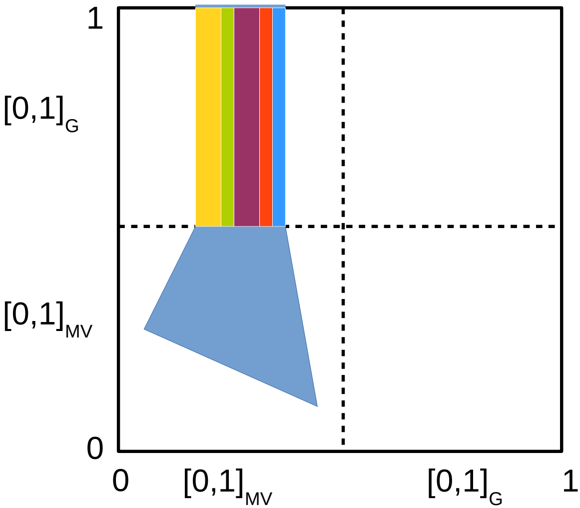
- *If* $\alpha_{\mathcal{MG}}(a, 1) = c \in [0, 1]_{MV} \setminus \{1\}$ *then* $\alpha_{\mathcal{MG}}(a, b) = c$ *for every* $b \in [0, 1]_G$,
- *If* $\alpha_{\mathcal{MG}}(a, 1) = 1$ *then there is a function* $g \in Free_{\mathcal{G}}(1)$ *such that* $\alpha_{\mathcal{MG}}(a, b) = g(b)$ *for every* $b \in [0, 1]_G$.

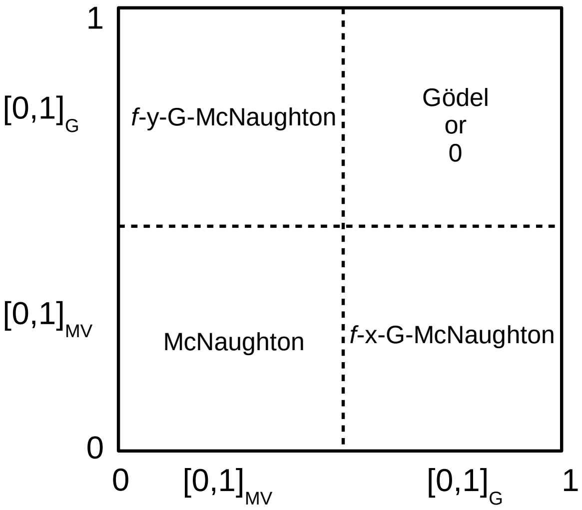
Definition

Let $f \in Free_{\mathcal{MV}}(2)$. If $A = \{x \in [0,1]_{MV} : f(x,1) = 1\}$ and $B = [0,1]_{MV} \setminus A$, we will say that $g : [0,1]_{MV} \times (0,1]_G \to \mathcal{MG}$ is an *f-y-G-McNaughton function* if:

1. For each $x_0 \in B$, $g(x_0, y) = f(x_0, 1)$, for every $y \in (0,1]_G$.
2. There is a regular triangulation Δ of A which determines the simplexes $\sigma_1, \ldots, \sigma_n$ and functions $g_1, \ldots, g_n \in Free_{\mathcal{G}}(1)$ such that $g(x,y) = g_i(y)$, fur every x in the interior of σ_i.


1
[0,1]G
[0,1]MV
0
0
[0,1]MV
[0,1]G
1


1
[0,1]$_G$
[0,1]$_{MV}$
0
0
[0,1]$_{MV}$
[0,1]$_G$
1


1
[0,1]$_G$
f-y-G-McNaughton
Gödel
or
0
[0,1]$_{MV}$
McNaughton
f-x-G-McNaughton
0
0
[0,1]$_{MV}$
[0,1]$_G$
1

$Free_{\mathcal{MG}}(n)$

Let $F \in Free_{\mathcal{MG}}(n)$. Then:

- For every $\bar{x} \in ([0,1]_{\mathbf{MV}})^n$,

$$F(\bar{x}) = f(\bar{x})$$

  where f is a function of $Free_{\mathcal{MV}}(n)$.

