A model-theoretic study of the category of **F**-structures for **mbC**

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mbC-structures

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mbC-structures

- The publication in 1963 of N. da Costa's Habilitation thesis *Inconsistent Formal Systems* constitutes a landmark in the history of paraconsistency. In that thesis, da Costa introduces the hierarchy C_n (for $n \geq 1$) of C-systems.
- The decidability of the calculi C_n was proved, for the first time, by M. Fidel in 1977 by means of a novel algebraic-relational class of structures called C_n-structures.

mbC-structures

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- The basic idea of **LFI**s, as in da Costa's *C*-systems, is that $\alpha, \neg\alpha \nvdash \beta$ in general, but $\alpha, \neg\alpha, \circ\alpha \vdash \beta$ always. The most basic **LFI** was studied by Carnielli, Coniglio and Marcos in 2007, is the logic **mbC**.

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- It is worth noting that **mbC** is obtained from a calculus for the positive cassical logic $\mathbf{CPL}^{+}$ by adding the following axioms:

$$\text{(A10)}\alpha \vee \neg\alpha$$

$$\text{(A111)} \circ \alpha \to (\alpha \to (\neg\alpha \to \beta))$$

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- A Fidel-structures for **mbC** and for several axiomatic extensions of it were presented in

W. A. Carnielli and M. E. Coniglio. Paraconsistent Logic: Consistency, contradiction and negation. Volume 40 of Logic, Epistemology, and the Unity of Science series. Springer, 2016.

mbC-structures

- In our talk, we will present our studies of the class of **F**-structures for **mbC** from the point of view of Model Theory and Category Theory. The basic point is that Fidel-structures for **mbC** (or **mbC**-structures) can be seen as first-order structures over the signature of Boolean algebras expanded by two binary predicate symbols N (for negation) and O (for the consistency connective).

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- In our talk, we will present our studies of the class of **F**-structures for **mbC** from the point of view of Model Theory and Category Theory. The basic point is that Fidel-structures for **mbC** (or **mbC**-structures) can be seen as first-order structures over the signature of Boolean algebras expanded by two binary predicate symbols N (for negation) and O (for the consistency connective).
- This perspective allows us to consider notions and results from Model Theory in order to analyze the class of **mbC**-structures. In particular, we will be interested in a Birkhoff-like representation theorem for **mbC**-structures as subdirect poducts in terms of subdirectly irreducible **mbC**-structures is obtained by adapting a general result for first-order structures due to Caicedo.

mbC-structures

- X. Caicedo, The subdirect decomposition theorem for classes of structures closed under direct limits. *Journal of the Australian Mathematical Society*, 30:171–179, 1981.

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- M. M. Fidel, The decidability of the calculi C_n. *Reports on Mathematical Logic*, 8:31–40, 1977.

mbC-structures

- Connectives: $\wedge$, $\vee$, $\rightarrow$ and $\sim$
  Predicates: N and O, and the symbol $\approx$ for the equality predicate which is always interpreted as the identity relation.
  Let us consider denumerable set $\mathcal{V}_{ind} = \{v_i \ : \ i \in \mathbb{N}\}$ of variables.

mbC-structures

Definition

An **F***-structure for* **mbC** *(***mbC***-structure) is a* Θ*-first order structure*

$$\mathcal{E} = \langle A, \sqcap^{\mathcal{E}}, \sqcup^{\mathcal{E}}, -^{\mathcal{E}}, \mathbf{0}^{\mathcal{E}}, \mathbf{1}^{\mathcal{E}}, N^{\mathcal{E}}, O^{\mathcal{E}} \rangle$$

such that:

(a) *the* Θ_{BA}*-reduct* $\mathcal{A} = \langle A, \sqcap^{\mathcal{E}}, \sqcup^{\mathcal{E}}, -^{\mathcal{E}}, \mathbf{0}^{\mathcal{E}}, \mathbf{1}^{\mathcal{E}} \rangle$ *is a Boolean algebra.*

(b) $\mathcal{E}$ *satisfies the following* Θ*-sentences:*

(i) $\forall u \exists w\, N(u, w)$,

(ii) $\forall u \exists w\, O(u, w)$,

(iii) $\forall u \forall w\, (N(u, w) \to (u \sqcup w \approx \mathbf{1}))$,

(iv) $\forall u \forall w\, (N(u, w) \to \exists z (O(u, z) \wedge ((u \sqcap w \sqcap z) \approx \mathbf{0}))$.

mbC-structures

Definition

A valuation *over an* **mbC***-structure* $\mathcal{E}$ *is a map* $v : For(\Sigma) \to A$ *satisfying the following properties, for every formulas* α *and* β*:*

(1) $v(\alpha\#\beta) = v(\alpha)\#v(\beta)$, *for* $\# \in \{\wedge, \vee, \to\}$*;*

(2) $v(\neg\alpha) \in N^{\mathcal{E}}_{v(\alpha)}$ *(that is,* $N^{\mathcal{E}}(v(\alpha), v(\neg\alpha))$*);*

