On the logic that preserves degrees of truth associated to involutive Stone algebras

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Pursuing to characterize five-valued Łukasiewicz-Moisil algebras as De Morgan algebras such that their complemented elements satisfy certain conditions and to obtain corresponding characterizations of four and three-valued cases, Cignoli and Sagastume, introduced the class of *involutive Stone algebras*, noted by **S**, in

The lattice structure of some Łukasiewicz algebras. Algebra Universalis. **13** (1981) 315–328.

Later, this same authors proved that **S** is a variety, provided a topological duality for them and they used this duality for determining the subdirectly irreducible algebras and the simple algebras, in

Dualities for some De Morgan algebras with operators and Łukasiewicz algebras. J. Austral. Math. Soc. (Series A) **34** (1983) 377–393.

Let A be a De Morgan algebra. Denote by $K(A)$ the set of all elements $a \in A$ such that the De Morgan negation of a, $\neg a$, coincides with the complement of a. For every $a \in A$, let

$$K_a = \{k \in K(A) : a \leq k\}$$

and, if K_a has a least element, denote by ∇a the least element of K_a. The class of all De Morgan algebras A such that for every $a \in A$, ∇a exists, and the map

$$a \mapsto \nabla a$$

is a lattice–homomorphism is called the class of involutive Stone algebras, denoted by $\mathbf{S}$.

In what follows, we are going to denote $\Box a$ instead of ∇a.

Theorem (Cignoli et al.)

$\mathbf{S}$ is an equational class. More over, if A is a De Morgan algebra $A \in \mathbf{S}$ if and only if there is an operator $\Box : A \to A$ such that

(IS1) $\Box 0 \approx 0$,

(IS2) $x \wedge \Box x \approx x$,

(IS3) $\Box(x \wedge y) \approx \Box x \wedge \Box y$,

(IS4) $\neg \Box x \wedge \Box x \approx 0$.

So, in what follows,

A involutive Stone algebra is an algebra $\langle A, \wedge, \vee, \neg, \Box, 0, 1\rangle$ of type $(2, 2, 1, 1, 0, 0)$ such that

- $\langle A, \wedge, \vee, \neg, \Box, 0, 1\rangle$ is a De Morgan algebra and
- $\Box$ satisfies (IS1)–(IS4).

Proposition

If A is an involutive Stone algebra then

(i) $\Box A = K(A)$,

(ii) $x \in \Box A$ iff $x = \Box x$,

Clearly, $\Box$ is a modal operator and it satisfies the following properties:

Proposition

In every involutive Stone algebra A the following identities hold:

(SI5)	$\Box 1 \approx 1$,	(SI6)	$\neg x \vee \Box x \approx 1$,
(SI7)	$\Box\Box x \approx \Box x$,	(SI8)	$\Box\neg\Box x \approx \neg\Box x$.
(SI9)	$x \wedge \neg\Box x \approx 0$,	(SI10)	$\Box(x \vee \Box y) \approx \Box x \vee \Box y$.

As usual, we can define the operator $\Diamond$ as follows

$$\Diamond a =_{def.} \neg\Box\neg a$$

And then,

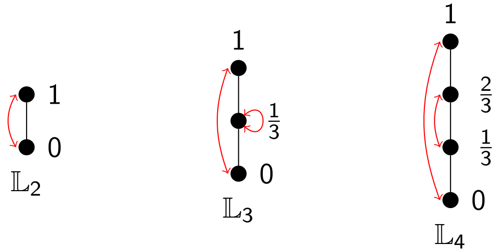
Lemma

In every involutive Stone algebra A the following identities hold:

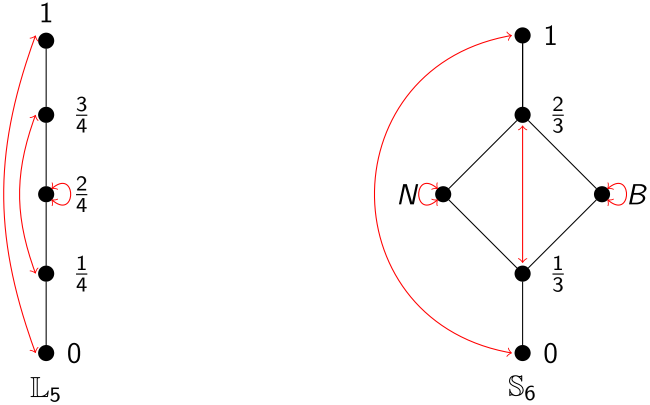
(SI11)	$\Diamond a \leq a$,	(SI12)	$\Diamond\Box a \approx \Box a$,
(SI13)	$\Box\Diamond a \approx \Diamond a$,	(SI14)	$\Diamond(a \wedge \neg a) \approx 0$.
(SI15)	$\Diamond(a \vee b) \approx \Diamond a \vee \Diamond b$,	(SI16)	$\Diamond(a \wedge b) \approx \Diamond a \wedge \Diamond b$.

Some examples of involutive Stone algebras are the following:

(i)



where $\Box x = 1$, for all $x \neq 0$

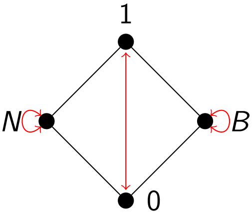


where $\Box x = 1$, for all $x \neq 0$.

(ii) Every Boolean algebra $\langle A, \wedge, \vee, \neg, 0, 1\rangle$ admits a structure of involutive Stone algebra. Indeed, taking $\Box x = x$ we have that $\langle A, \wedge, \vee, \neg, \Box 0, 1\rangle$ is a involutive Stone algebra.

(iii) (Cignoli et al.) If $\langle A, \wedge, \vee, \sim, \sigma_1, ..., \sigma_n, 0, 1\rangle$ is an n-valued Łukasiewicz-Moisil algebra, then the reduct $\langle A, \wedge, \vee, \sim, \sigma_n, 0, 1\rangle$ is a involutive Stone algebra.

Remark. Not every De Morgan algebra admits a structure of involutive Stone algebra. Indeed, consider



$\mathfrak{B}_4$

Theorem (Cignoli et al.)

The subdirectly irreducible algebras in the variety of involutive Stone algebras are $\mathbb{L}_i$ for $2 \leq i \leq 5$, and $\mathbb{S}_6$.

Theorem (Cignoli et al.)

The simple algebras in the variety of involutive Stone algebras are $\mathbb{L}_2$ and $\mathbb{L}_3$.

Corolary

$\mathbf{S} = \mathcal{V}(\mathbb{S}_6)$.

- H. P. Sankappanavar. *Principal congruences of pseudocomplemented De Morgan algebras*. Zeituchr. f. math. Logik und Grundlagen d. Math. Bd. 33, S. 3–11 (1987)

Now focus on the logic that preserves degrees of truth $\mathbb{L}_{\mathbf{S}}^{\leq}$ associated to involutive Stone algebras. This follows a very general pattern that can be considered for any class of truth structure endowed with an ordering relation; and which intend to exploit manyvaluedness focussing on the notion of inference that results from preserving lower bounds of truth values, and hence not only preserving the value 1.

Let $\mathfrak{Fm} = \langle Fm, \wedge, \vee, \neg, \Box, \bot, \top \rangle$ the absolutely free algebra of type $(2, 2, 1, 1, 0, 0)$ generated by some fixed denumerable set Var. As usual, the leters $p, q, \ldots$ denote propositional variables, the leters $\alpha, \beta, \ldots$ denote formulas and $\Gamma, \Delta, \ldots$ sets of formulas.

Definition

The logic that preserves degrees of truth over the variety **S** is $\mathbb{L}_{\mathbf{S}}^{\leq} = \langle Fm, \models_{\mathbf{S}}^{\leq} \rangle$ where, for every $\Gamma \cup \{\alpha\} \subseteq Fm$:

(i) If $\Gamma = \{\alpha_1, \ldots, \alpha_n\} \neq \emptyset$,

$$\alpha_1, \ldots, \alpha_n \models_{\mathbf{S}}^{\leq} \alpha \iff \begin{array}{c} \forall A \in \mathbf{S}, \forall v \in Hom_{\mathbf{S}}(\mathfrak{Fm}, A), \forall a \in A \\ \text{if } v(\alpha_i) \geq a, \text{ for all } i \leq n, \\ \text{then } v(\alpha) \geq a \end{array}$$