For the rest of the domain, the functions depend on this function $f : ([0,1]_{\mathbf{MV}})^n \to [0,1]_{\mathbf{MV}}$:

- On $([0,1]_{\mathbf{G}})^n$:
  1. If $f(\bar{1}) = 0$, then
  $$F(\bar{x}) = 0$$
  for every $\bar{x} \in ([0,1]_{\mathbf{G}})^n$.
  2. If $f(\bar{1}) = 1$, then
  $$F(\bar{x}) = g(\bar{x})$$
  for a function $g \in Free_{\mathcal{G}}(n)$, for every $\bar{x} \in ([0,1]_{\mathbf{G}})^n$.

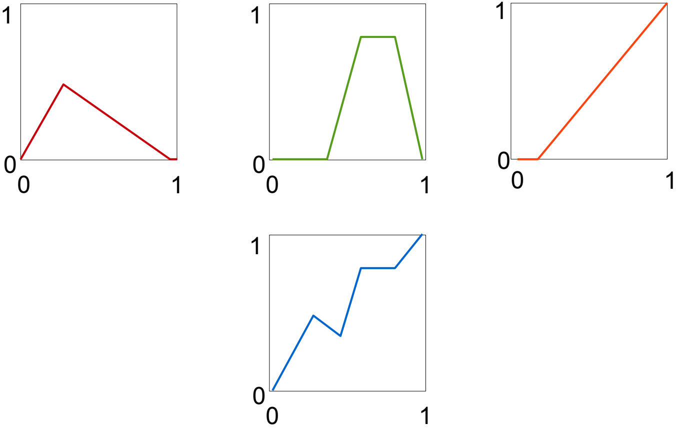
Let $B = \{x_{\sigma(1)}, \dots x_{\sigma(m)}\} \subsetneq \{x_1, \dots, x_n\}$ and R_B be the subset of $([0,1]_{MV} \oplus [0,1]_G)^n$ where $x_i \in B$ if and only if $x_i \in [0,1]_G$. For every $\bar{x} \in R_B$ we also define $\tilde{x}$ as:

$$\tilde{x}_i = \begin{cases} x_i & \text{if} \quad x_i \notin B \\ \\ 1 & \text{if} \quad x_i \in B \end{cases}$$

- 1. If $f(\tilde{x}) < 1$ then $F(\bar{x}) = f(\tilde{x})$.
  2. If $f(\tilde{x}) = 1$, then there is a regular triangulation Δ of $f^{-1}(1) \wedge R_B$ which determines the simplices $S_1, \dots, S_k$ and k Gödel functions $h_1, \dots, h_n$ in $n - m$ variables $x_{\sigma(m+1)}, \dots, x_{\sigma(n)}$ such that $F(\bar{x}) = h_i(x_{\sigma(m+1)}, \dots, x_{\sigma(n)})$ for each point $(x_{\sigma(1)}, \dots x_{\sigma(m)})$ in the interior of S_i.

co-hats

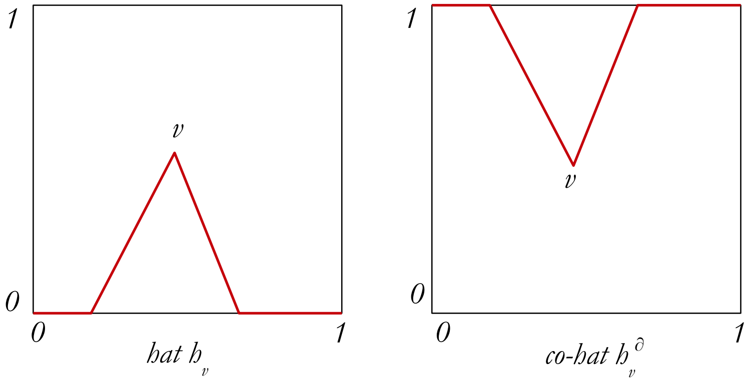
4



[4] Aguzzoli, S., Bova, S., The free *n*-generated BL-algebra, Ann. Pure Appl. Logic, 2010 vol. 161, N. 9, pag. 1144-1170.

