Epistemic BL-Algebras

An Algebraic Semantics for BL-Possibilistic Logic

Penélope Cordero

IMAL (CONICET - UNL)

XIV Congreso Dr. Antonio Monteiro, Bahía Blanca

Work in progress joint with M. Busaniche and R.O. Rodríguez

Monadic BL-Algebras

Definition (Castaño et al. 2016)

An algebra $\mathbf{A} = \langle A, \vee, \wedge, *, \to, \exists, \forall, 0, 1\rangle$ is called a *monadic BL-algebra* if the following are satisfied:

(M0) $\langle A, \vee, \wedge, *, \Rightarrow, 0, 1\rangle$ is a BL-algebra;

$$
\begin{array}{lrcl}
\text{(M1)} & \forall a \to a & = & 1 \\
\text{(M2)} & \forall(a \to \forall b) & = & \exists a \to \forall b \\
\text{(M3)} & \forall(\forall a \to b) & = & \forall a \to \forall b \\
\text{(M4)} & \forall(\exists a \vee b) & = & \exists a \vee \forall b \\
\text{(M5)} & \exists(a * a) & = & \exists a * \exists a
\end{array}
$$

Epistemic BL-Algebras

Definition
An algebra $\mathbf{A} = \langle A, \vee, \wedge, *, \to, \forall, \exists, 0, 1\rangle$ of type $(2, 2, 2, 2, 1, 1, 0, 0)$ is called a *Epistemic* BL-*algebra* (an EBL-algebra for short) if $\langle A, \vee, \wedge, *, \to, 0, 1\rangle$ is a BL-algebra that also satisfies:

Epistemic BL-Algebras

Definition
An algebra $\mathbf{A} = \langle A, \vee, \wedge, *, \rightarrow, \forall, \exists, 0, 1\rangle$ of type $(2, 2, 2, 2, 1, 1, 0, 0)$ is called a *Epistemic BL-algebra* (an EBL-algebra for short) if $\langle A, \vee, \wedge, *, \rightarrow, 0, 1\rangle$ is a BL-algebra that also satisfies:

$$
\begin{array}{llcl}
(\mathrm{E}\forall) & \forall 1 & = & 1 \\
(\mathrm{E}\exists) & \exists 0 & = & 0 \\
(\mathrm{E1}) & \forall a \rightarrow \exists a & = & 1 \\
(\mathrm{E2}) & \forall (a \rightarrow \forall b) & = & \exists a \rightarrow \forall b \\
(\mathrm{E3}) & \forall (\forall a \rightarrow b) & = & \forall a \rightarrow \forall b \\
(\mathrm{E4}) & \exists a \rightarrow \forall \exists a & = & 1 \\
(\mathrm{E4a}) & \forall (a \wedge b) & = & \forall a \wedge \forall b \\
(\mathrm{E4b}) & \exists (a \vee b) & = & \exists a \vee \exists b \\
(\mathrm{E5}) & \exists (a * \exists b) & = & \exists a * \exists b
\end{array}
$$

Epistemic BL-Algebras

Definition

An algebra $\mathbf{A} = \langle A, \vee, \wedge, *, \to, \forall, \exists, 0, 1\rangle$ of type $(2, 2, 2, 2, 1, 1, 0, 0)$ is called a *Epistemic* BL-*algebra* (an EBL-algebra for short) if $\langle A, \vee, \wedge, *, \to, 0, 1\rangle$ is a BL-algebra that also satisfies:

(E∀)	$\forall 1$	$=$	1				
(E∃)	$\exists 0$	$=$	0				
(E1)	$\forall a \to \exists a$	$=$	1	(M1)	$\forall a \to a$	$=$	1
(E2)	$\forall(a \to \forall b)$	$=$	$\exists a \to \forall b$				
(E3)	$\forall(\forall a \to b)$	$=$	$\forall a \to \forall b$				
(E4)	$\exists a \to \forall\exists a$	$=$	1				
(E4a)	$\forall(a \wedge b)$	$=$	$\forall a \wedge \forall b$				
(E4b)	$\exists(a \vee b)$	$=$	$\exists a \vee \exists b$				
(E5)	$\exists(a * \exists b)$	$=$	$\exists a * \exists b$				

