A topological duality for tense LM_n-algebras and applications

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Introduction

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- A year later, in

Gr.C. Moisil, *Recherches sur les logiques non-chrysippiennes, Ann. Sci. Univ. Iassy*, **26**, 1940, 431–466.

this author generalized these algebras by defining n-valued Łukasiewicz algebras (now these algebras are known as under the name of n-valued Łukasiewicz–Moisil or LM_n-algebras for short) and he studied them from the algebraic point of view.

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- R. Cignoli found algebraic counterparts for $n \geq 5$ and he called them proper n-valued Łukasiewicz algebras. See

R. Cignoli, *Proper n-valued Łukasiewicz Algebras as S-algebras of Łukasiewicz valued propositional calculi*. Studia Logica. **41**(1982), 3–16.

It is known that propositional logics, both classic or non–classic, usually do not incorporate the dimension of time. The tense logic is an extension of the classic logic which was introduced in order to formalize statements that include data about the moment in which they have occurred, that is, the tense logic allows us to differentiate whether an event occurs in the past, in the present, or in the future.

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- Then it can be defined new unary operators F (it will be the case) and P (it has been the case) by means of G and H as follows:

$F(x) = \neg G(\neg x)$ *and* $P(x) = \neg H(\neg x)$,

where $\neg x$ *denotes negation of the proposition* x.

- These tense operators were firstly introduced in the classic propositional logic in 1984. See

J. Burges, *Basic tense logic.* In: Gabbay, D.M., Günter, F. (eds) Handbook of Philosophical Logic, Vol. II, pp. 89–139. Reidel, Dordrecht (*1984*).

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- Thus tense algebras (or tense Boolean algebras) appeared, that is tense algebras are algebraic structures corresponding to the tense propositional logic.

Definition

An algebra $\langle A, \vee, \wedge, \neg, G, H, 0, 1\rangle$ *is a tense algebra if*

(i) $\langle A, \vee, \wedge, \neg, 0, 1\rangle$ *is a Boolean algebra,*

(ii) G, H *are two unary operators on* A *that satisfy the following axioms for all* $x, y \in A$*:*

(T1) $G(1) = 1$ *and* $H(1) = 1$,

(T2) $G(x \wedge y) = G(x) \wedge G(y)$ *and* $H(x \wedge y) = H(x) \wedge H(y)$,

(T3) $x \leq G(P(x))$ *and* $x \leq H(F(x))$,

where $P(x) := \neg H(\neg x)$ *and* $F(x) := \neg G(\neg x)$.

Taking into account that tense algebras constitute the algebraic basis for the bivalent tense logic, D. Diaconescu and G. Georgescu introduced in

D. Diaconescu and G. Georgescu, *Tense operators on MV-algebras and Łukasiewicz–Moisil algebras*, Fund. Inform. **81** (4), 379–408, (2007).

the tense MV-algebras and the tense Łukasiewicz–Moisil algebras (or tense n-valued Łukasiewicz–Moisil algebras) as algebraic structures for some many-valued tense logics.

- R. Cignoli in

R. Cignoli, *Moisil Algebras.* Notas de Lógica Matemática 27. Inst. Mat. Univ. Nacional del Sur, Bahía Blanca 1970,

defined the n-valued Łukasiewicz–Moisil algebras (or LM_n-algebras for short) in the following way:

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defined the n-valued Łukasiewicz–Moisil algebras (or LM_n-algebras for short) in the following way:

- In what follows n is a natural number and we use the notation $[n] := \{1, \ldots, n\}$.

Definition

An algebra $\langle A, \vee, \wedge, \sim, \{\varphi_i\}_{i\in[n-1]}, 0, 1\rangle$ *is an n-valued Łukasiewicz–Moisil algebra (or LM_n-algebra), where $n \geq 2$ is an integer number, if*

(i) $\langle A, \vee, \wedge, \sim, 0, 1\rangle$ *is a De Morgan algebra,*

(ii) φ_i, $i \in [n-1]$, *are unary operations on A which satisfy the following conditions for any $i, j \in [n-1]$ and $x, y \in A$:*

(L1) $\varphi_i(x \vee y) = \varphi_i(x) \vee \varphi_i(y)$,
(L2) $\varphi_i(x) \vee \sim \varphi_i(x) = 1$,
(L3) $\varphi_i(\varphi_j(x)) = \varphi_j(x)$,
(L4) $\varphi_i(\sim x) = \sim \varphi_{n-i}(x)$,
(L5) $i \leq j$ *implies* $\varphi_i(x) \leq \varphi_j(y)$,
(L6) $\varphi_i(x) = \varphi_i(y)$ *for all* $i \in [n-1]$, *implies* $x = y$.

