On an operation with regular elements

Rodolfo C. Ertola-Biraben (CLE/Unicamp - Brazil)

Talk XIV Congreso Dr. Antonio Monteiro

Baha Blanca, Argentina

May 31, 2017

Motivation

In [1] we find the following notion.

Definition 2.4.6 A Δ-algebra is a structure $\mathbf{L} = \langle L, *, \Rightarrow, \cap, \cup, 0, 1, \Delta\rangle$ which is a BL-algebra expanded by an unary operation Δ in which the following formulas are true:

$$\Delta x \cup (-)\Delta x = 1$$
$$\Delta(x \cup y) \leq \Delta x \cup \Delta y$$
$$\Delta x \leq x$$
$$\Delta x \leq \Delta\Delta x$$
$$(\Delta x) * (\Delta(x \Rightarrow y)) \leq \Delta y$$
$$\Delta 1 = 1.$$

[1] Hájek, P. *Metamathematics of Fuzzy Logic*, Trends in Logic, vol. 4, Kluwer (1998)

Question

In 2012 I asked Franco Montagna whether Δ could be defined in the context of a meet-complemented lattice as a maximum or minimum operation.

He suggested considering the maximum Boolean below, i.e. given a meet-complemented lattice $\mathbf{M} = (M; \wedge, \vee, \neg)$,

$$Bx = max\{y \in M : y \leq x \text{ and } y \vee \neg y = 1\}.$$

I acted accordingly and proved that we get an equational class:

$$Bx \preccurlyeq x,$$
$$Bx \vee \neg Bx \approx 1,$$
$$Bx \preccurlyeq B(x \vee y),$$
$$B1 \approx 1,$$
$$B(x \vee \neg x) \preccurlyeq Bx \vee \neg x \text{ (see [1])}.$$

[1] Ertola Biraben, R. C., Esteva, F., and Godo, L. Expanding FL ew with a Boolean connective. *Soft Computing* (2017) 21:97-111, DOI 10.1007/s00500-016-2275-y

Question for this talk

What happens if instead of the maximum Boolean below we consider the maximum regular below?

Regular elements only require $\wedge$ and $\neg$.

Accordingly, our setting will be meet-complemented meet-semilattices.

Meet-complemented meet-semilattices

Let $\mathbf{M} = (M; \leq)$ belong to the class $\mathbb{SL}$ of meet-semilattices and $a \in M$.

We say that $\mathbf{M}$ is a **meet-complemented meet-semilattice** iff for all $a \in M$ there exists

$$\neg a = max\{b \in M : a \wedge b \leq c, forall\ c \in M\},$$

where, as usual, we use $\wedge$ for the infimum in $\mathbf{M}$.

We use the notation $\mathbb{MSL}$ for the class of meet-complemented meet-semilattices.

Examples

We will use the examples of meet-complemented meet-semilattices given in the next slide, all of them being lattices and the last two even being Heyting algebras.

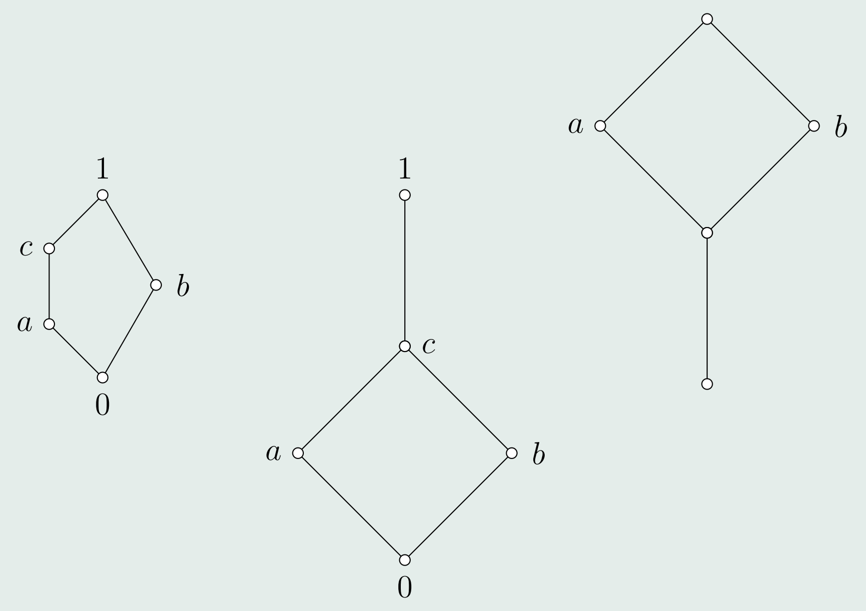

Figure 1: The pentagon (sometimes denoted $\mathbf{N_5}$), $\mathbf{2^2} \oplus \mathbf{1}$, and $\mathbf{1} \oplus \mathbf{2^2}$