Epistemic BL-Algebras

Definition

An algebra $\mathbf{A} = \langle A, \vee, \wedge, *, \rightarrow, \forall, \exists, 0, 1\rangle$ of type $(2, 2, 2, 2, 1, 1, 0, 0)$ is called a *Epistemic* BL-*algebra* (an EBL-algebra for short) if $\langle A, \vee, \wedge, *, \rightarrow, 0, 1\rangle$ is a BL-algebra that also satisfies:

(E$\forall$)	$\forall 1$	$=$	1				
(E$\exists$)	$\exists 0$	$=$	0				
(E1)	$\forall a \rightarrow \exists a$	$=$	1	(M1)	$\forall a \rightarrow a$	$=$	1
(E2)	$\forall(a \rightarrow \forall b)$	$=$	$\exists a \rightarrow \forall b$	(M2)	$\forall(a \rightarrow \forall b)$	$=$	$\exists a \rightarrow \forall b$
(E3)	$\forall(\forall a \rightarrow b)$	$=$	$\forall a \rightarrow \forall b$				
(E4)	$\exists a \rightarrow \forall \exists a$	$=$	1				
(E4a)	$\forall(a \wedge b)$	$=$	$\forall a \wedge \forall b$				
(E4b)	$\exists(a \vee b)$	$=$	$\exists a \vee \exists b$				
(E5)	$\exists(a * \exists b)$	$=$	$\exists a * \exists b$				

Epistemic BL-Algebras

Definition
An algebra $\mathbf{A} = \langle A, \vee, \wedge, *, \to, \forall, \exists, 0, 1\rangle$ of type $(2,2,2,2,1,1,0,0)$ is called a *Epistemic* BL-*algebra* (an EBL-algebra for short) if $\langle A, \vee, \wedge, *, \to, 0, 1\rangle$ is a BL-algebra that also satisfies:

(E$\forall$)	$\forall 1$	$=$	1				
(E$\exists$)	$\exists 0$	$=$	0				
(E1)	$\forall a \to \exists a$	$=$	1	(M1)	$\forall a \to a$	$=$	1
(E2)	$\forall(a \to \forall b)$	$=$	$\exists a \to \forall b$	(M2)	$\forall(a \to \forall b)$	$=$	$\exists a \to \forall b$
(E3)	$\forall(\forall a \to b)$	$=$	$\forall a \to \forall b$	(M3)	$\forall(\forall a \to b)$	$=$	$\forall a \to \forall b$
(E4)	$\exists a \to \forall\exists a$	$=$	1				
(E4a)	$\forall(a \wedge b)$	$=$	$\forall a \wedge \forall b$				
(E4b)	$\exists(a \vee b)$	$=$	$\exists a \vee \exists b$				
(E5)	$\exists(a * \exists b)$	$=$	$\exists a * \exists b$				

Epistemic BL-Algebras

Definition

An algebra $\mathbf{A} = \langle A, \vee, \wedge, *, \to, \forall, \exists, 0, 1\rangle$ of type $(2,2,2,2,1,1,0,0)$ is called a *Epistemic* BL-*algebra* (an EBL-algebra for short) if $\langle A, \vee, \wedge, *, \to, 0, 1\rangle$ is a BL-algebra that also satisfies:

(E$\forall$)	$\forall 1$	$=$	1				
(E$\exists$)	$\exists 0$	$=$	0				
(E1)	$\forall a \to \exists a$	$=$	1	(M1)	$\forall a \to a$	$=$	1
(E2)	$\forall(a \to \forall b)$	$=$	$\exists a \to \forall b$	(M2)	$\forall(a \to \forall b)$	$=$	$\exists a \to \forall b$
(E3)	$\forall(\forall a \to b)$	$=$	$\forall a \to \forall b$	(M3)	$\forall(\forall a \to b)$	$=$	$\forall a \to \forall b$
(E4)	$\exists a \to \forall \exists a$	$=$	1				
(E4a)	$\forall(a \wedge b)$	$=$	$\forall a \wedge \forall b$				
(E4b)	$\exists(a \vee b)$	$=$	$\exists a \vee \exists b$	(M4)	$\forall(\exists a \vee b)$	$=$	$\exists a \vee \forall b$
(E5)	$\exists(a * \exists b)$	$=$	$\exists a * \exists b$				