The axiom (L6) *is known as Moisil's determination principle.*

D. Diaconescu and G. Georgescu in

D. Diaconescu and G. Georgescu *Tense operators on MV-algebras and Łukasiewicz-Moisil algebras*, Fund. Inform. **81** (4), 379–408, (2007).

introduced the notion of tense n-valued Łukasiewicz–Moisil algebra as follows:

Definition

An algebra $\langle A, \vee, \wedge, \sim, \{\varphi_i\}_{i\in[n-1]}, G, H, 0, 1\rangle$ *is a tense n-valued Łukasiewicz–Moisil algebra (or tense LM_n-algebra) if*

(i) $\langle A, \vee, \wedge, \sim, \{\varphi_i\}_{i\in[n-1]}, 0, 1\rangle$ *is an LM_n-algebra,*

(ii) G, H *are two unary operators on A, which satisfy the following properties:*

(T1) $G(1) = 1$ *and* $H(1) = 1$,

(T2) $G(x \wedge y) = G(x) \wedge G(y)$ *and* $H(x \wedge y) = H(x) \wedge H(y)$,

(T3) $x \leq G(P(x))$ *and* $x \leq H(F(x))$,

(T4) $G(\varphi_i(x)) = \varphi_i(G(x))$ *and* $H(\varphi_i(x)) = \varphi_i(H(x))$,

for any x, $y \in A$ *and* $i \in [n-1]$,

where $P(x) = \sim H(\sim x)$ *and* $F(x) = \sim G(\sim x)$.

A topological duality for tense LM_n-algebras

- We have developed a topological duality for tense n-valued Łukasiewicz–Moisil algebras, taking into account the results established by A.V. Figallo, I. Pascual and A. Ziliani in

A. V. Figallo, I. Pascual and A. Ziliani, *Notes on monadic n-valued Łukasiewicz algebras*, Math. Bohem., 129 (3), 255–271, (2004),

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- and the results obtained by A.V. Figallo and G. Pelaitay in

A. V. Figallo and G. Pelaitay, *Tense operators on De Morgan algebras*. Log. J. IGPL, **22** (2), 255–267, (2014).

To this aim, we introduced the topological category $\boldsymbol{tLM_nS}$ *of tense* LM_n*-spaces and tense* LM_n*-functions. Specifically:*

Definition

A system $(X, g, \{f_i\}_{i\in[n-1]}, R)$ is a tense LM_n-space if the following conditions are satisfied for all $x, y \in X$ and $i, j \in [n-1]$:

(i) $(X, g, \{f_i\}_{i\in[n-1]})$ *is an LM_n-space, i.e.*

- (LP1) (X, g) *is an mP-space. More precisely,*
  - (mP1) X *is a Priestley space (or P-space),*
  - (mP2) $g : X \to X$ *is an involutive homeomorphism and an order anti-isomorphism,*
- (LP2) $f_i : X \longrightarrow X$ *is a continuous function,*
- (LP3) $x \leq y$ *implies* $f_i(x) = f_i(y)$ *for all* $i \in [n-1]$,
- (LP4) $i \leq j$ *implies* $f_i(x) \leq f_j(x)$,
- (LP5) $f_i \circ f_j = f_i$,
- (LP6) $f_i \circ g = f_i$,
- (LP7) $g \circ f_i = f_{n-i}$,
- (LP8) $X = \bigcup_{i=1}^{n-1} f_i(X)$.