(3) $v(\circ\alpha) \in O^{\mathcal{E}}_{v(\alpha)}$ *(that is,* $O^{\mathcal{E}}(v(\alpha), v(\circ\alpha))$*) is such that* $v(\alpha) \sqcap^{\mathcal{E}} v(\neg\alpha) \sqcap^{\mathcal{E}} v(\circ\alpha) = \mathbf{0}^{\mathcal{E}}$.

mbC-structures

Definition

Let $\Gamma \cup \{\alpha\} \subseteq For(\Sigma)$ *be a finite set of formulas.*
(i) Given a Fidel-structure $\mathcal{E}$ *for* **mbC***, we say that* α *is a semantical consequence of* Γ *(w.r.t.* $\mathcal{E}$*), denoted by* $\Gamma \Vdash_{\mathcal{E}}^{\mathbf{mbC}} \alpha$*, if, for every valuation* v *over* $\mathcal{E}$*:* $v(\alpha) = 1$ *whenever* $v(\gamma) = 1$ *for every* $\gamma \in \Gamma$*.*
(ii) We say that α *is a semantical consequence of* Γ *(w.r.t. Fidel-structures for* **mbC***), denoted by* $\Gamma \Vdash_{\mathbf{F}}^{\mathbf{mbC}} \alpha$*, if* $\Gamma \Vdash_{\mathcal{E}}^{\mathbf{mbC}} \alpha$ *for every* **F***-structure* $\mathcal{E}$ *for* **mbC***.*

mbC-structures

Theorem (Carnielli-Coniglio, 2016)

Let $\Gamma \cup \{\alpha\}$ *be a finite set of formulas in For*(Σ). *Then:*
$\Gamma \vdash_{\mathbf{mbC}} \alpha$ *iff* $\Gamma \Vdash^{\mathbf{mbC}}_{\mathbf{F}} \alpha$.

Homomorphisms and substructures

Definition

Let $\mathcal{E} = \langle \mathcal{A}, N^{\mathcal{E}}, O^{\mathcal{E}} \rangle$ and $\mathcal{E}' = \langle \mathcal{A}', N^{\mathcal{E}'}, O^{\mathcal{E}'} \rangle$ be two **mbC**-*structures. An* **mbC**-homomorphism *h from $\mathcal{E}$ to $\mathcal{E}'$ is a homomorphism $h : \mathcal{E} \to \mathcal{E}'$ in the category of Θ-structures.*

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Remark

By definition, an **mbC***-homomorphism $h : \mathcal{E} \to \mathcal{E}'$ is a function $h : A \to A'$ satisfying the following conditions, for every $a, b \in A$:*

(i) *$h : \mathcal{A} \to \mathcal{A}'$ is a homomorphism between Boolean algebras,*

(ii) *if $N^{\mathcal{E}}(a, b)$ then $N^{\mathcal{E}'}(h(a), h(b))$;*

(iii) *if $O^{\mathcal{E}}(a, b)$ then $O^{\mathcal{E}'}(h(a), h(b))$.*

Homomorphisms and substructures

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Definition

Let $\mathcal{E} = \langle \mathcal{A}, N^{\mathcal{E}}, O^{\mathcal{E}} \rangle$ and $\mathcal{E}' = \langle \mathcal{A}', N^{\mathcal{E}'}, O^{\mathcal{E}'} \rangle$ be two **mbC***-structures. The structure $\mathcal{E}$ is said to be a* substructure *of $\mathcal{E}'$, denoted by $\mathcal{E} \subseteq \mathcal{E}'$, if the following conditions hold:*

(i) *$\mathcal{A}$ is a Boolean subalgebra of $\mathcal{A}'$ (which will be denoted as $\mathcal{A} \subseteq \mathcal{A}'$),*

(ii) *$N^{\mathcal{E}} = N^{\mathcal{E}'} \cap (A')^2$ and $O^{\mathcal{E}} = O^{\mathcal{E}'} \cap (A')^2$.*

Homomorphisms and substructures

Proposition

The monomorphisms in **FmbC** *are precisely the* embeddings, *that is, the homomorphisms h which are injective mappings where (ii) and (iii) of Remark 6 are replaced by*

(ii)' $N^{\mathcal{E}}(a, b)$ *if and only if* $N^{\mathcal{E}'}(h(a), h(b))$;

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Proposition

(i) A homomorphism $h : \mathcal{E} \to \mathcal{E}'$ *is an epimorphism in* **FmbC** *if and only if h is onto as a mapping.*

(ii) A homomorphism $h : \mathcal{E} \to \mathcal{E}'$ *is an isomorphism in* **FmbC** *if and only if h is an embedding which is onto, that is, h is a bijective embedding.*

Congruences and quotient structures

Definition

Let θ be a relation on an **mbC***-structure $\mathcal{E} = \langle \mathcal{A}, N^{\mathcal{E}}, O^{\mathcal{E}} \rangle$. Then θ is said to be an* **mbC***-congruence over $\mathcal{E}$ if the following conditions hold:*

(i) *θ is a Boolean congruence over $\mathcal{A}$;*

(ii) *if $(x, x'), (y, y') \in \theta$ and $N^{\mathcal{E}}(x, y)$ then $N^{\mathcal{E}}(x', y')$;*

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- Let $\mathcal{A}$ be a Boolean algebra, and let $F \subseteq A$. Then F is a *filter over* $\mathcal{A}$ if the following holds: (i)$\mathbf{1}^{\mathcal{A}} \in F$; (ii) if $x, y \in F$ then $x \sqcap y \in F$; and (iii) if $x \in F$ and $x \leq y$ then $y \in F$. We denote by $F(\mathcal{A})$ the set of filters over $\mathcal{A}$.