Epistemic BL-Algebras

Definition

An algebra $\mathbf{A} = \langle A, \vee, \wedge, *, \rightarrow, \forall, \exists, 0, 1\rangle$ of type $(2,2,2,2,1,1,0,0)$ is called a *Epistemic BL-algebra* (an EBL-algebra for short) if $\langle A, \vee, \wedge, *, \rightarrow, 0, 1\rangle$ is a BL-algebra that also satisfies:

(E∀)	$\forall 1$	$=$	1				
(E∃)	$\exists 0$	$=$	0				
(E1)	$\forall a \rightarrow \exists a$	$=$	1	(M1)	$\forall a \rightarrow a$	$=$	1
(E2)	$\forall(a \rightarrow \forall b)$	$=$	$\exists a \rightarrow \forall b$	(M2)	$\forall(a \rightarrow \forall b)$	$=$	$\exists a \rightarrow \forall b$
(E3)	$\forall(\forall a \rightarrow b)$	$=$	$\forall a \rightarrow \forall b$	(M3)	$\forall(\forall a \rightarrow b)$	$=$	$\forall a \rightarrow \forall b$
(E4)	$\exists a \rightarrow \forall\exists a$	$=$	1				
(E4a)	$\forall(a \wedge b)$	$=$	$\forall a \wedge \forall b$				
(E4b)	$\exists(a \vee b)$	$=$	$\exists a \vee \exists b$	(M4)	$\forall(\exists a \vee b)$	$=$	$\exists a \vee \forall b$
(E5)	$\exists(a * \exists b)$	$=$	$\exists a * \exists b$	(M5)	$\exists(a * a)$	$=$	$\exists a * \exists a$

Epistemic BL-algebras and Monadic BL-algebras

As a consequence of the **Lemma 2.2** (Castaño et al. 2016), it can be seen that monadic BL-algebras satisfy the properties $(E\forall) - (E5)$, whereby

$\mathbb{MBL}$ is a subvariety of $\mathbb{EBL}$

Epistemic BL-algebra that is not Monadic

Theorem (Castaño et al. 2016)
If $\mathbf{A}$ is a totally ordered MMV-algebra, then $\exists a = a$ for every $a \in A$, that is, the quantifier on $\mathbf{A}$ is the identity.

Epistemic BL-algebra that is not Monadic

Theorem (Castaño et al. 2016)
If $\mathbf{A}$ is a totally ordered MMV-algebra, then $\exists a = a$ for every $a \in A$, that is, the quantifier on $\mathbf{A}$ is the identity.

Examples Consider the MV-chain $L_7 = \left\{0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1\right\}$

Epistemic BL-algebra that is not Monadic

Theorem (Castaño et al. 2016)
If $\mathbf{A}$ is a totally ordered MMV-algebra, then $\exists a = a$ for every $a \in A$, that is, the quantifier on $\mathbf{A}$ is the identity.

Examples Consider the MV-chain $L_7 = \left\{0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1\right\}$

$\forall a$	a	$\exists a$
1	1	1
$\frac{5}{6}$	$\frac{5}{6}$	$\frac{5}{6}$
$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
0	0	0

Table: Monadic BL-algebra

Epistemic BL-algebra that is not Monadic

Theorem (Castaño et al. 2016)
If $\mathbf{A}$ is a totally ordered MMV-algebra, then $\exists a = a$ for every $a \in A$, that is, the quantifier on $\mathbf{A}$ is the identity.

Examples Consider the MV-chain $L_7 = \left\{0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1\right\}$

$\forall a$	a	$\exists a$
1	1	1
$\frac{5}{6}$	$\frac{5}{6}$	$\frac{5}{6}$
$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
0	0	0

Table: Monadic BL-algebra

$\forall a$	a	$\exists a$
1	1	1
1	$\frac{5}{6}$	1
$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$
0	$\frac{1}{6}$	0
0	0	0

Table: Epistemic BL-algebra

Epistemic BL-Algebras. Properties.