(ii) $\emptyset \models_{\mathbf{S}}^{\leq} \alpha \iff \forall A \in \mathbf{S}, \forall v \in Hom_{\mathbf{S}}(\mathfrak{Fm}, A), v(\alpha) = 1.$

(iii) If $\Gamma \subseteq Fm$ is non–finite,

$$\Gamma \models_{\mathbf{S}}^{\leq} \alpha \iff \begin{array}{c} \text{there are } \alpha_1, \ldots, \alpha_n \in \Gamma \text{ such that} \\ \alpha_1, \ldots, \alpha_n \models_{\mathbf{S}}^{\leq} \alpha. \end{array}$$

From the definition we have that $\mathbb{L}_{\mathbf{S}}^{\leq}$ is a sentential logic, that is, $\models_{\mathbf{S}}^{\leq}$ is a finitary consequence relation over Fm.

Proposition

Let $\{\alpha_1, \ldots, \alpha_n, \alpha\} \subseteq Fm$, $n \geq 1$. Then, the following conditions are equivalente:

(i) $\alpha_1, \ldots, \alpha_n \models_{\mathbf{S}}^{\leq} \alpha$,

(ii) $\alpha_1 \wedge \cdots \wedge \alpha_n \models_{\mathbf{S}}^{\leq} \alpha$,

(iii) $\mathbf{S} \models \alpha_1 \wedge \cdots \wedge \alpha_n \preccurlyeq \alpha$.

Besides,

Proposition

Let $\alpha, \beta \in Fm$. Then,

(i) $\alpha \models_{\mathbf{S}}^{\leq} \beta \iff \mathbf{S} \models \alpha \preccurlyeq \beta$,

(ii) $\alpha \mathrel{=\!\mid\!\models_{\mathbf{S}}^{\leq}} \beta \iff \mathbf{S} \models \alpha \approx \beta$.

Since the variety of involutive Stone algebras is generated by the six–element algebra $\mathbb{S}_6$, i.e., $\mathbf{S} = \mathcal{V}(\mathbb{S}_6)$. This means that that every equation holds in $\mathbf{S}$ if and only if it holds in $\mathbb{S}_6$.

Then,

Proposition

For every $\Gamma \cup \{\alpha\} \subseteq Fm$, $\Gamma \models_{\mathbf{S}}^{\leq} \alpha$ if and only if there exists a finite set $\Gamma_0 \subseteq \Gamma$ such that for every $h \in Hom(\mathfrak{Fm}, \mathbb{S}_6)$, $\bigwedge\{h(\gamma) \ : \ \gamma \in \Gamma_0\} \leq h(\alpha)$. In particular, $\emptyset \models_{\mathbf{S}}^{\leq} \alpha$ if and only if $h(\alpha) = 1$ for all $h \in Hom(\mathfrak{Fm}, \mathbb{S}_6)$.

Recall that, a *generalized matrix* is a pair $\langle \mathbf{A}, \mathcal{C} \rangle$ where $\mathbf{A}$ is an algebra and $\mathcal{C}$ is a family of subsets of A that is a closed set system of subsets of A. That is,

(C1) $A \in \mathcal{C}$,

(C2) if $\{X_i\}_{i \in I} \subseteq \mathcal{C}$ then $\bigcup_{i \in I} X_i \in \mathcal{C}$.

If $G = \langle A, \mathcal{C} \rangle$ is a generalized matrix. The logic $\mathbb{L}_G = \langle \mathfrak{Fm}, \models_G \rangle$ determined by G is defined by follows: if $\Gamma \cup \{\alpha\} \subseteq Fm$

$$\Gamma \models_G \alpha \iff \begin{array}{c} \forall v \in Hom_{\mathbf{S}}(Fm, A),\ \forall F \in \mathcal{C} \\ \text{if } v(\gamma) \in F, \text{ for all } \gamma \in F, \text{ then } v(\alpha) \in F. \end{array}$$

Now, consider the family of logic matrices

$$\mathcal{F} = \{\langle A, F \rangle : A \in \mathbf{S} \text{ and } F \text{ is a lattice filter of } A\}$$

and the family of generalized matrices

$$\mathcal{F}_g = \{\langle A, Fi(A)\rangle : A \in \mathbf{S}\}$$

where $Fi(A)$ is the set of all filters of A.