Congruences and quotient structures

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Given an **mbC**-*structure* $\mathcal{E} = \langle \mathcal{A}, N^{\mathcal{E}}, O^{\mathcal{E}} \rangle$, *a set* $F \subseteq A$, *is said to be an* **mbC**-filter *if the following conditions hold:*

(i) *F is a filter over the Boolean algebra $\mathcal{A}$;*

(ii) *$R(F)$ verifies conditions (ii) and (iii) of Definition above, where* $R(F) = \{(x, y) \in A^2 \ : \ x \sqcap z = y \sqcap z \text{ for some } z \in F\}$.

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Theorem

Let $\mathcal{E} = \langle \mathcal{A}, N^{\mathcal{E}}, O^{\mathcal{E}} \rangle$ *be an* **mbC***-structure. Then, there exists a lattice isomorphism between* $F_{\mathbf{mbC}}(\mathcal{E})$ *and* $Con_{\mathbf{mbC}}(\mathcal{E})$.

Congruences and quotient structures

- Now, we are going to define quotient **mbC**-structures. Let $\mathcal{E}$ be an **mbC**-structure, and let θ be an **mbC**-congruence on it. Consider the following relations over A/θ induced from $\mathcal{E}$:

$$N^{\mathcal{E}/\theta} \stackrel{\text{def}}{=} \{([x]_\theta, [y]_\theta) \in A/\theta \times A/\theta \ : \ (x, y) \in N^{\mathcal{E}}\}$$

and

$$O^{\mathcal{E}/\theta} \stackrel{\text{def}}{=} \{([x]_\theta, [y]_\theta) \in A/\theta \times A/\theta \ : \ (x, y) \in O^{\mathcal{E}}\}.$$

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- It follows that $(x, y) \in N^{\mathcal{E}}$ if and only if $([x]_\theta, [y]_\theta) \in N^{\mathcal{E}/\theta}$; the same holds for the predicate O.

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- It follows that $(x, y) \in N^{\mathcal{E}}$ if and only if $([x]_\theta, [y]_\theta) \in N^{\mathcal{E}/\theta}$; the same holds for the predicate O.
- It is easy to check that $\mathcal{E}/\theta = \langle \mathcal{A}/\theta, N^{\mathcal{E}/\theta}, O^{\mathcal{E}/\theta} \rangle$ is an **mbC**-structure.

Congruences and quotient structures

- The canonical projection $q : \mathcal{E} \to \mathcal{E}/\theta$ is an **mbC**-homomorphism wich is an epimorphism in **FmbC** .

Congruences and quotient structures

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- It is well-known that given a Boolean homomorphism $h : \mathcal{A} \rightarrow \mathcal{A}'$, the relation $Ker(h) = \{(x, y) \in A \times A \ : \ h(x) = h(y)\}$ is a Boolean congruence.

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Definition

Let $h : \mathcal{E} \to \mathcal{E}'$ be an **mbC**-*homomorphism. Then h is said to be* congruential *if it satisfies the following:*

$$\textit{If } h(x) = h(x'),\ h(y) = h(y') \textit{ and } N^{\mathcal{E}}(x, y) \textit{ then } N^{\mathcal{E}}(x', y'),$$

$$\textit{If } h(x) = h(x'),\ h(y) = h(y') \textit{ and } O^{\mathcal{E}}(x, y) \textit{ then } O^{\mathcal{E}}(x', y').$$

Congruences and quotient structures

Theorem (First Isomorphism theorem)

Let $h : \mathcal{E} \to \mathcal{E}'$ be an **mbC***-homomorphism which is congruential. Then, there is a unique* **mbC***-monomomorphism $\overline{h} : \mathcal{E}\,/\,Ker(h) \to \mathcal{E}'$ such that $\overline{h} \circ q = h$. In particular, if h is surjective then $\overline{h}$ is an isomorphism between $\mathcal{E}\,/\,Ker(h)$ and $\mathcal{E}'$.*

A Birkhoff representation theorem for **mbC**-structures

Definition

Let $\mathcal{E}_i = \langle \mathcal{A}_i, N^{\mathcal{E}_i}, O^{\mathcal{E}_i} \rangle$ *(for* $i \in I$*) be an* **mbC***-structure. The* direct product *of the family* $\{\mathcal{E}_i\}_{i \in I}$ *is the structure* $\prod_{i \in I} \mathcal{E}_i = \langle \prod_{i \in I} \mathcal{A}_i, N^{\prod_{i \in I} \mathcal{E}_i}, O^{\prod_{i \in I} \mathcal{E}_i} \rangle$ *defined as follows:*

(i) $\prod_{i \in I} \mathcal{A}_i$ *is the standard product of the family* $\{\mathcal{A}_i\}_{i \in I}$ *of Boolean algebras;*

