Selfextensional logics with a nearlattice term

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Objectives

We propose a definition of when a ternary term m can be considered as a distributive nearlattice term (DN-term) for a sentential logic $\mathcal{S}$. We show that a selfextensional logics with a DN-term m can be characterized as the logics $\mathcal{S}$ for which there exists a class of algebras K such that the $\{m\}$-reduct of the algebras of K are distributive nearlattices (DN-algebras) and the consequence relation of $\mathcal{S}$ can be defined using the order induced by m.

Distributive nearlattice

Definition

A **distributive nearlattice** (**DN-algebra**) is an algebra $\langle A, m \rangle$ of type (3) such that the following identities hold:

1. $m(x, y, x) = x$,
2. $m(m(x, y, z), m(y, m(u, x, z), z), w) = m(w, w, m(y, m(x, u, z), z))$
3. $m(x, x, m(y, z, w)) = m(m(x, x, y), m(x, x, z), w)$.

Proposition

Let $\langle A, m \rangle$ be a DN-algebra. If we define

$$x \vee y := m(x, x, y),$$

then $\langle A, \vee \rangle$ is a join-semilattice such that for every $a \in A$, $\langle \uparrow a, \wedge_a, \vee \rangle$ is a distributive lattice. Moreover

$$m(x, y, z) = (x \vee z) \wedge_z (y \vee z).$$

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Let $\mathcal{S} = \langle Fm(\mathcal{L}), \vdash_{\mathcal{S}} \rangle$ be a sentential logic. The **Frege relation** $\Lambda\mathcal{S}$ of $\mathcal{S}$ is the binary relation on $Fm(\mathcal{L})$ defined as:

$$(\varphi, \psi) \in \Lambda\mathcal{S} \iff \varphi \vdash_{\mathcal{S}} \psi \text{ and } \psi \vdash_{\mathcal{S}} \varphi.$$

Definition

A sentential logic $\mathcal{S}$ is said to be **selfextensional** if $\Lambda\mathcal{S}$ is a congruence on Fm.

Definition

Let $\mathcal{S}$ be a selfextensional logic. Let us denote:

- $\mathrm{Alg}(\mathcal{S})$ the canonical class of algebras associated with $\mathcal{S}$;
- $\mathrm{K}_{\mathcal{S}} = \mathbb{V}(Fm/\Lambda\mathcal{S})$ called the intrinsic variety of $\mathcal{S}$.

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DN-based logics

Let $\mathcal{L}$ be an arbitrary similarity type.

Definition

A class of algebras K is called **DN-class** if there is a ternary term m and for every algebra A of K, the m-reduct $\langle A, m^A \rangle$ is a DN-algebra.

Let m be a ternary term of $\mathcal{L}$. We consider:

- $x \vee y := m(x, x, y)$;

and for every natural number n, we define inductively $m^{n-1}(x_1, \ldots, x_n, y)$ as follows:

- $m^0(x_1, y) := m(x_1, x_1, y) = x_1 \vee y$;
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A sentential logic $\mathcal{S}$ is said to be **DN-based** if and only if there is a ternary term m and DN-class of algebras K and it holds that

$$\varphi_1, \ldots, \varphi_n \vdash_{\mathcal{S}} \varphi \iff (\forall A \in \mathrm{K})(\forall h \in \mathrm{Hom}(Fm, A))$$
$$m^{n-1}(h\varphi_1, \ldots, h\varphi_n, h\varphi) \leq h\varphi$$
$$(h\varphi_1 \vee h\varphi) \wedge_{h\varphi} \cdots \wedge_{h\varphi} (h\varphi_n \vee h\varphi) \leq h\varphi.$$

It follows that

$$\varphi_1, \ldots, \varphi_n \vdash_{\mathcal{S}} \varphi \iff \mathrm{K} \models m^{n-1}(\varphi_1, \ldots, \varphi_n, \varphi) \approx \varphi$$
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and since $m^0(\varphi, \psi) = \varphi \vee \psi$, we have

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Let $\mathcal{S}$ be a DN-based logic. Then, for all $\varphi, \psi, \chi \in Fm$, the following properties hold:

(B1) $\varphi \vee \psi \vdash_{\mathcal{S}} \chi$ if and only if $\varphi \vdash_{\mathcal{S}} \chi$ and $\psi \vdash_{\mathcal{S}} \chi$;

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Proof.