Let $\mathbf{A} \in \mathbb{EBL}$ and $a, b \in A$:

(E6)	$\forall 0$	$=$	0
(E7)	$\exists 1$	$=$	1
(E8)	$\forall\forall a$	$=$	$\forall a$
(E9)	$\exists\forall a$	$=$	$\forall a$
(E10)	$\exists\exists a$	$=$	$\exists a$
(E11)	$\forall\exists a$	$=$	$\exists a$

(E12)	$\exists(\exists a \vee \exists b)$	$=$	$\exists a \vee \exists b$
(E13)	$\exists(\exists a \star \exists b)$	$=$	$\exists a \star \exists b$
(E14)	$\forall(\exists a \rightarrow b)$	$=$	$\exists a \rightarrow \forall b$
(E15)	$\exists(\exists a \rightarrow \exists b)$	$=$	$\exists a \rightarrow \exists b$
(E16)	$\exists(\exists a \wedge \exists b)$	$=$	$\exists a \wedge \exists b$
(M$\forall$)	If $a \rightarrow b = 1$	then	$\forall a \rightarrow \forall b = 1$
(M$\exists$)	If $a \rightarrow b = 1$	then	$\exists a \rightarrow \exists b = 1$

Epistemic BL-Algebras. Properties.

Let $\mathbf{A} \in \mathbb{EBL}$ and $a, b \in A$:

(E6)	$\forall 0$	$=$	0
(E7)	$\exists 1$	$=$	1
(E8)	$\forall\forall a$	$=$	$\forall a$
(E9)	$\exists\forall a$	$=$	$\forall a$
(E10)	$\exists\exists a$	$=$	$\exists a$
(E11)	$\forall\exists a$	$=$	$\exists a$

(E12)	$\exists(\exists a \vee \exists b)$	$=$	$\exists a \vee \exists b$
(E13)	$\exists(\exists a \star \exists b)$	$=$	$\exists a \star \exists b$
(E14)	$\forall(\exists a \to b)$	$=$	$\exists a \to \forall b$
(E15)	$\exists(\exists a \to \exists b)$	$=$	$\exists a \to \exists b$
(E16)	$\exists(\exists a \wedge \exists b)$	$=$	$\exists a \wedge \exists b$
(M$\forall$)	If $a \to b = 1$	then	$\forall a \to \forall b = 1$
(M$\exists$)	If $a \to b = 1$	then	$\exists a \to \exists b = 1$

Theorem

If $\mathbf{A} \in \mathbb{EBL}$, then $\forall A = \exists A$ and $\exists\mathbf{A}$ is a BL-*subalgebra of* $\mathbf{A}$.

Epistemic BL-Algebras. Filters and Congruences.

Definition

A subset F of a EBL-algebra $\mathbf{A}$ is *a epistemic* BL-*filter* if F is a BL-implicative filter (i.e. a filter of its BL-reduct) and

Epistemic BL-Algebras. Filters and Congruences.

Definition
A subset F of a EBL-algebra $\mathbf{A}$ is *a epistemic* BL-*filter* if F is a BL-implicative filter (i.e. a filter of its BL-reduct) and

$$x \to y \in F \text{ implies } \forall x \to \forall y \in F \text{ and } \exists x \to \exists y \in F.$$

Epistemic BL-Algebras. Filters and Congruences.