Definition

(ii) *R is a binary relation on X and R^{-1} is the converse of R such that for all x, $y \in X$:*

(tS1) $(x, y) \in R$ *implies* $(g(x), g(y)) \in R$,

(tS2) *for each* $x \in X$, $R(x)$ *and* $R^{-1}(x)$ *are closed in* X

(tS3) *for each* $x \in X$, $R(x) = \downarrow R(x) \cap \uparrow R(x)$,

(tS4) $(x, y) \in R$ *implies* $(f_i(x), f_i(y)) \in R$ *for any* $i \in [n-1]$,

(tS5) $(f_i(x), y) \in R$, $i \in [n-1]$, *implies that there exists* $z \in X$ *such that* $(x, z) \in R$ *and* $f_i(z) \leq y$,

(tS6) $(y, f_i(x)) \in R$, $i \in [n-1]$, *implies that there exists* $z \in X$ *such that* $(z, x) \in R$ *and* $f_i(z) \leq y$,

(tS7) *for each* $U \in D(X)$, $G_R(U), H_{R^{-1}}(U) \in D(X)$, *where*

$$G_R(U) = \{x \in X \mid R(x) \subseteq U\},$$

$$H_{R^{-1}}(U) = \{x \in X \mid R^{-1}(x) \subseteq U\},$$

and $D(X)$ *is the lattice of increasing and clopen subsets of* X.

We emphasize the following properties of LM_n-spaces which were useful to obtain the duality for tense LM_n-algebras:

(LP9) *for each* $x \in X$ *there is an index* $i \in [n-1]$, *such that* $x = f_i(x)$,

(LP10) *if* Y, Z *are subsets of* X *and* $f_i^{-1}(Y) = f_i^{-1}(Z)$ *for all* $i \in [n-1]$, *then* $Y = Z$,

(LP11) *every* LM_n*-space* $(X, g, \{f_i\}_{i\in[n-1]})$ *is the cardinal sum of a family of chains, each of which has at most,* $n-1$ *elements,*

(LP12) *if* $(X, g, \{f_i\}_{i\in[n-1]})$ *is an* LM_n*-space,* $x \in X$ *and* C_x *denotes the unique maximum chain containing* x, *then* $C_x = \{f_i(x) : i \in [n-1]\}$.

Definition

A tense LM_n-function f from a tense LM_n-space $(X, g, \{f_i\}_{i\in[n-1]}, R)$ into another $(X', g', \{f'_i\}_{i\in[n-1]}, R')$, is a function $f : X \longrightarrow X'$ such that:

(i) $f : X \longrightarrow X'$ *is an LM_n-function. More precesily,*

- (Pf) f *is a continuous and increasing function (P-function),*
- (mPf) $f \circ g = g' \circ f$,
- (LPf) $f'_i \circ f = f \circ f_i$ *for all $i \in [n-1]$,*

(ii) *for any $U \in D(X_2)$, the following properties hold:*

- (tf6) $f^{-1}(G_{R_2}(U)) = G_{R_1}(f^{-1}(U))$,
- (tf7) $f^{-1}(H_{R_2^{-1}}(U)) = H_{R_1^{-1}}(f^{-1}(U))$.

Then, using the usual procedures we proved the following result:

Theorem

The category $\boldsymbol{tLM_nS}$ of tense LM_n-spaces and tense LM_n-functions is naturally equivalent to the dual of the category $\boldsymbol{tLM_nA}$ of tense LM_n-algebras and tense LM_n-homomorphism.

Simple and subdirectly irreducible tense LM_n-algebras

In this section, our first objective was the characterization of the congruences on a tense LM_n-algebra by means of the duality that we have obtained for these algebras. Later, this result was taken into account to describe the simple and subdirectly irreducible tense LM_n-algebras, as we will indicate below:

As a consequence of the duality that we have developed for tense LM_n-algebras, we introduced the following unary operators on these algebras:

Definition

Let $(A, \sim, \{\varphi_i\}_{i\in[n-1]}, G, H)$ *be a tense* LM_n*-algebra and let* $d : A \longrightarrow A$ *be a function defined for all* $a \in A$ *by the prescription:*

$$d(a) = G(a) \wedge a \wedge H(a),$$

For all $n \in \omega$*, let* $d^n : A \longrightarrow A$ *be a function defined for all* $a \in A$ *by:*

$$d^0(a) = a,$$

$$d^{n+1}(a) = d(d^n(a)).$$

Lemma

Let $(A, \sim, \{\varphi_i\}_{i\in[n-1]}, G, H)$ *be a tense* LM_n*-algebra Then, for all* $n \in \omega$ *and* a, $b \in A$, *the following conditions are satisfied:*

(d1) $d^n(1) = 1$ *and* $d^n(0) = 0$,

(d2) $d^{n+1}(a) \leq d^n(a)$,

(d3) $d^n(a \wedge b) = d^n(a) \wedge d^n(b)$,

(d4) $a \leq b$ *implies* $d^n(a) \leq d^n(b)$,

(d5) $d^n(a) \leq a$,

(d6) $d^{n+1}(a) \leq G(d^n(a))$ *and* $d^{n+1}(a) \leq H(d^n(a))$,

(d7) *for all* $i \in [n-1]$ *and* $n \in \omega$, $d^n(\varphi_i(a)) = \varphi_i(d^n(a))$.