Some well-known properties

In $\mathbb{MSL}$ we have

Extension of Double Negation: $\circ \preccurlyeq \neg\neg$,
Anti-Monotonicity of Negation: $x \preccurlyeq y \Rightarrow \neg y \preccurlyeq \neg x$,
$x \preccurlyeq \neg y \Rightarrow y \preccurlyeq \neg x$,
Triple Negation: $\neg\neg\neg \approx \neg$,
$y \preccurlyeq \neg[x \wedge (z \wedge \neg z)]$,
$x \wedge \neg(x \wedge y) \preccurlyeq \neg y$,
$\neg\neg(x \wedge y) \approx \neg\neg x \wedge \neg\neg y$,
$\neg x \approx 1 \Leftrightarrow x \approx 0$,
$\neg 0 \approx 1$,
$\neg\neg 1 \approx 1$,
$\neg 1 \approx 0$,
$\neg\neg 0 \approx 0$.

An equational class

$\mathbb{MSL}$ is an equational class taking a set of identities proving that the class $\mathbb{SL}$ is an equational class and adding the following:

$$x \wedge \neg x \preccurlyeq y,$$

$$y \preccurlyeq \neg(x \wedge \neg x), \text{ and}$$

$$x \wedge \neg(x \wedge y) \preccurlyeq \neg y.$$

Conservative expansions

We will be interested in the concept of conservative expansion.

The class $\mathbb{MSL}$ is a conservative expansion of the class $\mathbb{SL}$, that is, no identity (or quasi-identity) involving only the infimum that does not hold in $\mathbb{SL}$ holds in the expansion $\mathbb{MSL}$.

This follows from the fact that the class $\mathbb{ML}$ of meet-complemented lattices is a conservative expansion of the class $\mathbb{L}$ of lattices.

This is so, because every lattice $\mathbf{L}$ can be immersed in $\mathbf{1} \oplus \mathbf{L} \oplus \mathbf{1}$, which clearly has $\neg$, where $\mathbf{1} \oplus \mathbf{L} \oplus \mathbf{1}$ is the lattice that results from $\mathbf{L}$ adding new elements acting as bottom and top.

Double Negation

We will be interested in the behaviour of $\neg\neg$.

In $\mathbb{MSL}$ we have the following facts, which we will take as saying that $\neg\neg$ is a **grounded closure operator**:

$$\text{Extension of Double Negation: } \circ \preccurlyeq \neg\neg,$$

$$\text{Monotonicity of Double Negation: } x \preccurlyeq y \Rightarrow \neg\neg x \preccurlyeq \neg\neg y,$$

$$\text{Idempotency of Double Negation: } \neg\neg\neg\neg \approx \neg\neg,$$

$$\text{Groundness of Double Negation: } \neg\neg 0 \approx 0.$$

Note that we also seen that $\neg\neg$ is **multiplicative**, i.e.

$$\neg\neg(x \wedge y) \approx \neg\neg x \wedge \neg\neg y$$

and that $\neg\neg 1 \approx 1$.

On Boolean elements

Now let us remind the following concept.

Given an $\mathbf{M} \in \mathbb{ML}$ and $a \in M$, a is said to be *Boolean* iff there is an element $b \in M$ such that

$$a \wedge b = 0 \text{ and } a \vee b = 1,$$

where, as usual, we use $\wedge$, $\vee$, 0, and 1 for the infimum, the supremum, bottom, and top of $\mathbf{M}$, respectively.

The given definition is easily seen to be equivalent to saying that

$$a \text{ is Boolean iff } a \vee \neg a = 1.$$

On regular elements

Now let us remind the following concept, which, from a logical point of view, says that to be regular is to be a negation.

Let $\mathbf{M} = (M; \wedge, \neg) \in \mathbb{MSL}$ and $a \in M$. Then, we say that

a is **regular** iff there is a $b \in M$ such that $a = \neg b$.

Let $\mathbf{M} \in \mathbb{MSL}$ and $a, b \in M$. Then,

a is regular iff $\neg\neg a = a$,

if a and b are regular, then $a \wedge b$ is regular,

$\neg a$ is regular, and

0 and 1 are regular.