Definition
A subset F of a EBL-algebra $\mathbf{A}$ is *a epistemic* BL-*filter* if F is a BL-implicative filter (i.e. a filter of its BL-reduct) and

$$x \to y \in F \text{ implies } \forall x \to \forall y \in F \text{ and } \exists x \to \exists y \in F.$$

Theorem

Let F be a epistemic BL-*filter of a* EBL-*algebra* $\mathbf{A}$. *Then the binary relation $\equiv_F$ on A defined by $a \equiv_F b$ if and only if $a \to b \in F$ and $b \to a \in F$ is a congruence relation. Moreover, $F = \{a \in A |\ a \equiv_F 1\}$.*

Conversely, if $\equiv$ is a congruence on A, then $F_{\equiv} = \{a \in A |\ a \equiv 1\}$ is a epistemic BL-*filter, and $a \equiv b$ if and only if $a \to b \equiv 1$ and $b \to a \equiv 1$.*

Therefore, the correspondence $F \mapsto \equiv_F$ is a bijection from the set of epistemic BL-*filters of* $\mathbf{A}$ *onto the set of congruences on* $\mathbf{A}$.

Epistemic BL-Algebras. Subdirect Representation

Theorem

Every non trivial EBL-algebra is a subdirect product of EBL-chains.

Complex Epistemic BL-Algebras

Considering a $\Pi\mathbf{A}$-frame $\mathcal{P} = \langle W, \pi \rangle$ and remembering that $\pi \in \mathbf{A}^W$, we can define its associated epistemic $\mathbf{A}$-algebra $\mathcal{A}^{\mathcal{P}} = \langle \mathbf{A}^W, \forall^{\mathcal{P}}, \exists^{\mathcal{P}} \rangle$ where $\mathbf{A}^W$ is the product algebra, and for each map $f \in \mathbf{A}^W$:

$$\begin{aligned} \forall^{\mathcal{P}}(f) &= \inf_{w \in W} \{\pi(w) \Rightarrow f(w)\} \\ \exists^{\mathcal{P}}(f) &= \sup_{w \in W} \{\pi(w) * f(w)\} \end{aligned}$$

We call an algebra with universe $\mathbf{A}^W$ a complex Epistemic $\mathbf{A}$-algebra.

Complex Epistemic BL-Algebras

Considering a $\Pi\mathbf{A}$-frame $\mathcal{P} = \langle W, \pi \rangle$ and remembering that $\pi \in \mathbf{A}^W$, we can define its associated epistemic $\mathbf{A}$-algebra $\mathcal{A}^{\mathcal{P}} = \langle \mathbf{A}^W, \forall^{\mathcal{P}}, \exists^{\mathcal{P}} \rangle$ where $\mathbf{A}^W$ is the product algebra, and for each map $f \in \mathbf{A}^W$:

$$\begin{aligned} \forall^{\mathcal{P}}(f) &= \inf_{w \in W} \{\pi(w) \Rightarrow f(w)\} \\ \exists^{\mathcal{P}}(f) &= \sup_{w \in W} \{\pi(w) * f(w)\} \end{aligned}$$

We call an algebra with universe $\mathbf{A}^W$ a complex Epistemic $\mathbf{A}$-algebra.

Theorem

Let $\mathcal{P} = \langle W, \pi, e \rangle$ be a $\Pi\mathcal{A}$-model. Then the set $\boldsymbol{E} = \{\tilde{e}(\varphi) | \varphi \in \mathcal{L}\} \subseteq \mathbf{A}^W$ is the universe of a complex Epistemic BL*-algebra, and, hence, there is a one to one correspondence between $\Pi\mathcal{A}$-models, and complex Epistemic* BL*-algebras.*

Complex Epistemic BL-algebra. Example

Consider $\mathbf{A} = [0,1]_{MV}$ and the $\Pi\mathbf{A}$-frame $\langle W, \pi \rangle$ such that $W = \mathbb{N}$ and $\pi : \mathbb{N} \to [0,1]$ defined by

$$\pi(n) = \frac{n}{n+1}$$

$f \in [0,1]^{\mathbb{N}}$	$\inf\limits_{n\in\mathbb{N}}\{f(n)\}$	$\inf\limits_{n\in\mathbb{N}}\{\pi(n) \Rightarrow f(n)\}$	$\sup\limits_{n\in\mathbb{N}}\{f(n)\}$	$\sup\limits_{n\in\mathbb{N}}\{\pi(n) * f(n)\}$