- It is worth mentioning that the above unary operators d^n, $n \in \omega$, were previously defined by T. Kowalski for tense algebras in

T. Kowalski, *Varieties of tense algebras*, *Rep. Math. Logic*, **32**, 53–95, (1998).

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- Also these operators were defined by D. Diaconescu and G. Georgescu for tense MV-algebras in

D. Diaconescu and G. Georgescu, Tense operators on MV-algebras and Łukasiewicz-Moisil algebras, *Fund. Inform.* **81** (4), 379–408, (2007).

- and by C. Chiriță for tense θ-valued Łukasiewicz–Moisil algebras in

C. Chiriţă, *Tense θ-valued Łukasiewicz–Moisil algebras*, J. Mult. Valued Logic Soft Comput., **17** (1), 1–24, (2011).

- The unary operators d^n, $n \in \omega$, defined on tense LM_n-algebras, play a fundamental role in the characterization of congruences on these algebras and the description of the simple and subdirectly irreducible tense LM_n-algebras.

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- Let's first observe the following fact:

If $(A, \sim, \{\varphi_i\}_{i\in[n-1]}, G, H)$ is a tense LM_n-algebra, then for any $a \in A$, $(d^n(a))_{n\in\omega}$ is a decreasing succession in A such that

$$d^n(a) \leq a \text{ for all } n \in \omega.$$

By means of the invariance properties of the operator d defined on tense LM_n-algebras, we obtained the following result:

Lemma

Let $(A, \sim, \{\varphi_i\}_{i\in[n-1]}, G, H)$ *be a tense* LM_n*-algebra and*

$$\mathcal{C}(A) := \{a \in A : \ a \text{ is a } d\text{-invariant element}\}$$

$$= \{a \in A : \ d(a) = a\} = \{a \in A : \ d^n(a) = a, \ \text{for all } n \in \omega\}.$$

Then,

(i) $\langle \mathcal{C}(A), \vee, \wedge, \sim, \{\varphi_i\}_{i\in[n-1]}, 0, 1\rangle$ *is an* LM_n*-algebra,*

(ii) $\langle \mathcal{B}(\mathcal{C}(A)), \vee, \wedge, \sim, G, H, 0, 1\rangle$ *is a tense Boolean algebra,*

where $\mathcal{B}(\mathcal{C}(A))$ *is the set of all complemented elements of* $\mathcal{C}(A)$.

By means of the above duality, we obtained the following characterization of the congruences on a tense LM_n-algebra:

Proposition

Let $(A, \sim, \{\varphi_i\}_{i\in[n-1]}, G, H)$ be a tense LM_n-algebra. Then, the following conditions are equivalent for all $\theta \subseteq A \times A$:

(i) $\theta \in Con_{tLM_n}(A)$,

(i) *there is a tense Stone filter S of A such that $\theta = \Theta(S)$,*

where $\Theta(S)$ is the lattice congruence associated with the filter S, a filter S of A is a tense filter if $d^n(a) \in S$ for all $a \in S$ and for all $n \in \omega$, and S is an Stone filter if $\varphi_i(a) \in S$ for all $a \in S$ and for all $i \in [n-1]$.

Finally, the above duality allowed us to characterize the simple and subdirectly irreducible tense LM_n-algebras as we will indicate below:

Theorem

Let $(A, \sim, \{\varphi_i\}_{i\in[n-1]}, G, H)$ be a tense LM_n-algebra. Then, the following conditions are equivalent:

(i) $(A, \sim, \{\varphi_i\}_{i\in[n-1]}, G, H)$ *is a simple tense LM_n-algebra,*

(ii) *for each $b \in \mathcal{B}(A) \setminus \{1\}$, there is $n_b \in \omega$ such that $d^{n_b}(b) = 0$,*

(iii) $\mathcal{F}_{TS}(A) = \{A, \{1\}\}$, *where $\mathcal{F}_{TS}(A)$ is the set of all tense Stone filters of A.*

As a consequence of the previous theorem we obtained that:

Corollary

If $(A, \sim, \{\varphi_i\}_{i\in[n-1]}, G, H)$ *is a simple tense* LM_n*-algebra, then*

$(\mathcal{C}(A), \sim, \{\varphi_i\}_{i\in[n-1]})$ *is a simple* LM_n*-algebra,*

where $\mathcal{C}(A) = \{a \in A : a = d(a)\}$.