(ii) $(x, y) \in N^{\prod_{i \in I} \mathcal{E}_i}$ *if and only if* $(x(i), y(i)) \in N^{\mathcal{E}_i}$ *for every* $i \in I$*;*

(iii) $(x, y) \in O^{\prod_{i \in I} \mathcal{E}_i}$ *if and only if* $(x(i), y(i)) \in O^{\mathcal{E}_i}$ *for every* $i \in I$*.*

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(iii) $(x, y) \in O^{\prod_{i \in I} \mathcal{E}_i}$ *if and only if* $(x(i), y(i)) \in O^{\mathcal{E}_i}$ *for every* $i \in I$*.*

- Observe that the canonical i-projection $\pi_i : \prod_{i \in I} \mathcal{E}_i \to \mathcal{E}_i$, $\pi_i(x) = x(i)$ is an **mbC**-homomorphism which is onto.

A Birkhoff representation theorem for **mbC**-structures

- The product has the following universal property.: for any family $\{f_i : \mathcal{E} \to \mathcal{E}_i\}_{i \in I}$ of **mbC**-homomorphisms, there exists a unique **mbC**-homomorphism $g : \mathcal{E} \to \prod_{i \in I} \mathcal{E}_i$ such that $f_i = \pi_i \circ g$ for every $i \in I$. Clearly, $g(a)(i) = f_i(a)$ for every $a \in |\mathcal{E}|$ and $i \in I$.

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Theorem

The category **FmbC** *has arbitrary products.*

A Birkhoff representation theorem for **mbC**-structures

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Theorem

The category **FmbC** *has arbitrary products.*

- Observe that the product of the empty family of **mbC**-structures is the terminal object $\mathbf{1}_\perp = \langle \mathbb{A}_\perp, N^{\mathbf{1}_\perp}, O^{\mathbf{1}_\perp} \rangle$ given by the one-element Boolean algebra $\mathbb{A}_\perp$ with domain $A_\perp = \{*\}$, and where $N^{\mathbf{1}_\perp} = O^{\mathbf{1}_\perp} = \{(*, *)\}$. Note that $\mathbf{0}^{\mathbb{A}_\perp} = \mathbf{1}^{\mathbb{A}_\perp} = *$.

A Birkhoff representation theorem for **mbC**-structures

Definition

For $i \in I$ let $\mathcal{E}_i = \langle \mathcal{A}_i, N^{\mathcal{E}_i}, O^{\mathcal{E}_i} \rangle$ be an **mbC***-structure. A* subdirect product *of the family $\{\mathcal{E}_i\}_{i \in I}$ is a monomorphism $h : \mathcal{E} \to \prod_{i \in I} \mathcal{E}_i$ (for some* **mbC***-structure $\mathcal{E}$) such that $\pi_i \circ h$ is onto for every $i \in I$. It is also called a* subdirect decomposition *of $\mathcal{E}$.*

Definition

An **mbC***-structure $\mathcal{E}$ is said to be* subdirectly irreducible *(s.i.) in* **FmbC** *if for every subdirect descomposition $h : \mathcal{E} \to \prod_{i \in I} \mathcal{E}_i$ in* **FmbC** *with $I \neq \emptyset$, there is an $i \in I$ such that $\pi_i \circ h$ is an isomorphism in* **FmbC**.

A Birkhoff representation theorem for **mbC**-structures

Theorem

Let $\mathcal{E} = \langle \mathcal{A}, N^{\mathcal{E}}, O^{\mathcal{E}} \rangle$ be an **mbC**-*structure such that $\mathcal{E} \neq \mathbf{1}_{\perp}$. Then, $\mathcal{E}$ is subdirectly irreducible in* **FmbC** *if and only if there exists a predicate $P \in \{N, O, \approx\}$ and $(x, y) \in A^2$ such that $(x, y) \notin P^{\mathcal{E}}$, and for every onto homomorphism $h : \mathcal{E} \to \mathcal{E}'$ in* **FmbC** *which is not an isomorphism, $(h(x), h(y)) \in P^{\mathcal{E}'}$.*

A Birkhoff representation theorem for **mbC**-structures

Remark

It follows from the previous result that an **mbC***-structure* $\mathcal{E} = \langle \mathcal{A}, N^{\mathcal{E}}, O^{\mathcal{E}} \rangle \neq \mathbf{1}_{\perp}$ *is not subdirectly irreducible in* **FmbC** *if only if the following conditions hold:*

1. *For every* $(x, y) \in A^2$, $(x, y) \notin N^{\mathcal{E}}$ *implies that there exists an onto homomorphism* $h : \mathcal{E} \to \mathcal{E}'$ *in* **FmbC** *which is not an isomorphism, such that* $(h(x), h(y)) \notin N^{\mathcal{E}'}$ *(hence* $\mathcal{E}' \neq \mathbf{1}_{\perp}$*);*
2. *For every* $(x, y) \in A^2$, $(x, y) \notin O^{\mathcal{E}}$ *implies that there exists an onto homomorphism* $h : \mathcal{E} \to \mathcal{E}'$ *in* **FmbC** *which is not an isomorphism, such that* $(h(x), h(y)) \notin O^{\mathcal{E}'}$ *(hence* $\mathcal{E}' \neq \mathbf{1}_{\perp}$*);*
3. *For every* $(x, y) \in A^2$, $x \neq y$ *implies that there exists an onto homomorphism* $h : \mathcal{E} \to \mathcal{E}'$ *in* **FmbC** *which is not an isomorphism, such that* $h(x) \neq h(y)$ *(hence* $\mathcal{E}' \neq \mathbf{1}_{\perp}$*).*