Theorem

The logic $\mathbb{L}_{\mathbf{S}}^{\leq}$ coincides with the logic $\mathbb{L}_{\mathcal{F}}$ and also with the logic $\mathbb{L}_{\mathcal{F}_g}$.

On the other hand, the filters of $\mathbb{S}_6$ are precisely the principal filters generated by each element of $\mathbb{S}_6$. Then, $Fi(\mathbb{S}_6) = \{[0), [\frac{1}{3}), [N), [B), [\frac{2}{3}), [1)\}$. Now, consider the following family of matrices
$\mathcal{F}^6 = \{\langle \mathbb{S}_6, [0)\rangle, \langle \mathbb{S}_6, \left[\frac{1}{3}\right)\rangle, \langle \mathbb{S}_6, [N)\rangle, \langle \mathbb{S}_6, [B)\rangle, \langle \mathbb{S}_6, \left[\frac{2}{3}\right)\rangle, \langle \mathbb{S}_6, [1)\rangle\}$
and the generalized matrix
$\mathcal{F}_g^6 = \langle \mathbb{S}_6, Fi(\mathbb{S}_6)\rangle$

Then,

Theorem

The logics $\mathbb{L}_{\mathcal{F}^6}$, $\mathbb{L}_{\mathcal{F}^6_g}$ and $\mathbb{L}_{\mathbf{S}}^{\leq}$ coincide.

On the other hand,

Lemma

Let $h : \mathfrak{Fm} \to \mathbb{S}_6$, $\mathcal{V}' \subseteq Var$ and $h' : \mathfrak{Fm} \to \mathbb{S}_6$ such that, for every $p \in \mathcal{V}'$,

$$h'(p) = \begin{cases} N & \text{if } h(p) = B \\ B & \text{if } h(p) = N \\ h(p) & \text{otherwise} \end{cases}. \text{ Then } h'(\alpha) = \begin{cases} N & \text{if } h(\alpha) = B \\ B & \text{if } h(\alpha) = N \\ h(\alpha) & \text{otherwise} \end{cases}$$

for all $\alpha \in \mathfrak{Fm}$ such that $Var(\alpha) \subseteq \mathcal{V}'$.

Lemma

The matrices $\langle \mathbb{S}_6, [N) \rangle$ and $\langle \mathbb{S}_6, [B) \rangle$ are isomorphic.

And then

Theorem

The logic that preserves degrees of truth associated to involutive Stone algebras, $\mathbb{L}_{\mathbf{S}}^{\leq}$, is a six–valued logic determined by the matrices $\langle \mathbb{S}_6, \left[\frac{1}{3}\right) \rangle$, $\langle \mathbb{S}_6, [N) \rangle$, $\langle \mathbb{S}_6, \left[\frac{2}{3}\right) \rangle$ and $\langle \mathbb{S}_6, [1) \rangle$.

In what follows, we denote by **Six** the logic $\mathbb{L}_{\mathbf{S}}^{\leq}$. We have

Lemma

Six is not functionally complete.

Proposition

Six is non–trivial and non–explosive.

Proposition

Six is paracomplete.

On the other hand, if we consider in the language of **Six** the set $\bigcirc(p) = \{\Diamond p \vee \Diamond\neg p\}$.

Proposition

Six is finitely gently explosive with respect to $\bigcirc(p)$ and $\neg$.

From this, we have

Theorem

Six is a **LFI** with respect to $\neg$ and the consistency operator $\circ$ defined by $\circ\alpha = \Diamond\alpha \vee \Diamond\neg\alpha$, for every $\alpha \in Fm$.

The operator $\circ$ propagates through the others connectives.

Lemma

The following conditions hold:

(i) $\models_{\textbf{Six}} \circ\bot$ (ii) $\circ\alpha \models_{\textbf{Six}} \circ\Box\alpha$

(iii) $\circ\alpha \models_{\textbf{Six}} \circ\neg\alpha$ (iv) $\circ\alpha, \circ\beta \models_{\textbf{Six}} \circ(\alpha\#\beta)$ for $\# \in \{\wedge, \vee\}$.