Complex Epistemic BL-algebra. Example

Consider $\mathbf{A} = [0,1]_{MV}$ and the $\Pi\mathbf{A}$-frame $\langle W, \pi \rangle$ such that $W = \mathbb{N}$ and $\pi : \mathbb{N} \to [0,1]$ defined by

$$\pi(n) = \frac{n}{n+1}$$

$f \in [0,1]^{\mathbb{N}}$	$\inf_{n\in\mathbb{N}}\{f(n)\}$	$\inf_{n\in\mathbb{N}}\{\pi(n) \Rightarrow f(n)\}$	$\sup_{n\in\mathbb{N}}\{f(n)\}$	$\sup_{n\in\mathbb{N}}\{\pi(n) * f(n)\}$
$\frac{1}{n}$	0	0	1	$\frac{1}{2}$

Complex Epistemic BL-algebra. Example

Consider $\mathbf{A} = [0,1]_{MV}$ and the $\Pi\mathbf{A}$-frame $\langle W, \pi \rangle$ such that $W = \mathbb{N}$ and $\pi : \mathbb{N} \to [0,1]$ defined by

$$\pi(n) = \frac{n}{n+1}$$

$f \in [0,1]^{\mathbb{N}}$	$\inf_{n\in\mathbb{N}}\{f(n)\}$	$\inf_{n\in\mathbb{N}}\{\pi(n) \Rightarrow f(n)\}$	$\sup_{n\in\mathbb{N}}\{f(n)\}$	$\sup_{n\in\mathbb{N}}\{\pi(n) * f(n)\}$
$\frac{1}{n}$	0	0	1	$\frac{1}{2}$
$\left(\frac{1}{2}\right)^n$	0	0	$\frac{1}{2}$	0

Complex Epistemic BL-algebra. Example

Consider $\mathbf{A} = [0,1]_{MV}$ and the $\Pi\mathbf{A}$-frame $\langle W, \pi \rangle$ such that $W = \mathbb{N}$ and $\pi : \mathbb{N} \to [0,1]$ defined by

$$\pi(n) = \frac{n}{n+1}$$

$f \in [0,1]^{\mathbb{N}}$	$\inf_{n\in\mathbb{N}}\{f(n)\}$	$\inf_{n\in\mathbb{N}}\{\pi(n) \Rightarrow f(n)\}$	$\sup_{n\in\mathbb{N}}\{f(n)\}$	$\sup_{n\in\mathbb{N}}\{\pi(n) * f(n)\}$
$\frac{1}{n}$	0	0	1	$\frac{1}{2}$
$\left(\frac{1}{2}\right)^n$	0	0	$\frac{1}{2}$	0
$1 - \left(\frac{1}{n}\right)^n$	0	$\frac{1}{2}$	1	1

Complex Epistemic BL-algebra. Example

Consider $\mathbf{A} = [0,1]_{MV}$ and the $\Pi\mathbf{A}$-frame $\langle W, \pi \rangle$ such that $W = \mathbb{N}$ and $\pi : \mathbb{N} \to [0,1]$ defined by

$$\pi(n) = \frac{n}{n+1}$$

$f \in [0,1]^{\mathbb{N}}$	$\inf_{n\in\mathbb{N}}\{f(n)\}$	$\inf_{n\in\mathbb{N}}\{\pi(n) \Rightarrow f(n)\}$	$\sup_{n\in\mathbb{N}}\{f(n)\}$	$\sup_{n\in\mathbb{N}}\{\pi(n) * f(n)\}$
$\frac{1}{n}$	0	0	1	$\frac{1}{2}$
$\left(\frac{1}{2}\right)^n$	0	0	$\frac{1}{2}$	0
$1 - \left(\frac{1}{n}\right)^n$	0	$\frac{1}{2}$	1	1
$\begin{cases} 1 & n = 1 \\ \frac{1}{2} + \frac{1}{n} & n > 1 \end{cases}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1