A Birkhoff representation theorem for **mbC**-structures

Proposition

(i) *The terminal* **mbC***-structure* $\mathbf{1}_{\perp} = \langle \mathbb{A}_{\perp}, N^{\mathbf{1}_{\perp}}, O^{\mathbf{1}_{\perp}} \rangle$ *is subdirectly irreducible in* **FmbC**.

(ii) *Each* **mbC***-structure defined over the two-element Boolean algebra* $\mathbb{A}_2$ *is subdirectly irreducible in* **FmbC**.

A Birkhoff representation theorem for **mbC**-structures

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(ii) *Each* **mbC***-structure defined over the two-element Boolean algebra* $\mathbb{A}_2$ *is subdirectly irreducible in* **FmbC***.*

- We are going to consider the boolean algebra with four element $FOUR = \{0, a, b, 1\}$. It is well know that if we have onto homomorphim form $FOUR$ to A' which is not isomorphism, them A' have to be the terminal object or two-element booalean algebra.

A Birkhoff representation theorem for **mbC**-structures

- Let us begin by analyzing the predicate N. Let $X = \{(x, y) \in FOUR^2 \ : \ x \sqcup y = 1\}$.

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- Observe that $N^{\mathcal{E}} \subseteq X$ for every **mbC**-structure $\mathcal{E} = \langle \mathbb{A}_4, N^{\mathcal{E}}, O^{\mathcal{E}} \rangle$ defined over the four-element Boolean algebra $\mathbb{A}_4$. Moreover, $(0, 1) \in N^{\mathcal{E}}$ for every $\mathcal{E}$. Taking this into account, the set of relevant points for the predicate N in order to apply Caicedo's Theorem is $X' = X \setminus \{(0, 1)\} = \{(1, 0), (1, 1), (\mathrm{a}, \mathrm{b}), (\mathrm{b}, \mathrm{a}), (\mathrm{a}, 1), (\mathrm{b}, 1), (1, \mathrm{a}), (1, \mathrm{b})\}$.

A Birkhoff representation theorem for **mbC**-structures

- Let us begin by analyzing the predicate N. Let $X = \{(x, y) \in FOUR^2 \ : \ x \sqcup y = 1\}$.
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- We denote by $(x_1, y_1)|_N(x_2, y_2)$ iff "either $(x_1, y_1) \in N^{\mathcal{E}}$ or $(x_2, y_2) \in N^{\mathcal{E}}$".

A Birkhoff representation theorem for **mbC**-structures

- We denote by $((x_1, y_1)|_N(x_2, y_2))\&((x'_1, y'_1)|_N(x'_2, y'_2))$ iff the condition " both $(x_1, y_1)|_N(x_2, y_2)$ and $(x'_1, y'_1)|_N(x'_2, y'_2)$ hold".
- Let us consider the following names for conditions $C_N(x, y)$ for each $(x, y) \in X'$, by using the notation introduced in items (1) and (2) above:
  1. $C_N(1, 0) \stackrel{\text{def}}{=} ((\mathrm{a}, \mathrm{b})|_N(1, \mathrm{b}))\&((\mathrm{b}, \mathrm{a})|_N(1, \mathrm{a}))$;
  2. $C_N(1, 1) \stackrel{\text{def}}{=} ((1, \mathrm{a})|_N(\mathrm{a}, 1))\&((1, \mathrm{b})|_N(\mathrm{b}, 1))$;
  3. $C_N(\mathrm{a}, \mathrm{b}) \stackrel{\text{def}}{=} (1, 0)|_N(1, \mathrm{b})$;
  4. $C_N(\mathrm{b}, \mathrm{a}) \stackrel{\text{def}}{=} (1, 0)|_N(1, \mathrm{a})$;
  5. $C_N(\mathrm{a}, 1) \stackrel{\text{def}}{=} (1, 1)|_N(1, \mathrm{a})$;
  6. $C_N(\mathrm{b}, 1) \stackrel{\text{def}}{=} (1, 1)|_N(1, \mathrm{b})$;
  7. $C_N(1, \mathrm{a}) \stackrel{\text{def}}{=} ((1, 1)|_N(\mathrm{a}, 1))\&((1, 0)|_N(\mathrm{b}, \mathrm{a}))$;
  8. $C_N(1, \mathrm{b}) \stackrel{\text{def}}{=} ((1, 0)|_N(\mathrm{a}, \mathrm{b}))\&((1, 1)|_N(\mathrm{b}, 1))$.