Comparing Boolean and regular elements

Note that in the pentagon all elements are Boolean.

However, the atom non-coatom is not regular (and it is the only non-regular element in the pentagon).

So, being Boolean does not imply being regular.

It is also the case that being regular does not imply being Boolean, as can be seen taking any atom in the lattice $2^2 \oplus 1$.

On the greatest Boolean below (1)

In a previous work we have studied, in the context of residuated lattices, the operation B given by the greatest Boolean below a given element. In particular, our results hold for the class $\mathbb{ML}$ of meet-complemented lattices.

Accordingly, given an $\mathbf{M} \in \mathbb{ML}$, we have postulated that for all $a \in M$, there exists

$$Ba = max\{b \in M : b \leq a \text{ and } b \vee \neg b = 1\},$$

using the notation $\mathbb{ML}^B$ for the corresponding class.

[1] Ertola Biraben, R. C., Esteva, F., and Godo, L. Expanding FL ew with a Boolean connective. Soft Computing (2017) 21:97-111, DOI 10.1007/s00500-016-2275-y

On the greatest Boolean below (2)

It is equivalent to postulate that the following hold:

$$Bx \preccurlyeq x,$$

$$Bx \vee \neg Bx \approx 1, \text{ and}$$

$$\text{if } y \leq x \;\&\; y \vee \neg y \approx 1 \;\Rightarrow\; y \preccurlyeq Bx.$$

On the greatest regular below (1)

Now we deal with the operation that results when substituting the notion of regular for the notion of Boolean.

Given an $\mathbf{M} \in \mathbb{MSL}$, we may postulate that for all $a \in M$, there exists

$$Ra = max\{b \in M : b \leq a \text{ and } \neg\neg b = b\},$$

using the notation $\mathbb{MSL}^R$ for the corresponding class.

On the greatest regular below (2)

The following concept is equivalent.

An algebra $\langle M; \wedge, \neg, R\rangle$ is a **meet-negated meet-semilattice with R** iff $\langle M; \wedge, \neg\rangle$ is a meet-negated meet-semilattice and the following identities and quasi-identity hold:

$$Rx \preccurlyeq x,$$

$$\neg\neg Rx \approx Rx, \text{ and}$$
$$y \preccurlyeq x \;\&\; \neg\neg y \approx y \;\Rightarrow y \;\preccurlyeq Rx.$$

The notation $\mathbb{MSL}^R$ stands for the class of meet-negated meet-semilattices with R.

The set $\{\wedge, \vee, \rightarrow, \neg, R\}$ is independent.

On other similar operations

Note that the minimum regular above is not a new operation, as it equals $\neg\neg$.

On the other hand, operation R seems to deserve inspection.

A grounded operator

We have the following facts, which we take as saying that R is a **multiplicative grounded interior operator**. In $\mathbb{MSL}^R$ we have

(i) $R \preccurlyeq \circ$,

(ii) Monotonicity of R: $x \preccurlyeq y \Rightarrow Rx \preccurlyeq Ry$,

(iii) $RR \approx R$,

(iv) $R1 \approx 1$, and

(v) $R(x \wedge y) \approx Rx \wedge Ry$.

We also have the following facts. In $\mathbb{MSL}^{\mathbb{R}}$ it holds

(i) $Rx \approx x \Leftrightarrow \neg\neg x \approx x$,

(ii) $R\neg \approx \neg$, and

(iii) $R0 \approx 0$.

Adjointness

The next proposition says that R is right adjoint of $\neg\neg$.

In $\mathbb{MSL}^R$ it holds that

$$\neg\neg x \preccurlyeq y \Leftrightarrow x \preccurlyeq Ry.$$

An equational class

$\mathbb{MSL}^R$ is an equational class.

Indeed, take identities proving that $\mathbb{MSL}$ is an equational class and add the following:

$$Rx \preccurlyeq x,$$

$$\neg\neg Rx \approx Rx,$$

$$R(x \wedge y) \preccurlyeq Rx, \text{ and}$$

$$\neg x \preccurlyeq R\neg x.$$

The given set of equations is independent.

Non-compatibility

Operation R is not compatible, that is, the congruences of $\mathbb{MSL}$ and $\mathbb{MSL}^R$ are not the same.