co-hats

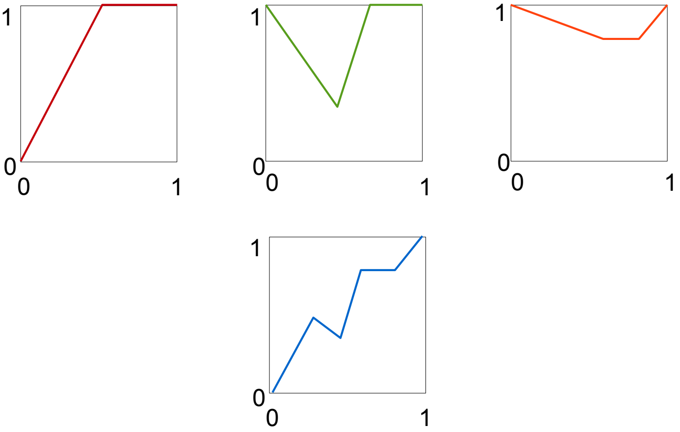
4



[4] Aguzzoli, S., Bova, S., The free *n*-generated BL-algebra, Ann. Pure Appl. Logic, 2010 vol. 161, N. 9, pag. 1144-1170.

co-hats

[4]



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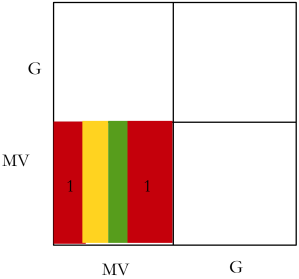
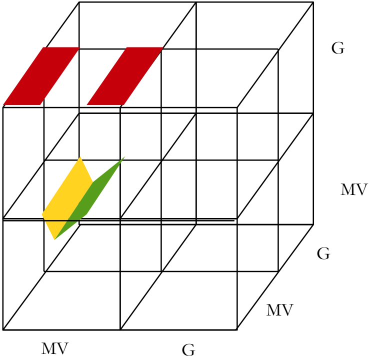
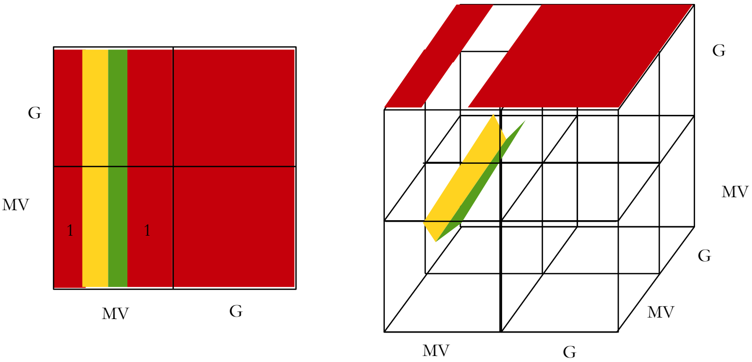
Basic Functions





Basic Functions

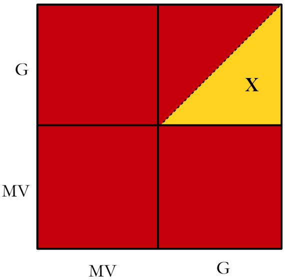


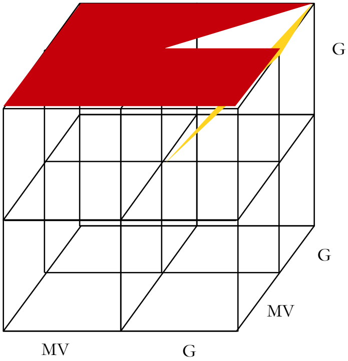
Basic Functions

{x}

|

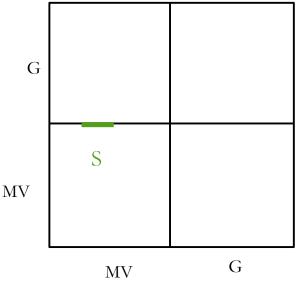
{y}

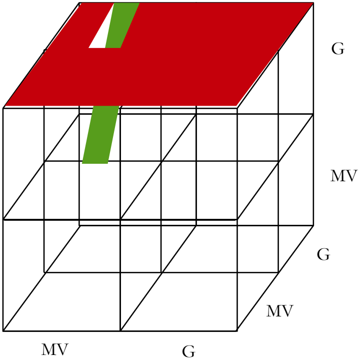




Basic Functions

{y}





Thank you!