A Birkhoff representation theorem for **mbC**-structures

Proposition

Let $\mathcal{E} = \langle \mathbb{A}_4, N^{\mathcal{E}}, O^{\mathcal{E}} \rangle$ be an **mbC***-structure defined over the four-element Boolean algebra $\mathbb{A}_4$. Let $(x, y) \in X'$ such that $(x, y) \notin N^{\mathcal{E}}$.*

(1) If $C_N(x, y)$ holds then $\mathcal{E}$ is subdirectly irreducible in **FmbC***.*

(2) If $C_N(x, y)$ does not hold then there exists an onto homomorphism $h : \mathcal{E} \to \mathcal{E}'$ which is not an isomorphism in **FmbC***, such that $(h(x), h(y)) \notin N^{\mathcal{E}'}$.*

A Birkhoff representation theorem for **mbC**-structures

- Let $X'' = FOUR^2 \setminus X = \{(0, a), (a, 0), (0, b), (b, 0), (a, a), (b, b), (0, 0)\}$. Going through a similar analysis, it is easy to prove the following:

Proposition

*Let $\mathcal{E} = \langle \mathbb{A}_4, N^{\mathcal{E}}, O^{\mathcal{E}} \rangle$ be an **mbC**-structure defined over the four-element Boolean algebra $\mathbb{A}_4$. Let $(x, y) \in X''$ (hence $(x, y) \notin N^{\mathcal{E}}$). Then, there exists an onto homomorphism $h : \mathcal{E} \to \mathcal{E}'$ which is not an isomorphism in **FmbC**, such that $(h(x), h(y)) \notin N^{\mathcal{E}'}$.*

A Birkhoff representation theorem for **mbC**-structures

- We denote by $(x_1, y_1)|_O(x_2, y_2)|_O(x_3, y_3)$ iff "either $(x_1, y_1) \in O^{\mathcal{E}}$ or $(x_2, y_2) \in O^{\mathcal{E}}$ or $(x_3, y_3) \in O^{\mathcal{E}}$".
- We denote by

$$((x_1, y_1)|_O(x_2, y_2)|_O(x_3, y_3))\&((x_1', y_1')|_O(x_2', y_2')|_O(x_3', y_3')).$$

iff "both

$$(x_1, y_1)|_O(x_2, y_2)|_O(x_3, y_3)$$

and

$$(x_1', y_1')|_O(x_2', y_2')|_O(x_3', y_3')$$

hold".

A Birkhoff representation theorem for **mbC**-structures

- Let us consider the following names for conditions $C_O(x,y)$ for each $(x,y) \in FOUR^2$, by using the notation introduced in items (1) and (2) above:

1. $$C_O(1,0) \stackrel{\text{def}}{=} ((\mathrm{a},\mathrm{b})|_O(1,\mathrm{b})|_O(\mathrm{a},0))\&((\mathrm{b},\mathrm{a})|_O(1,\mathrm{a})|_O(\mathrm{b},0));$$
2. $$C_O(0,1) \stackrel{\text{def}}{=} ((\mathrm{b},\mathrm{a})|_O(\mathrm{b},1)|_O(0,\mathrm{a}))\&((\mathrm{a},\mathrm{b})|_O(\mathrm{a},1)|_O(0,\mathrm{b}));$$
3. $$C_O(1,1) \stackrel{\text{def}}{=} ((1,\mathrm{a})|_O(\mathrm{a},1)|_O(\mathrm{a},\mathrm{a}))\&((1,\mathrm{b})|_O(\mathrm{b},1)|_O(\mathrm{b},\mathrm{b}));$$
4. $$C_O(0,0) \stackrel{\text{def}}{=} ((\mathrm{b},\mathrm{b})|_O(0,\mathrm{b})|_O(\mathrm{b},0))\&((\mathrm{a},\mathrm{a})|_O(0,\mathrm{a})|_O(\mathrm{a},0));$$
5. $$C_O(\mathrm{a},\mathrm{b}) \stackrel{\text{def}}{=} ((1,0)|_O(\mathrm{a},0)|_O(1,\mathrm{b}))\&((0,1)|_O(0,\mathrm{b})|_O(\mathrm{a},1));$$
6. $$C_O(\mathrm{b},\mathrm{a}) \stackrel{\text{def}}{=} (0,1)|_O(0,a)|_O(\mathrm{b},1))\&((1,0)|_O(\mathrm{b},0)|_O(1,\mathrm{a}));$$
7. $$C_O(\mathrm{a},1) \stackrel{\text{def}}{=} ((1,1)|_O(1,\mathrm{a})|_O(\mathrm{a},\mathrm{a}))\&((0,1)|_O(\mathrm{a},\mathrm{b})|_O(0,\mathrm{b}));$$