In fact, R is not compatible even in the case of the class of Gödel algebras, i.e. Heyting algebras extended with

$$(x \to y) \vee (y \to x) \approx 1.$$

Modalities

Taking *modality* to mean a finite combination of unary operators, i.e. $\neg$ and R, the next statement shows how many different modalities there are in $\mathbb{MSL}^R$, where we use $\circ$ for the identity, we omit arguments for better readability, and by positive and negative modalities we mean modalities with an even and odd number of negations, respectively.

Proposition. *In* $\mathbb{MSL}^R$ *there are five different modalities. They may be ordered as follows: on the one hand, the positive modalities* $R \preccurlyeq \circ \preccurlyeq \neg\neg$ *and, on the other hand, the negative modalities* $\neg \preccurlyeq \neg R$ *(see Figure).*

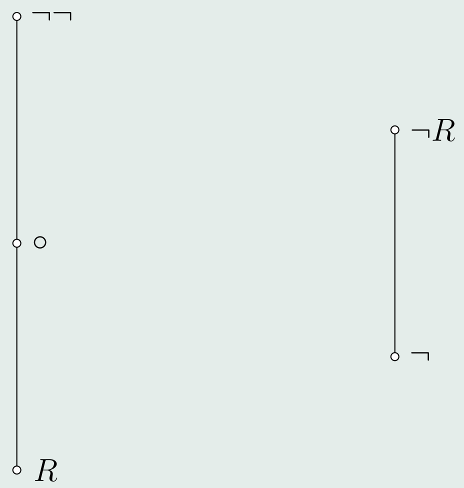

Figure 1: Positive and negative modalities in $\mathbb{MSL}^R$

Non-conservativity

The next result says that $\mathbb{MSL}^R$ is not conservative of $\mathbb{MSL}$ regarding quasi-identities.

Proposition. *The quasi-identity*

$$\neg\neg x \preccurlyeq z \;\;\&\;\; \neg\neg y \preccurlyeq z \;\Rightarrow\; \neg(\neg x \wedge \neg y) \preccurlyeq z$$

holds in $\mathbb{MSL}^R$*, but does not hold in* $\mathbb{MSL}$*.*

Corollary. *The following quasi-identities hold in* $\mathbb{MSL}^R$*, but do not hold in* $\mathbb{MSL}$*:*

$$\neg x \preccurlyeq y \;\;\&\;\; \neg\neg x \preccurlyeq y \;\Rightarrow\; 1 \preccurlyeq y,$$

$$\neg x \preccurlyeq z \;\;\&\;\; \neg y \preccurlyeq z \;\Rightarrow\; \neg(x \wedge y) \preccurlyeq z.$$

Meet-negated lattices with R

In $\mathbb{ML}$ the following identities and quasi-identity hold:

($\vee$I) $x \preccurlyeq x \vee y$, $y \preccurlyeq x \vee y$, and

($\vee$E) $x \preccurlyeq z \;\;\&\;\; y \leq z \;\Rightarrow\; x \vee y \preccurlyeq z$.

Regarding $\neg$, in $\mathbb{ML}$ we also have

(T) $\neg\neg(x \vee \neg x) \approx 1$,

(DM1) $\neg(x \vee y) \approx \neg x \wedge \neg y$, and

$\neg x \vee \neg y \preccurlyeq \neg(x \wedge y)$.

Non-conservativity (1)

As in $\mathbb{ML}^R$ the equation $R(x \wedge y) \approx Rx \wedge Ry$ holds and R is right adjoint of $\neg\neg$, it is natural to expect the following result, which is not the case for Heyting algebras, let alone for $\mathbb{ML}$.

Theorem. *The identity* **(J)** $\neg\neg(x \vee y) \approx \neg\neg x \vee \neg\neg y$ *holds in* $\mathbb{ML}^R$.

From a logical point of view, the given result means that it is not conservative to expand the logic corresponding to $\mathbb{ML}$ with a connective corresponding to R.

Non-conservativity (2)

Note that in $\mathbb{ML}^{\mathbb{R}}$ grounded closure operator $\neg\neg$ has turned into an additive (due to **(J)**) grounded closure operator.

It is well known that the next result follows.

Corollary. *In* $\mathbb{ML}^{R}$ *we have*

(i) **(S)** $\neg x \vee \neg\neg x \approx 1$ *and*

(ii) **(DM2)** $\neg(x \wedge y) \approx \neg x \vee \neg y$.

Note that $\mathbb{ML}_{DM2} = \mathbb{ML}_{J}$.