More over, $\models_{\textbf{Six}} \circ\neg^n\circ\alpha$, for all $n \geq 0$. In particular, $\models_{\textbf{Six}} \circ\circ\alpha$. In this way, **Six** validates al the axioms $(cc)_n$ of the logic **mCi**. Como es usual en el marco de las **LFI**s, es posible definir un operador de inconsistencia $\bullet$ sobre **Six** del siguiente modo:

$$\bullet\alpha =_{def} \neg \circ \alpha.$$

Lemma

It holds:

(i) $\alpha \wedge \neg\alpha \models_{\mathbf{Six}} \bullet\alpha$ but $\bullet\alpha \not\models_{\mathbf{Six}} \alpha \wedge \neg\alpha$,

(ii) $\bullet\alpha \models_{\mathbf{Six}} \bullet\neg\alpha$ and $\bullet\neg\alpha \models_{\mathbf{Six}} \bullet\alpha$,

(iii) $\bullet(\alpha\#\beta) \models_{\mathbf{Six}} \bullet\alpha \vee \bullet\beta$ for $\# \in \{\wedge, \vee\}$; the converse doesn't hold.

As it is usual in **LFI**s, it is interesting explore the possibility of reproducing the classical logic inside **Six**.
We have the following *Derivability Adjustment Theorem* (DAT) with respect to**CPL**.

Theorem

Let $\Gamma \cup \{\alpha\}$ be a finite set of formulas in **CPL**. Then, $\Gamma \vdash_{\mathbf{CPL}} \alpha$ if and only if, $\Gamma, \circ p_1, \ldots, \circ p_n \models_{\mathbf{Six}} \alpha$ where $\mathrm{Var}(\Gamma \cup \{\alpha\}) = \{p_1, \ldots, p_n\}$.

The sequent calculus $\mathfrak{S}$

Axioms

(structural ax.) $\alpha \Rightarrow \alpha$

(1st modal ax.) $\alpha \Rightarrow \Box\alpha$

(2nd modal ax) $\Rightarrow \Box\alpha \vee \neg\Box\alpha$

Structurales rules

(left weak.) $\dfrac{\Gamma \Rightarrow \Delta}{\Gamma, \alpha \Rightarrow \Delta}$

(right weak.) $\dfrac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \alpha}$

(cut) $\dfrac{\Gamma \Rightarrow \Delta, \alpha \qquad \alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$

Logical rules

$$(\wedge \Rightarrow) \quad \frac{\Gamma, \alpha, \beta \Rightarrow \Delta}{\Gamma, \alpha \wedge \beta \Rightarrow \Delta} \qquad (\Rightarrow \wedge) \quad \frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \wedge \beta}$$

$$(\vee \Rightarrow) \quad \frac{\Gamma, \alpha \Rightarrow \Delta \quad \Gamma, \beta \Rightarrow \Delta}{\Gamma, \alpha \vee \beta \Rightarrow \Delta} \qquad (\Rightarrow \vee) \quad \frac{\Gamma \Rightarrow \Delta, \alpha, \beta}{\Gamma \Rightarrow \Delta, \alpha \vee \beta}$$

$$(\neg) \quad \frac{\alpha \Rightarrow \beta}{\neg\beta \Rightarrow \neg\alpha}$$

$$(\neg\neg \Rightarrow) \quad \frac{\Gamma, \alpha \Rightarrow \Delta}{\Gamma, \neg\neg\alpha \Rightarrow \Delta} \qquad (\Rightarrow \neg\neg) \quad \frac{\Gamma \Rightarrow \alpha, \Delta}{\Gamma \Rightarrow \neg\neg\alpha, \Delta}$$

$$(\Box) \quad \frac{\Gamma, \alpha \Rightarrow \Box\Delta}{\Gamma, \Box\alpha \Rightarrow \Box\Delta} \qquad (\neg\Box \Rightarrow) \quad \frac{\Gamma, \neg\Box\alpha \Rightarrow \Delta}{\Gamma, \Box\neg\Box\alpha \Rightarrow \Delta}$$

The notion of *proof* (or $\mathfrak{S}$–*proof*) in the sequent calculus $\mathfrak{S}$ is the usual.

We say that $\Gamma \Rightarrow \Delta$ is *probable* in $\mathfrak{S}$, denoted by $\mathfrak{S} \vdash \Gamma \Rightarrow \Delta$, if there exists a proof in $\mathfrak{S}$ for it. If α is a formula we write $\mathfrak{S} \vdash \alpha$ to express that $\mathfrak{S} \vdash \Rightarrow \alpha$, i.e., the sequent $\Rightarrow \alpha$ is probable in $\mathfrak{S}$.