A Birkhoff representation theorem for **mbC**-structures

1. $C_O(b,1) \stackrel{\text{def}}{=} ((0,1)|_O(b,a)|_O(0,a))\&((1,1)|_O(b,b)|_O(b,1));$
2. $C_O(1,a) \stackrel{\text{def}}{=} ((1,1)|_O(a,1)|_O(a,a))\&((1,0)|_O(b,a)|_O(b,0));$
3. $C_O(1,b) \stackrel{\text{def}}{=} ((1,0)|_O(a,b)|_O(a,0))\&((1,1)|_O(b,b)|_O(b,1));$
4. $C_O(0,a) \stackrel{\text{def}}{=} ((0,1)|_O(b,a)|_O(b,1))\&((0,0)|_O(a,a)|_O(a,0));$
5. $C_O(0,b) \stackrel{\text{def}}{=} ((0,0)|_O(b,b)|_O(b,0))\&((0,1)|_O(a,b)|_O(a,1));$
6. $C_O(a,0) \stackrel{\text{def}}{=} ((1,0)|_O(a,b)|_O(1,b))\&((0,0)|_O(a,a)|_O(0,a));$
7. $C_O(b,0) \stackrel{\text{def}}{=} ((0,0)|_O(b,b)|_O(0,b))\&((1,0)|_O(b,a)|_O(1,a));$
8. $C_O(a,a) \stackrel{\text{def}}{=} ((1,1)|_O(1,a)|_O(a,1))\&((0,0)|_O(0,a)|_O(a,0));$
9. $C_O(b,b) \stackrel{\text{def}}{=} ((0,0)|_O(b,0)|_O(0,b))\&((1,1)|_O(b,1)|_O(1,b)).$

A Birkhoff representation theorem for **mbC**-structures

Proposition

Let $\mathcal{E} = \langle \mathbb{A}_4, N^{\mathcal{E}}, O^{\mathcal{E}} \rangle$ be an **mbC***-structure defined over the four-element Boolean algebra $\mathbb{A}_4$. Let $(x, y) \in FOUR^2$ such that $(x, y) \notin O^{\mathcal{E}}$.*

(1) If $C_O(x, y)$ holds then $\mathcal{E}$ is subdirectly irreducible in **FmbC***.*

(2) If $C_O(x, y)$ does not hold then there exists an onto homomorphism $h : \mathcal{E} \to \mathcal{E}'$ which is not an isomorphism in **FmbC***, such that $(h(x), h(y)) \notin O^{\mathcal{E}'}$.*

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Proposition

Let $\mathcal{E} = \langle \mathbb{A}_4, N^{\mathcal{E}}, O^{\mathcal{E}} \rangle$ be an **mbC***-structure defined over the four-element Boolean algebra $\mathbb{A}_4$. Let $(x, y) \in FOUR$ such that $x \neq y$. Then, there exists an onto homomorphism $h : \mathcal{E} \to \mathcal{E}'$ which is not an isomorphism in* **FmbC***, such that $h(x) \neq h(y)$.*

A Birkhoff representation theorem for **mbC**-structures

Theorem

Let $\mathcal{E} = \langle \mathbb{A}_4, N^{\mathcal{E}}, O^{\mathcal{E}} \rangle$ be an **mbC***-structure defined over the four-element Boolean algebra $\mathbb{A}_4$.*
(1) Suppose that for some $(x, y) \in X'$, $(x, y) \notin N^{\mathcal{E}}$ and $C_N(x, y)$ holds. Then, $\mathcal{E}$ is subdirectly irreducible in **FmbC***.*

(2) Suppose that for every $(x, y) \in X'$, if $(x, y) \notin N^{\mathcal{E}}$ then $C_N(x, y)$ does not hold. Then: $\mathcal{E}$ is not subdirectly irreducible in **FmbC** *if and only if, for every $(x', y') \in FOUR^2$, if $(x', y') \notin O^{\mathcal{E}}$ then $C_O(x', y')$ does not hold.*

A Birkhoff representation theorem for **mbC**-structures

- A partially ordered set $(I, \leq)$ is said to be *directed* if for every $i, j \in I$ there exists $k \in I$ such that $k \geq i, j$. A *directed diagram* of **mbC**-structures over a directed set $(I, \leq)$ is a family of **mbC**-structures $\{\mathcal{E}_i\}_{i \in I}$ and a family of **mbC**-homomorphism $\{h_{ij} : \mathcal{E}_i \to \mathcal{E}_j\}_{i \leq j}$ such that $h_{ii} = id_{\mathcal{E}_i}$ and $h_{ik} = h_{jk} \circ h_{ij}$ whenever $i \leq j \leq k$.

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- Now, we are going to consider a family $\{A_i\}_{\{i \in I\}}$ of Boolean algebras. Then, the direct limit the family is defined as follow:

A Birkhoff representation theorem for **mbC**-structures

- $\underset{\rightarrow}{\mathcal{A}} = (\underset{\rightarrow}{A}, \underset{\rightarrow}{\sqcap}, \underset{\rightarrow}{\sqcup}, \underset{\rightarrow}{-}, \underset{\rightarrow}{\mathbf{0}}, \underset{\rightarrow}{\mathbf{1}})$ is the Boolean algebra defined as follows: $\underset{\rightarrow}{A} = \left(\bigsqcup_{i \in I} A_i\right) / \sim$ is the disjoint union of the A_i's divided by the following equivalence relation: $(a, i) \sim (b, j)$ if and only if there exists $k \geq i, j$ such that $h_{ik}(a) = h_{jk}(b)$. If $[(a, i)]$ denotes the equivalence class of $(a, i) \in \bigsqcup_{i \in I} A_i$, then the operations in $\underset{\rightarrow}{\mathcal{A}}$ are defined as follows: $[(a, i)] \underset{\rightarrow}{\#} [(b, j)] \overset{\text{def}}{=} [h_{ik}(a) \# h_{jk}(b)]$, where $\# \in \{\sqcap, \sqcup\}$ and $k \geq i, j$; $\underset{\rightarrow}{-}[(a, i)] \overset{\text{def}}{=} [(-a, i)]$; $\underset{\rightarrow}{\mathbf{0}} \overset{\text{def}}{=} [(\mathbf{0}^{\mathcal{E}_i}, i)]$ for any $i \in I$; and $\underset{\rightarrow}{\mathbf{1}} \overset{\text{def}}{=} [(\mathbf{1}^{\mathcal{E}_i}, i)]$ for any $i \in I$.