Theorem

Let $(A, \sim, \{\varphi_i\}_{i\in[n-1]}, G, H)$ be a tense LM_n-algebra. Then, the following conditions are equivalent:

(i) $(A, \sim, \{\varphi_i\}_{i\in[n-1]}, G, H)$ *is a subdirectly irreducible tense LM_n-algebra,*

(ii) *there is $b \in \mathcal{B}(A) \setminus \{1\}$ such that for every $a \in \mathcal{B}(A) \setminus \{1\}$, $d^{n_a}(a) \leq b$ for some $n_a \in \omega$,*

(iii) *there is $T \in \mathcal{F}_{TS}(A)$, $T \neq \{1\}$ such that $T \subseteq S$ for all $S \in \mathcal{F}_{TS}(A)$, $S \neq \{1\}$, where $\mathcal{F}_{TS}(A)$ is the set of all tense Stone filters of A.*

Complete and finite simple and subdirectly irreducible tense LM_n-algebras

We were interested in the characterization of the simple and subdirectly irreducible complete tense LM_n-algebras (i.e for all $S \subseteq A$, $\bigwedge_{a \in S} a \in A$ or equivalently $\bigvee_{a \in S} a \in A$), whose filters are complete. For this, we took into account the following property:

If A is a complete lattice whose prime filters are complete, and $\sigma_A : A \longrightarrow D(\mathfrak{X}(A))$ is defined by

$$\sigma_A(a) = \{Q \in \mathfrak{X}(A) : a \in Q\},$$

then for all $S \subseteq A$,

$$\sigma_A\left(\bigwedge_{a \in S} a\right) = \bigwedge_{a \in S} \sigma_A(a) = \bigcap_{a \in S} \sigma_A(a).$$

First, we obtained that the notion of $d-$invariant element of a complete tense LM_n-algebra and particularly the notion of $d-$invariant element of a finite tense LM_n-algebra has several equivalent formulations, as we will indicate next.

Proposition

Let (A, G, H) be a complete tense LM_n-algebra or a finite LM_n-algebra. Then, the following conditions are equivalent for any $a \in A$:

(i) $a = d(a)$,

(ii) $a = d^n(a)$ *for all* $n \in \omega$,

(iii) $a = \bigwedge_{n\in\omega} d^n(a)$,

(iv) $a = \bigwedge_{n\in\omega} d^n(b)$ *for some* $b \in A$.

Therefore $C(A) = \left\{ \bigwedge_{n\in\omega} d^n(a) : a \in A \right\}$.

Theorem

Let $(A, \sim, \{\varphi_i\}_{i\in[n-1]}, G, H)$ be a complete tense LM_n-algebra, whose filters are complete or a finite complete tense LM_n-algebra. Then, the following conditions are equivalent:

(i) $(A, \sim, \{\varphi_i\}_{i\in[n-1]}, G, H)$ *is a simple tense LM_n-algebra,*

(ii) $(A, \sim, \{\varphi_i\}_{i\in[n-1]}, G, H)$ *is a subdirectly irreducible tense LM_n-algebra,*

(iii) $(\mathcal{C}(A), \sim, \{\varphi_i\}_{i\in[n-1]})$ *is a simple tense LM_n-algebra,*

(iv) $\mathcal{B}(\mathcal{C}(A)) = \{0, 1\}$.

A representation theorem for tense LM_n-algebras

As an application of the categorical equivalence obtained, we proved a representation theorem for tense LM_n-algebras, which was formulated and proved by a different method by D. Diaconescu and G. Georgescu in

D. Diaconescu and G. Georgescu, *Tense operators on MV-algebras and Łukasiewicz-Moisil algebras*, Fund. Inform. **81** (4), 379–408, (2007).

The following examples play a fundamental role in a representation theorem for tense LM_n-algebras, which will be formulated next.