The following are some technical results that are useful to prove soundness and completeness with respect to **S**.

Lemma

Let $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m \in Fm$. The following conditions are equivalent.

(i) $\alpha_1, \ldots, \alpha_n \Rightarrow \beta_1, \ldots, \beta_m$ is probable in $\mathfrak{S}$,

(ii) $\alpha_1 \wedge \cdots \wedge \alpha_n \Rightarrow \beta_1, \ldots, \beta_m$ is probable in $\mathfrak{S}$

(iii) $\alpha_1, \ldots, \alpha_n \Rightarrow \beta_1 \vee \cdots \vee \beta_m$ is probable in $\mathfrak{S}$

(iv) $\alpha_1 \wedge \cdots \wedge \alpha_n \Rightarrow \beta_1 \vee \cdots \vee \beta_m$ is probable in $\mathfrak{S}$.

Lemma

Let $\alpha, \beta, \gamma \in Fm$ such that $\alpha \Leftrightarrow \beta$. Then,

(i) $\alpha \vee \gamma \Leftrightarrow \beta \vee \gamma$, (iii) $\neg\alpha \Leftrightarrow \neg\beta$,
(ii) $\alpha \wedge \gamma \Leftrightarrow \beta \wedge \gamma$, (v) $\Box\alpha \Leftrightarrow \Box\beta$.

Lemma

Let $\alpha, \beta, \gamma \in Fm$. Then, $\alpha \vee (\beta \wedge \gamma) \Leftrightarrow (\alpha \vee \beta) \wedge (\alpha \vee \gamma)$

Lemma

Let $\alpha, \beta \in Fm$. The following sequents are probable in $\mathfrak{S}$.

(i) $\alpha \Rightarrow \neg\neg\alpha$, $\neg\neg\alpha \Rightarrow \alpha$,
(ii) $\alpha \vee \beta \Rightarrow \neg\neg\alpha \vee \neg\neg\beta$,
(iii) $\neg\neg\alpha \wedge \neg\neg\beta \Rightarrow \alpha \vee \beta$,
(iv) $\neg(\alpha \vee \beta) \Rightarrow \neg\alpha \wedge \neg\beta$,
(v) $\neg\alpha \vee \neg\beta \Rightarrow \neg(\alpha \wedge \beta)$,
(vi) $\neg\alpha \wedge \neg\beta \Rightarrow \neg(\alpha \vee \beta)$,
(vii) $\neg(\alpha \wedge \beta) \Rightarrow \neg\alpha \vee \neg\beta$,

Lemma

Let $\alpha, \beta \in Fm$. The following sequents are probable in $\mathfrak{S}$.

(i) $\Box(\alpha \vee \beta) \Leftrightarrow \Box\alpha \vee \Box\beta$,

(ii) $\Box(\alpha \wedge \beta) \Leftrightarrow \Box\alpha \wedge \Box\beta$,

(iii) $\Box\Box\alpha \Leftrightarrow \Box\alpha$,

(iv) $\Box\neg\Box\alpha \Leftrightarrow \neg\Box\alpha$,

Let $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m \in Fm$. We say that the sequent $\alpha_1, \ldots, \alpha_n \Rightarrow \beta_1, \ldots, \beta_m$ is valid if for every $h \in Hom(Fm, \mathbb{S}_6)$ we have

$$h(\bigwedge_{i=1}^{n} \alpha_i) \leq h(\bigvee_{j=1}^{m} \beta_j)$$

If $n = 0$, we consider $\bigwedge_{i=1}^{n} \alpha_i = \top$ and if $m = 0$, $\bigvee_{j=1}^{m} \beta_j = \bot$.

Besides, we say that the rule

$$\frac{\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_k \Rightarrow \Delta_k}{\Gamma \Rightarrow \Delta}$$

preserves validity if it holds:

$$\Gamma_i \Rightarrow \Delta_i \text{ is valid for } 1 \leq i \leq k \;\text{ implies }\; \Gamma \Rightarrow \Delta \text{ is valid.}$$

Then,

Theorem (Soundness)

Every sequent $\Gamma \Rightarrow \Delta$ probable in $\mathfrak{S}$ is valid.