On the other hand, $\mathbb{ML}_S \neq \mathbb{ML}_J$, as the (non-distributive) lattice in Figure 3 belongs to $\mathbb{ML}_S$, but it is not the case that $\neg\neg(p \vee q) \leq \neg\neg p \vee \neg\neg q$.

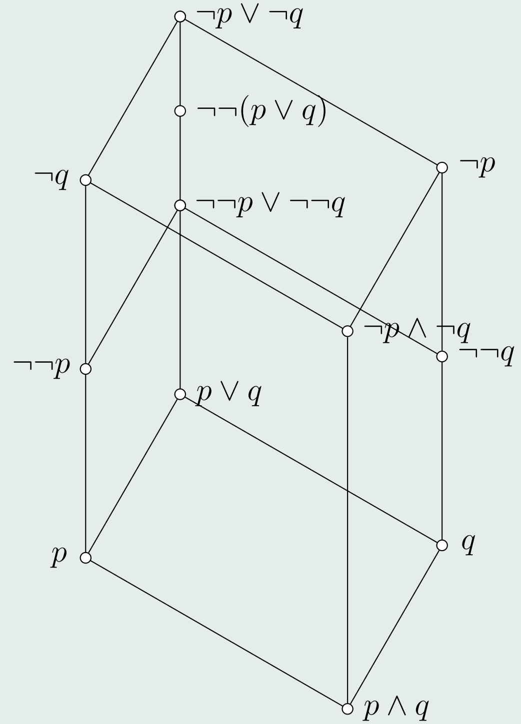

Figure 1: Lattice in $\mathbb{ML}_S$ not satisfying $\neg\neg(x \vee y) \preccurlyeq \neg\neg x \vee \neg\neg y$

Further consequences (1)

We have seen in Theorem that the existence of R implies schema **(J)**.

It is natural to ask whether R exists in every finite algebra in the class $\mathbb{ML}_J$, which, as expected, stands for the class of meet-negated lattices extended with **(J)**.

Let us see that the answer is yes.

Proposition. *Let* $\mathbf{M} \in \mathbb{ML}_J$ *with* M *finite. Then, for all* $a \in M$, *there exists* Ra.

Further consequences (2)

Next, as we have that **(S)** holds in $\mathbb{ML}^R$, we have the following fact.

Lemma. *Let* $\mathbf{M} \in \mathbb{ML}^R$ *and* $a \in M$. *Then, if* a *is regular, then* a *is Boolean.*

Remark. *It is still not the case that being Boolean implies being regular, as the pentagon belongs to* $\mathbb{ML}^R$ *and its atom non-coatom element is Boolean, but not regular.*

Note that so far we have *never* used distributivity.

R in the distributive case

In $\mathbb{ML}_{dS}$ to be Boolean is equivalent to be regular.

So, $\mathbb{ML}_d^R = \mathbb{ML}_{dS}^B$.

In $\mathbb{ML}_d^R$ we have $B = R$ and in $\mathbb{ML}_{dS}^B$ we have $R = B$.

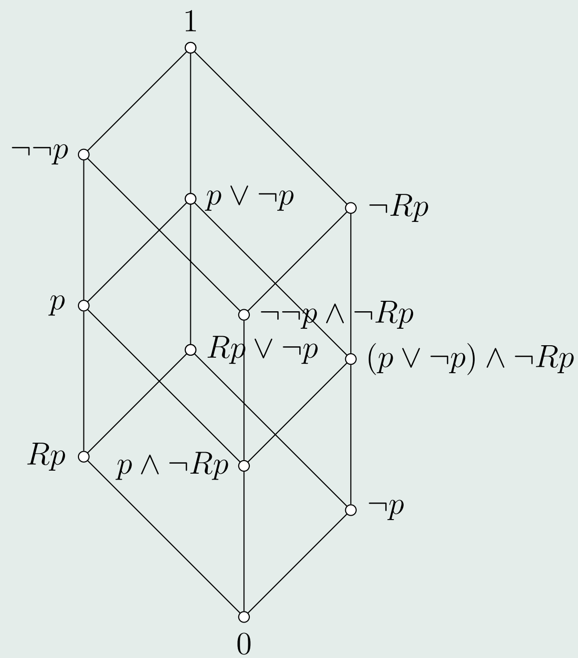

Figure 1: The one generated free algebra of $\mathbb{ML}_d^R$

Franco Montagna (1948-2015)