Example

An example of an LM_n-algebra is the chain of n rational fractions

$$L_n = \left\{ \frac{j}{n-1} : 0 \leq j \leq n-1 \right\},$$

in which $n \geq 2$ is an integer number, endowed with the natural lattice structure and the unary operations $\sim$ and φ_i, defined as follows:

$$\sim \left(\tfrac{j}{n-1}\right) = 1 - \tfrac{j}{n-1},$$

$\varphi_i\left(\frac{j}{n-1}\right) = 0$ *if* $i + j < n$ *or* $\varphi_i\left(\frac{j}{n-1}\right) = 1$ *if* $i + j \geq n$.

Example

Let (X, R) be a frame (i.e. X is a nonempty set and R is a binary relation on X) and let G^*, H^* : $L_n^X \longrightarrow L_n^X$ be defined as follows:

$$G^*(p)(x) = \bigwedge\{p(y) |\, y \in X,\, xRy\},$$

$$H^*(p)(x) = \bigwedge\{p(y) |\, y \in X,\, yRx\},$$

for all $p \in L_n^X$ and $x \in X$. Then,

$$\langle L_n^X, \wedge, \vee, \sim, \{\varphi_i\}_{i\in[n-1]}, G^*, H^*, 0, 1\rangle,$$

is a tense LM_n-algebra, where the operations of the LM_n-algebra $\langle L_n^X, \wedge, \vee, \sim, \{\varphi_i\}_{i[n-1]}, 0, 1\rangle$ are defined pointwise.

- The above example was given in

D. Diaconescu and G. Georgescu, *Tense operators on MV-algebras and Łukasiewicz-Moisil algebras*, Fund. Inform. **81** (4), 379–408, (2007).

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- Also, these authors formulated and proved the following important theorem:

Theorem

For any tense LM_n-algebra (A, G, H), there exists a frame (X, R) and an injective morphism of tense LM_n-algebras from (A, G, H) into (L_n^X, G^*, H^*).

- The above example was given in

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- Also, these authors formulated and proved the following important theorem:

Theorem

For any tense LM_n-algebra (A, G, H), there exists a frame (X, R) and an injective morphism of tense LM_n-algebras from (A, G, H) into (L_n^X, G^*, H^*).

- We offer an alternative proof of this result by means of the above duality for tense LM_n-algebras, as we will indicate below.

Given a tense LM_n-algebra (A, G, H), our first objective was to find a frame (X, R) such that there is a surjective tense LM_n-function from the tense LM_n-space associated with tense LM_n-algebra (L^X, G^*, H^*) onto the tense LM_n-space associated with (A, G, H).

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- To this aim, we showed the following results:

Lemma

Let $(A, \sim, \{\varphi_i\}_{i\in[n-1]}, G, H)$ *be a tense* LM_n*-algebra and* $(\mathfrak{X}(A), g_A, \{f_i^A\}_{i\in[n-1]}, R^A)$ *be the tense* LM_n*-space associated with* A*. If* $X = \max \mathfrak{X}(A)$ *and* $R = R^A\mid_X$*, then for all* $M \in X$*,*

$$R(M) = \{\, T \in X : G^{-1}(M) \subseteq T\} = \{\, T \in X : T \subseteq F^{-1}(M)\}$$
$$= \{T \in X : H^{-1}(T) \subseteq M\} = \{T \in X : M \subseteq P^{-1}(T)\},$$

where the binary relation R^A *is defined by the prescription:*

$$(S, T) \in R^A \Longleftrightarrow G^{-1}(S) \subseteq T \subseteq F^{-1}(S),$$

or by the prescription:

$$(S, T) \in R^A \Longleftrightarrow H^{-1}(T) \subseteq S \subseteq P^{-1}(T).$$

Therefore (X, R) *is a frame.*

Theorem

For any tense LM_n-algebra (A, G, H) there exists a surjective tense LM_n-function from $\mathfrak{X}\left(L_n^X\right)$ onto $\mathfrak{X}(A)$, where $\mathfrak{X}(A)$ is the tense LM_n-space associated with (A, G, H), $\mathfrak{X}\left(L_n^X\right)$ is the tense LM_n-space associated with (L_n^X, G^*, H^*) and the frame $(X, R) = \left(\max \mathfrak{X}(A), R^A\mid_{\max \mathfrak{X}(A)}\right)$.