	Functional Monadic BL-algebra		Complex Epistemic BL-algebra	
	$\forall_\wedge f$	$\exists_\vee f$	$\forall f$	$\exists f$
$\frac{1}{n}$	0	1	0	$\frac{1}{2}$
$\left(\frac{1}{2}\right)^n$	0	0	$\frac{1}{2}$	0
$1-\left(\frac{1}{n}\right)^n$	0	$\frac{1}{2}$	1	1
$\begin{cases} 1 & n=1 \\ \frac{1}{2}+\frac{1}{n} & n>1 \end{cases}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1

c-EBL-algebras

Definition

Given a EBL-algebra $\mathbf{A}$, we will be called a c-EBL-algebra, if there is an element $c \in A$ which satisfies:

$$c = \inf_{a \in A} \{(\forall a \to a) \wedge (a \to \exists a)\}.$$

c-EBL-algebras

Definition
Given a EBL-algebra $\mathbf{A}$, we will be called a c-EBL-algebra, if there is an element $c \in A$ which satisfies:

$$c = \inf_{a \in A} \{(\forall a \to a) \wedge (a \to \exists a)\}.$$

Theorem

Let $\mathbf{A}$ *be a c-EBL-algebra such that* $\forall c = 1$ *and let* B *be the subalgebra* $\forall \mathbf{A} = \exists \mathbf{A}$*, then*

$$\forall a = \max\{b \in B : b \leq c \to a\} \quad \textit{and} \quad \exists a = \min\{b \in B : c * a \leq b\}$$

Example of c-EBL-algebra

Consider the following EBL-algebras on $L_7 = \left\{0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1\right\}$

$\forall a$	a	$\exists a$
1	1	1
$\frac{1}{2}$	$\frac{5}{6}$	1
$\frac{2}{2}$	$\frac{2}{3}$	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
0	$\frac{1}{3}$	$\frac{1}{2}$
0	$\frac{1}{6}$	$\frac{1}{2}$
0	0	0

Example of c-EBL-algebra

Consider the following EBL-algebras on $L_7 = \{0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1\}$

$\forall a$	a	$\exists a$
1	1	1
$\frac{1}{2}$	$\frac{5}{6}$	1
$\frac{2}{2}$	$\frac{2}{3}$	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
0	$\frac{1}{3}$	$\frac{1}{2}$
0	$\frac{1}{6}$	$\frac{1}{2}$
0	0	0

$\forall a$	a	$\exists a$
1	1	1
1	$\frac{5}{6}$	1
$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$
0	$\frac{1}{6}$	0
0	0	0

Example of c-EBL-algebra

Consider the following EBL-algebras on $L_7 = \{0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1\}$

$\forall a$	a	$\exists a$
1	1	1
$\frac{1}{2}$	$\frac{5}{6}$	1
$\frac{2}{2}$	$\frac{2}{3}$	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
0	$\frac{1}{3}$	$\frac{1}{2}$
0	$\frac{1}{6}$	$\frac{1}{2}$
0	0	0

$c = 1$

$\forall a$	a	$\exists a$
1	1	1
1	$\frac{5}{6}$	1
$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$
0	$\frac{1}{6}$	0
0	0	0

$c = \frac{5}{6}$

Example of c-EBL-algebra

We take the subalgebra $\forall L_7 = \{0, \frac{1}{2}, 1\}$ and $c = 1$ for the first case, then

a	$c \to a$	$\max\{b \in \forall A : b \leq c \to a\}$	$c * a$	$\min\{b \in \forall A : c * a \leq b\}$
1	1	1	1	1
$\frac{5}{6}$	$\frac{5}{6}$	$\frac{1}{2}$	$\frac{5}{6}$	1
$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{2}$
$\frac{1}{6}$	$\frac{1}{6}$	0	$\frac{1}{6}$	$\frac{1}{2}$
0	$\frac{1}{9}$	0	0	0

For the second case, with $c = \frac{5}{6}$, we obtain

a	$c \to a$	$\max\{b \in \forall A : b \leq c \to a\}$	$c * a$	$\min\{b \in \forall A : c * a \leq b\}$
1	1	1	$\frac{5}{6}$	1
$\frac{5}{6}$	1	1	$\frac{2}{3}$	1
$\frac{2}{3}$	$\frac{5}{6}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$
$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{2}$
$\frac{1}{6}$	$\frac{1}{3}$	0	0	0
0	$\frac{1}{6}$	0	0	0