$$\varphi \vdash_{\mathcal{S}} \psi \iff (\forall A \in \mathrm{K})(\forall h \in \mathrm{Hom}(Fm, A))(h(\varphi) \leq h(\psi))$$ A

- Since for all $A \in \mathrm{K}$, $\vee$ is a join operation on A;
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Characterization of DN-based logics

Definition

Let $\mathcal{S}$ be a sentential logic. A ternary term m is said to be a **DN-term** of $\mathcal{S}$ if $\mathcal{S}$ satisfies properties (B1)-(B4) with respect to m. B

Theorem

Let $\mathcal{S}$ be logic and m a ternary term of $\mathcal{S}$. Then, $\mathcal{S}$ is a DN-based logic relative to m if and only if $\mathcal{S}$ is selfextensional and m is a DN-term.

Proof.

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Proof.

($\Leftarrow$) Since $\mathcal{S}$ is selfextensional, $\Lambda\mathcal{S}$ is a congruence on Fm and thus $Fm^* := Fm/\Lambda\mathcal{S}$ is an algebra.
By properties (B1)-(B4) we have that $\langle Fm^*, m^* \rangle$, with

$$m^*(\overline{\varphi}, \overline{\psi}, \overline{\chi}) = \overline{m(\varphi, \psi, \chi)},$$

is a DN-algebra and moreover, $\mathcal{S}$ is a DN-based logic with respect to $\{Fm^*\}$. $\square$

Properties of DN-based logics

Theorem

Let $\mathcal{S}$ be a DN-based logic. Then:

1. $\mathrm{Alg}(\mathcal{S}) = \mathrm{K}_{\mathcal{S}}$;
2. $\mathrm{Alg}(\mathcal{S})$ is a variety;
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Bijective correspondence

Let $\mathcal{L}$ be an algebraic language and m a ternary term of $\mathcal{L}$.

- $\mathbb{S}_m(\mathcal{L}) := \{\mathcal{S} : \mathcal{S}$ is a DN-based and non-pseudo axiomatic logic$\}$.
- $\mathbb{K}_m(\mathcal{L}) := \{\mathrm{K} : \mathrm{K}$ is a subvariety of the variety over $\mathcal{L}$ axiomatized by the DN-algebra equations$\}$.

Theorem

$\mathbb{S}_m(\mathcal{L})$ and $\mathbb{K}_m(\mathcal{L})$ are dual order-isomorphic.

$$\mathcal{S} \mapsto \mathrm{K}_{\mathcal{S}}$$

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$$\varphi_1, \ldots, \varphi_n \vdash_{\mathcal{S}_{\mathrm{K}}} \varphi \iff (\forall A \in \mathrm{K})(\forall h \in \mathrm{Hom}(Fm, A)) \; m^{n-1}(h\varphi_1, \ldots, h\varphi_n, h\varphi) \leq h\varphi$$

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Bijective correspondence

Let $\mathcal{L}$ be an algebraic language and m a ternary term of $\mathcal{L}$.

- $\mathbb{S}_m(\mathcal{L}) := \{\mathcal{S} : \mathcal{S}$ is a DN-based and non-pseudo axiomatic logic$\}$.
- $\mathbb{K}_m(\mathcal{L}) := \{\mathrm{K} : \mathrm{K}$ is a subvariety of the variety over $\mathcal{L}$ axiomatized by the DN-algebra equations$\}$.

Theorem

$\mathbb{S}_m(\mathcal{L})$ and $\mathbb{K}_m(\mathcal{L})$ are dual order-isomorphic.

$$\mathcal{S} \mapsto \mathrm{K}_{\mathcal{S}}$$

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$$\varphi_1, \ldots, \varphi_n \vdash_{\mathcal{S}_{\mathrm{K}}} \varphi \iff (\forall A \in \mathrm{K})(\forall h \in \mathrm{Hom}(Fm, A))\; m^{n-1}(h\varphi_1, \ldots, h\varphi_n, h\varphi) \leq h\varphi$$

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Muchas Gracias!

References

[1] I. Chajda, R. Halaš, and J. Kühr.
Semilattice structures, volume 30.
Heldermann, Lemgo, 2007.

[2] R. Jansana.
Selfextensional logics with implication.
In J-Y. Beziau, editor, *Logica Universalis*, pages 65–88.
Springer, 2005.

[3] R. Jansana.
Selfextensional logics with a conjunction.
Studia Logica, 84(1):63–104, 2006.