- Finally, the previous theorem and the topological duality for tense LM_n-algebras enabled us to formulated and proved the following theorem:

Theorem

(*Representation Theorem for tense LM_n-algebras*) *For any tense LM_n-algebra (A, G, H) there exist a frame (X, R) and an injective morphism of tense LM_n-algebras from (A, G, H) into the tense LM_n-algebra (L_n^X, G^*, H^*).*

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(*Representation Theorem for tense LM_n-algebras*) *For any tense LM_n-algebra (A, G, H) there exist a frame (X, R) and an injective morphism of tense LM_n-algebras from (A, G, H) into the tense LM_n-algebra (L_n^X, G^*, H^*).*

- This last theorem reduces the calculus in an arbitrary tense LM_n-algebra (A, G, H) to the calculus in $\left(L_n^X, G^*, H^*\right)$.

In what follows we will give a brief description of the steps of the proof of Representation Theorem for tense LM_n-algebras.

First step:

From a tense LM_n-algebra $(A, \sim, \{\varphi_i\}_{i\in[n-1]}, G, H)$, we got a frame (X, R). More precisely,

Lemma

If $(A, \sim, \{\varphi_i\}_{i\in[n-1]}, G, H)$ is a tense LM_n-algebra, $(\mathfrak{X}(A), g_A, \{f_i^A\}_{i\in[n-1]}, R^A)$ is the tense LM_n-space associated with A, $X = \max \mathfrak{X}(A)$ and $R = R^A \mid_X$, then (X, R) is a frame.

Then, we proposed to prove that for any tense LM_n-algebra (A, G, H) there is a continuous function with certain additional properties from the tense LM_n-space $\mathfrak{X}\left(L_n^X\right)$ associated with the tense LM_n-algebra $\left(L_n^X, G^*, H^*\right)$ to the tense LM_n-space $\mathfrak{X}(A)$ associated with (A, G, H), where (X, R) is the frame that we had obtained.

To achieve our goal, we took into account the following known fact:

$\mathfrak{D} = \left\{ S \times {L_n}^{X \setminus \{x\}} : S \in \mathfrak{X}(L_n), x \in X \right\}$ *is a subset of* $\mathfrak{X}(L_n^X)$,

where for each $x \in X$,

$$ {L_n}^{X \setminus \{x\}} = \{ f : X \setminus \{x\} \longrightarrow L_n \}. $$

Also, we took into account the following theorem of extension of continuous functions:

Theorem

Let (X, τ_X) be a topological space, $D \subseteq X$ dense in X, (Y, τ_Y) be a regular topological space and $f : D \longrightarrow Y$ be a continuous function. Then f has a continuous extension $F : X \longrightarrow Y$ if and only if, for every $x \in X$ and all nets $(x_i)_{i \in I} \subseteq D$ which converge to x, the nets $(f(x_i))_{i \in I}$ converge to the same limit in Y. If F exists, then F is the unique continuous extension of f.

Second step:

We obtained a dense subset of the tense LM_n-space $\mathfrak{X}(L_n)$ associated with $\left(L_n^X, G^*, H^*\right)$. That is,

Lemma

Let $(L_n^X, \sim, \{\varphi_i^{L_n^X}\}_{i\in[n-1]}, G^*, H^*)$ *be the tense* LM_n*-algebra described in the previous example,* $\mathfrak{X}\left(L_n^X\right)$ *be the tense* LM_n*-space associated with* L_n^X *and* $\mathfrak{X}\left(L_n\right)$ *be the* LM_n*-space associated with the* LM_n*-algebra* L_n*. If*

$$\mathfrak{D} = \left\{S \times {L_n}^{X\setminus\{x\}} : S \in \mathfrak{X}\left(L_n\right), x \in X\right\},$$

where for each $x \in X$, ${L_n}^{X\setminus\{x\}} = \{f : X \setminus \{x\} \longrightarrow L_n\}$, *then* $\mathfrak{D}$ *is a dense subset of* $\mathfrak{X}\left(L_n^X\right)$.