A Birkhoff representation theorem for **mbC**-structures

Definition

Let $D = (\{\mathcal{E}_i\}_{i \in I}, \{h_{ij} : i \leq j\})$ *be a directed diagram in* **FmbC**. *The* direct limit *of* D *is the* Θ*-structure* $\lim\limits_{\rightarrow} D = \langle \underset{\rightarrow}{\mathcal{A}}, \underset{\rightarrow}{N}, \underset{\rightarrow}{O} \rangle$ *where*

(i) $\underset{\rightarrow}{\mathcal{A}}$ *direct limit of family of boolean algebras* $\{A_i\}_{i \in I}$,

(ii) *For every* $[(a, i)], [(b, j)] \in \underset{\rightarrow}{A}$ *we have:*

(a) $([(a, i)], [(b, j)]) \in \underset{\rightarrow}{N}$ *if and only if there is* $k \geq i, j$ *such that* $(h_{ik}(a), h_{jk}(b)) \in N^{\mathcal{E}_k}$.

(b) $([(a, i)], [(b, j)]) \in \underset{\rightarrow}{O}$ *if and only if there is* $k \geq i, j$ *such that* $(h_{ik}(a), h_{jk}(b)) \in O^{\mathcal{E}_k}$.

A Birkhoff representation theorem for **mbC**-structures

Theorem

The category **FmbC** *is closed under direct limits.*

A Birkhoff representation theorem for **mbC**-structures

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Corollary

Any non-trivial **mbC**-*structure* $\mathcal{E} = \langle \mathcal{A}, N^{\mathcal{E}}, O^{\mathcal{E}} \rangle$ *is a subdirect product of at most* $\aleph_0 + ||A||$ *non-trivial subdirectly irreducible structures, where* $||A||$ *denotes the cardinal number of the domain* A *of the Boolean algebra* $\mathcal{A}$.

Fidel's decomposition theorem for **mbC**-structures

- Fidel showed that every C_n-structure is weakly isomorphic to a weak substructure of a product of a special C_n-structure defined over the two-element Boolean algebra $\mathbb{A}_2$. By a weak isomorphism we mean a homomorphism which is bijective as a mapping.

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Definition

Let $\mathcal{E} = \langle \mathcal{A}, N^{\mathcal{E}}, O^{\mathcal{E}} \rangle$ and $\mathcal{E}' = \langle \mathcal{A}', N^{\mathcal{E}'}, O^{\mathcal{E}'} \rangle$ be two **mbC***-structures. We say that $\mathcal{E}$ is a* weak substructure *of $\mathcal{E}'$, denoted by $\mathcal{E} \subseteq_W \mathcal{E}'$, if $\mathcal{A}$ is a Boolean subalgebra of $\mathcal{A}'$, $N^{\mathcal{E}} \subseteq N^{\mathcal{E}'}$, and $O^{\mathcal{E}} \subseteq O^{\mathcal{E}'}$. This is equivalent to say that the inclusion map $i : A \to A'$ defines an injective homomorphism $i : \mathcal{E} \to \mathcal{E}'$.*

Fidel's decomposition theorem for **mbC**-structures

Definition

Let $h : \mathcal{E} \to \mathcal{E}'$ *be an* **mbC***-homomorphism. Then h is said to be a* weak isomorphism *if h is a bijective mapping.*

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Definition

Let $h : \mathcal{E} \to \mathcal{E}'$ be an **mbC***-homomorphism. Then h is said to be a* weak isomorphism *if h is a bijective mapping.*

Theorem (Weak subdirect decomposition theorem for mbC-structures)

Let $\mathcal{E}$ be an **mbC***-structure. Then, there exists a set I such that $\mathcal{E}$ is weakly isomorphic to a weak substructure of $\prod_{i \in I} \mathcal{E}_i$, where each $\mathcal{E}_i$ is defined over $\mathbb{A}_2$ for every $i \in I$.*

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Definition

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Theorem (Weak subdirect decomposition theorem for mbC-structures)

Let $\mathcal{E}$ be an **mbC***-structure. Then, there exists a set I such that $\mathcal{E}$ is weakly isomorphic to a weak substructure of $\prod_{i \in I} \mathcal{E}_i$, where each $\mathcal{E}_i$ is defined over $\mathbb{A}_2$ for every $i \in I$.*

- It is interesting to compare this result with a related one obtained by Caicedo for first-order structures. So, we can conclude that the **mbC**-structures over the 2-element Boolean algebra plays the role of subdirectly irreducible structures.