Subalgebra c-relatively complete

Definition

Let $\mathbf{A}$ be a BL-algebra and $\mathbf{B}$ a BL-subalgebra of $\mathbf{A}$. We say that a pair (B, c) is a c-relatively complete subalgebra, with a fixed c, if the following conditions hold:

Subalgebra c-relatively complete

Definition

Let $\mathbf{A}$ be a BL-algebra and $\mathbf{B}$ a BL-subalgebra of $\mathbf{A}$. We say that a pair (B, c) is a c-relatively complete subalgebra, with a fixed c, if the following conditions hold:

e1 For every $a \in A$, the subset $\{b \in B : b \leq c \to a\}$ has a greatest element and the subset $\{b \in B : c * a \leq b\}$ has a least element.

Subalgebra c-relatively complete

Definition

Let $\mathbf{A}$ be a BL-algebra and $\mathbf{B}$ a BL-subalgebra of $\mathbf{A}$. We say that a pair (B, c) is a c-relatively complete subalgebra, with a fixed c, if the following conditions hold:

- e1 For every $a \in A$, the subset $\{b \in B : b \leq c \to a\}$ has a greatest element and the subset $\{b \in B : c * a \leq b\}$ has a least element.
- e2 $\{x \in A : c^2 \leq x\} \cap B = \{1\}$.

Subalgebra c-relatively complete

Definition

Let **A** be a BL-algebra and **B** a BL-subalgebra of **A**. We say that a pair (B, c) is a c-relatively complete subalgebra, with a fixed c, if the following conditions hold:

e1 For every $a \in A$, the subset $\{b \in B : b \leq c \to a\}$ has a greatest element and the subset $\{b \in B : c * a \leq b\}$ has a least element.

e2 $\{x \in A : c^2 \leq x\} \cap B = \{1\}$.

Theorem

Given a BL-*algebra* $\boldsymbol{A}$ *and a c-relatively complete subalgebra* $(\boldsymbol{B}, c)$*, if we define on* $\boldsymbol{A}$ *the operations:*

$$\forall a := \max\{b \in B : b \leq c \to a\} \qquad \exists a := \min\{b \in B : c * a \leq b\}$$

then $\langle \boldsymbol{A}, \forall, \exists \rangle$ *is an epistemic BL-algebra such that* $\forall A = \exists A = B$*. Conversely, if* $\boldsymbol{A}$ *is a c-EBL-algebra such that* $\forall c = 1$*, then* $(\forall A, c)$ *is a c-relatively complete subalgebra of* $\boldsymbol{A}$*.*

Building EBL-algebras from c-relatively complete subalgebras

Consider L_7 and $c = 1$ fixed. If we take $L_4 \hookrightarrow L_7$ is clear that (L_4, c) is a c-relatively complete subalgebra of L_7 and by the above, we can construct an EBL-algebra structure on L_7:

Building EBL-algebras from c-relatively complete subalgebras

Consider L_7 and $c = 1$ fixed. If we take $L_4 \hookrightarrow L_7$ is clear that (L_4, c) is a c-relatively complete subalgebra of L_7 and by the above, we can construct an EBL-algebra structure on L_7:

a	$c \to a$	$\max\{b \in L_4 : b \leq c \to a\}$	$c * a$	$\min\{b \in L_4 : c * a \leq b\}$
1	1	1	1	1
$\frac{5}{6}$	$\frac{5}{6}$	$\frac{2}{3}$	$\frac{5}{6}$	1
$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$
$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$\frac{1}{6}$	$\frac{1}{6}$	0	$\frac{1}{6}$	$\frac{1}{3}$
0	0	0	0	0

Building EBL-algebras from c-relatively complete subalgebras

$(L_4, 1) \hookrightarrow L_7$ determines the following EBL-algebra:

$\forall a$	a	$\exists a$
1	1	1
$\frac{2}{3}$	$\frac{5}{6}$	1
$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$
$\frac{1}{3}$	$\frac{1}{2}$	$\frac{3}{3}$
$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
0	$\frac{1}{6}$	$\frac{1}{3}$
0	0	0

Thank you for your attention