Third step:

We defined a function from the dense subset $\mathfrak{D}$ of $\mathfrak{X}(L_n^X)$ that we had obtained to the tense LM_n-space $\mathfrak{X}(A)$ associated with the given tense LM_n-algebra A, and we showed that this function satisfies the conditions established in the aforementioned theorem of extension of continuous functions, as we will indicate below:

Proposition

Let $(A, \sim, \{\varphi_i^A\}_{i\in[n-1]}, G, H)$ *be a tense* LM_n*-algebra,* $(\mathfrak{X}(A), g_A, \{f_i^A\}_{i\in[n-1]}, R^A)$ *be the tense* LM_n*-space associated with* A, $X = \max \mathfrak{X}(\mathrm{A})$ *and*

$$\mathfrak{D} = \left\{ Q \times {L_n}^{X\setminus\{M\}} : \; Q \in \mathfrak{X}\left(L_n\right), \, M \in X \right\}.$$

Proposition

If $f : \mathfrak{D} \longrightarrow \mathfrak{X}(A)$ is defined for each $Q \in \mathfrak{X}(L_n)$ and for each $M \in X$, by the prescription:

$$f\left(Q \times {L_n}^{X\setminus\{M\}}\right) = \varphi_i^{A^{-1}}(M), \text{ if } Q = \varphi_i^{-1}(Q),$$
$$\text{for some } i \in [n-1],$$

then, f satisfies the following properties:

(i) *f is continuous, considering $\mathfrak{D}$ as subspace of $\mathfrak{X}(L_n^X)$.*

(ii) *If $(T_d)_{d\in\mathcal{D}} \subseteq \mathfrak{D}$ is a net such that $T_d \underset{d\in\mathcal{D}}{\longrightarrow} T$ for some $T \in \mathfrak{X}\left(L_n^X\right) \setminus \mathfrak{D}$, then the net $(f(T_d))_{d\in\mathcal{D}}$ converges in $\mathfrak{X}(A)$.*

(iii) *If the nets $(T_d)_{d\in\mathcal{D}} \subseteq \mathfrak{D}$ and $(S_d)_{d\in\mathcal{D}} \subseteq \mathfrak{D}$ converge to the same element $T \in \mathfrak{X}\left(L_n^X\right) \setminus \mathfrak{D}$, then the nets $(f(T_d))_{d\in\mathcal{D}}$ and $(f(S_d))_{d\in\mathcal{D}}$ converge to the same element in $\mathfrak{X}(A)$.*

Fourth step:

The above results allowed us to define the surjective tense LM_n-function from $\mathfrak{X}(L_n^X)$ onto $\mathfrak{X}(A)$ we were looking for. Specifically:

Theorem

Let $(A, \sim, \{\varphi_i^A\}_{i\in[n-1]}, G, H)$ be a tense LM_n-algebra, $\left(\mathfrak{X}(A), g_A, \{f_i^A\}_{i\in[n-1]}, R^A\right)$ be the tense LM_n-space associated with A, and $(\mathfrak{X}\left(L_n^X\right), g_{L_n^X}, \{f_i^{L_n^X}\}_{i\in[n-1]}, R^{L_n^X})$ be the tense LM_n-space associated with $(L_n^X, \sim, \{\varphi_i^{L_n^X}\}_{i\in[n-1]}, G^*, H^*)$, where the frame $(X, R) = \left(\max \mathfrak{X}(A), R^A\mid_{\max \mathfrak{X}(A)}\right)$.

Theorem

Let $\Phi : \mathfrak{X}\left(L_n^X\right) \longrightarrow \mathfrak{X}(A)$ *be defined by the prescription:*

for all $T \in \mathfrak{D}$,

$$\Phi(T) = f(T) = f(Q \times {L_n}^{X \setminus \{M\}}) = \phi_i^A(M) \text{ if } Q = \phi_i^{-1}(Q),$$

for all $T \in \mathfrak{X}\left(L_n^X\right) \setminus \mathfrak{D}$,

$$\Phi(T) = S \text{ if and only if } f(T_d) \xrightarrow[d \in \mathcal{D}]{} S$$

$$\text{for any net } (T_d)_{d \in \mathcal{D}} \subseteq \mathfrak{D} \text{ such that } T_d \xrightarrow[d \in \mathcal{D}]{} T,$$

where $\mathfrak{D} = \left\{Q \times {L_n}^{X \setminus \{M\}} : Q \in \mathfrak{X}\left(L_n\right), M \in X\right\}$.
Then Φ *is a surjective tense* LM_n*-function.*

Fith step:

Finally, the last theorem and the topological duality previously obtained for tense LM_n-algebras allowed us to formulate and prove the following theorem:

Theorem

(*Representation Theorem for tense LM_n-algebras*) *For any tense LM_n-algebra (A, G, H) there exist a frame (X, R) and an injective morphism of tense LM_n-algebras from (A, G, H) into the tense LM_n-algebra (L_n^X, G^*, H^*).*

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